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TABLE OF CONTENTS

VOLUME 11 1910

PAGES

BECK, H., of Charlottenburg, Prussia. Ein Seitenstück zur Mobius'schen Geometrie der Kreisverwandtschaften 414

BIRKHOFF, G. D., of Princeton, N. J. A simplified treatment of the regular singular point 199

BLICHFELDT, H. F., of Stanford University, Cal. Theorems on simple groups 1

BLISS, GILBERT AMES, of Chicago, Ill. (with MAX MASON). Fields of extremals in space 325

CRAIG, CLYDE FIRMAN, of Ithaca, N. Y. On a class of hyperfuchsian functions 37

EISENHART, LUTHER PFAHLER, of Princeton, N. J. Congruences of the elliptic type 351

——— Surfaces with isothermal representation of their lines of curvature and their transformations (second memoir) 475

EVANS, GRIFFITH C., of Cambridge, Mass. Volterra's integral equation of the second kind, with discontinuous kernel 393

HOSKINS, L. M., of Stanford University, Cal. The strain of a gravitating, compressible elastic sphere 203

——— The strain of a non-gravitating sphere of variable density . 494

INGOLD, LOUIS, of Columbia, Mo. Vector interpretation of symbolic differential parameters 449

KASNER, EDWARD, of New York, N. Y. The theorem of Thomson and Tait and natural families of trajectories 121

LYTLE, ERNEST B., of Urbana, Ill. Proper multiple integrals over iterable fields 25

MACMILLAN, WILLIAM DUNCAN, of Chicago, Ill. Periodic orbits about an oblate spheroid 55

MASON, MAX, of Madison, Wis. (with G. A. BLISS). Fields of extremals in space 325

MILLER, G. A., of Urbana, Ill. Groups generated by two operators (s_1, s_2) satisfying the equation $s_1 s_2^2 = s_2 s_1^2$ 341

NEIKIRK, LEWIS IRVING, of Urbana, Ill. Groups of rational transformations in a general field 294

OWENS, F. W., of Ithaca, N. Y. The introduction of ideal elements and a new definition of projective n-space 141

198114

PAGES

RANUM, ARTHUR, of Ithaca, N. Y. The groups of classes of congruent quadratic integers with respect to a composite ideal modulus . . . 172

SMITH, PERCEY F., of New Haven, Conn. On osculating element-bands associated with loci of surface-elements 301

SNYDER, VIRGIL, of Ithaca, N. Y. Infinite discontinuous groups of birational transformations which leave certain surfaces invariant . 15

——— Conjugate line-congruences contained in a bundle of quadric surfaces 371

STUDY, EDUARD, of Bonn, Germany. Die natürlichen Gleichungen der analytischen Curven im Euclidischen Raume 249

WAHLIN, G. E., of Urbana, Ill. On the base of a relative number-field, with an application to the composition of fields 487

WESTLUND, JACOB, of Lafayette, Ind. On the fundamental number of the algebraic number-field $(k(\sqrt[p]{m})$ 388

YOUNG, JOHN WESLEY, of Lawrence, Kans. Two-dimensional chains and the associated collineations 280

THEOREMS ON SIMPLE GROUPS*

BY

H. F. BLICHFELDT

Introduction and Terminology.

§ 1. It is our purpose in this paper to state and prove a somewhat more precise theorem on simple groups than that given by FROBENIUS in his article *Ueber auflösbare Gruppen. V.*† This theorem of FROBENIUS we shall state in the following form: "Let H be a group whose order is divisible by p^λ, p being a prime. If H is simple, then there is in H a substitution T_1 whose order is prime to p, which is commutative with some subgroup Q of H of order p^μ and not commutative with every subgroup of Q. In the contrary case, H has an invariant subgroup of index p^λ, containing all the substitutions of H whose orders are prime to p."

The theorem which we shall here establish (divided for convenience into four parts, I, II, III, IV) defines the group Q more explicitly, and the nature of the relation of T_1 to Q in the most important case, namely, when Q is a Sylow subgroup of H. Several applications are added in the form of corollaries.

The phraseology of ordinary group theory will be employed. The following abbreviations are used:

$A = B$ means "the groups (substitutions) A and B are identical";

$A < B$ means "the group (substitution) A is contained in the group B, though it is not identical with B."

The letters H, P, P_1 and T have the following significance throughout:

H represents a group of order $p^\lambda n$, p being a prime number not dividing n;

P represents any given subgroup of H of order p^λ;

P_1 represents any given subgroup of P of order $p^{\lambda-1}$; and

T is the general symbol for a substitution of H whose order is prime to p.

Though it is contrary to general usage, we shall proceed from right to left in a succession of substitutions indicated. A function subjected to a substitution or a succession of such shall be written to the right of the substitutions indi-

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* The present paper is an extension of two papers read before the San Francisco section, viz., *A theorem concerning the Sylow subgroups of simple groups*, September 29, 1906; and *A theorem on simple groups*, September 26, 1908.

†Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, 1901, p. 1324.

cated, enclosed in parentheses. Thus, if A and B represent substitutions, I a function, the symbol $AB(I)$ represents the function obtained by first subjecting I to the substitution B, and then subjecting the result to the substitution A.

Theorems and corollaries.

§ 2. **Theorem I.** *Among the substitutions contained in P, but not in P_1, let S be any one of those of lowest order. Then, if H is simple, there are in H a group Q (either $= P$ or $< P$, containing S) and a substitution T, say T_1, of the following nature:*

1°) *If $Q = P$, this substitution T_1 is commutative with P but not with S; more generally, the commutator $S^{-1}T_1^{-1}ST_1$ is not in P_1.*

2°) *If $Q < P$, then T_1 is commutative with Q, but not with every subgroup of Q of index p.*

Theorem II. *If $Q < P$, there is a group $Q_1 \leqq Q$ containing S, and there is a certain series of groups*

$$Q_1,\ Q_2,\ Q_3,\ \cdots,\ Q_{\lambda-\nu+1} = P$$

of orders $p^\nu, p^{\nu+1}, p^{\nu+2}, \cdots, p^\lambda$, respectively, each a subgroup of the one to the right of it, possessing the following property. Let R_i be that group whose substitutions are common to all the subgroups of Q_i of index p. Then, if A_i be any substitution of Q_i but not of Q_{i-1}, $i > 1$, the substitution

$$(SA_i)^p A_i^{-p},$$

though a substitution of Q_{i-1}, does not belong to R_{i-1}.

Corollary 1. *Every substitution of P commutative with S must belong to Q.* More generally, if $p > 2$, every substitution A of P, such that the substitutions A and SAS^{-1} are commutative, must belong to Q. Hence, *if $p > 2$, every abelian subgroup of P which is transformed into itself by S, must belong to Q.*

We prove this corollary from Theorem II by making use of the facts that R_{i-1} is invariant in Q_i; that every commutator of Q_{i-1} belongs to R_{i-1}; that the pth powers of the substitutions of Q_{i-1} belong to R_{i-1}.

Theorem III.

α) *If $Q_1 < P$, then the order p^λ of P is $\geqq p^{p+1}$, or is $\geqq p^{p^2}$, according as the order of Q_1 is $= p^{\lambda-1}$, or is $< p^{\lambda-1}$.*

β) *If the order of P is $< p^{2p-1}$, then there is in P some subgroup P_1 and substitution S for which the corresponding group $Q = P$.*

Corollary 2. *The group Q is certainly $= P$ in the cases where P*

1°) *is abelian* (Theorem II),

2°) *or contains no substitutions of order p^2* (Theorem II),

3°) *or is of order p^p at most* (Theorem III).

In any of these cases H will therefore contain a subgroup M which contains P invariantively. Any one of the substitutions S of P (defined in Theorem I) is non-commutative with some substitution T of M.

In particular, if P fulfills both 1°) and 2°), then no one of its substitutions (except identity) is invariant in M.

Corollary 3. *Let p be the lowest prime which divides the order of A, and let it be given that the corresponding subgroup P is abelian.* Let a set of independent generating substitutions of P be constructed; then, if one of these generators be of order p^a, there must be at least three generators of order p^a if $p > 2$; at least two if $p = 2$.*

Hence, if $p > 2$, and if the order of P be $\leqq p^8$, then P must be of type $(1, 1, \cdots, 1)$ in every case, with the exception of the possibility $(2, 2, 2)$. Again, if $p = 2$, and the order of P be $\leqq p^6$, the only permissible types are the following: $(1, 1)$, $(1, 1, 1)$, $(1, 1, 1, 1)$, $(1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 1, 1)$, $(2, 2)$, $(2, 2, 1, 1)$, $(2, 2, 2)$, $(3, 3)$.

To prove this corollary, let $A_1, A_2, \cdots$ be the generators of order p^a; $B_1, B_2, \cdots$ the generators whose orders, $p^{b_1}, p^{b_2}, \cdots$ are $> p^a$. The group K, generated by the substitutions $B_1^{p^{b_1-1}}, B_2^{p^{b_2-1}}, \cdots, A_1^{p^{a-1}}, A_2^{p^{a-1}}, \cdots$, is a characteristic subgroup of P.† Moreover, the group K', generated by $B_1^{p^{b_1-1}}, B_2^{p^{b_2-1}}, \cdots$ (leaving out of K the generators $A_1^{p^{a-1}}, \cdots$), is a characteristic subgroup of P. Let the factor-group K/K' be denoted by L, and its generating substitutions by $a_1, a_2, \cdots$, corresponding to $A_1, A_2 \cdots$.

In applying Theorem I, 1°), let P_1 be that group generated by those generators of P whose orders are higher or lower than p^a, and by $A_1^p, A_2, \cdots$. Then we may put $S = A_1$, and we must have

$$T_1^{-1} A_1 T_1 = A_1^x C, \qquad x \not\equiv 1 \pmod{p},$$

where C belongs to P_1.

Accordingly,

$$T_1^{-1} A_1^{p^{a-1}} T_1 = (A_1^{p^{a-1}})^x C^{p^{a-1}}.$$

Since T_1 is commutative with L, we have correspondingly

$$T_1^{-1} a_1 T_1 = a_1^x c,$$

c belonging to that subgroup of L generated by $a_2, \cdots$. Hence, if there were only two generators, a_1, a_2, in L, then the order of T_1 would be a factor of $p^2 - 1$. But this number is divisible by no prime $> p$ except when $p = 2$.

* The more general case where P is not abelian, if it be only such that, for every P_1 and S, $Q = P$ (cf. Cor. 2), is embodied in this corollary when $B_1, \cdots, A_1, \cdots$, instead of being the generators of P, are the generators of the factor-group P/W, W being the commutator subgroup of P.

† FROBENIUS, *Ueber auflösbare Gruppen.* II, Sitzungsberichte der Akademie der Wissenschaften zu Berlin (1895), pp. 1028–1029. BURNSIDE, *Theory of Groups*, pp. 233–235.

At the end of his paper on soluble groups, referred to in the introduction,* FROBENIUS proves that every group of order $p^\lambda qr$ is soluble, if $\lambda \leqq 4$; p, q, r being odd primes. Using his arguments in conjunction with Theorems I–III we find the

Corollary 4. *Every group of order $p^\lambda qr$ is soluble, if $\lambda \leqq p^2 - 1$; p, q, r being different odd primes.*

Theorem IV. *If the conditions as specified for a simple group H in Theorems* I, II *and* III *are not all fulfilled with reference to S and P, then H has an invariant subgroup H_1 of index p, which does not contain S, but contains every substitution T of H.*

Proof of the theorems.

§ 3. Let P' be the largest subgroup of H whose substitutions are commutative with P, and which, besides, transform S into itself or into $S\times$ (a substitution of P_1). This group P' is of order $p^\lambda n'$, n' being a factor of n. We shall prove that

A) The group P' contains an invariant subgroup P'_1 of order $p^{\lambda-1}$, to which S does not belong; further, no substitution of P, not belonging to P'_1, is of lower order than S.

We write P' as a regular group in $p^\lambda n'$ letters

$$x_1,\ x_2,\ \cdots,\ x_{p^\lambda n'}.$$

The substitutions of P_1 will transform x_1 into $p^{\lambda-1}$ different letters, say

$$x_1,\ x_2,\ \cdots,\ x_{p^{\lambda-1}}.$$

The functions

$$(1) \qquad I_1 = x_1 + x_2 + \cdots + x_{p^{\lambda-1}},\ S(I_1),\ S^2(I_1),\ \cdots,\ S^{p-1}(I_1)$$

are all absolute invariants for the group P_1. Let θ be a root different from 1, of the equation

$$\theta^p - 1 = 0,$$

and let us consider the function

$$I_2 = I_1 + \theta^{-1}S(I_1) + \theta^{-2}S^2(I_1) + \cdots + \theta^{-p+1}S^{p-1}(I_1).$$

It is not identically zero, no two of the functions (1) containing the same letter x_i. It is an absolute invariant for the group P_1, and we have

$$S(I_2) = \theta I_2.$$

We shall introduce the following abbreviations:

ζ means a power of θ; and

* Loc. cit., p. 1329.

ϕ means a power of θ, different from 1.

Then if A is any substitution of P,

$$A(I_2) = \zeta I_2, \qquad S(I_2) = \phi I_2.$$

§ 4. When the group P' is written symbolically in the form

$$P' = P + B_2 P + B_3 P + \cdots + B_{n'} P,$$

the function

$$I_3 = I_2 \cdot B_2(I_2) \cdot B_3(I_2) \cdot \cdots \cdot B_{n'}(I_2) \tag{2}$$

is seen to be a relative invariant for P'. We find that

$$S(I_3) = \theta^{n'} I_3 = \phi I_3.$$

It follows that all the substitutions of P' for which I_3 is an absolute invariant form an invariant subgroup P'' of order $p^{\lambda-1}n'$, not containing S. This subgroup has a subgroup P_1' of order $p^{\lambda-1}$, also invariant in P'. The first part of (A), § 3, has thus been proved.

To prove the second part, we remark first that, as P is invariant in P', all the factors of I_3 are relative invariants for P. Hence, if A be a substitution of P and not of P_1', the condition

$$A(I_3) = \phi I_3$$

necessitates that, for some factor $B_i(I_2)$ of the right-hand member of (2), the relation shall subsist:

$$AB_i(I_2) = \phi B_i(I_2).$$

This gives

$$B_i^{-1} A B_i(I_2) = \phi I_2.$$

It follows that $B_i^{-1}AB_i$, though a substitution of P, does not belong to P_1. Hence its order is not lower than that of S (cf. Theorem I). Hence the order of A is not lower than that of S.

§ 5. We shall now write H as a regular group in $h = p^\lambda n$ letters, $x_1, x_2, \cdots, x_h$. The substitutions of P'' will transform x_1 into $p^{\lambda-1}n'$ letters, say $x_1, x_2, \cdots, x_{p^{\lambda-1}n'}$. Let $I = x_1 + x_2 + \cdots + x_{p^{\lambda-1}n'}$, and let us consider the function

$$J = I + \theta^{-1}S(I) + \theta^{-2}S^2(I) + \cdots + \theta^{-p+1}S^{p-1}(I),$$

which is not identically zero.

B) The group P' contains all the substitutions of H for which J is a relative invariant.

Let W transform x_a into x_b, both occurring in J. Since the letters involved in J form a transitive set for the group P', we can find a substitution in the latter, say V, which also transforms x_a into x_b. The substitution VW^{-1} will

therefore leave x_b unchanged. Since H is written in regular form, it follows that VW^{-1} is identity. *If J be a relative invariant for W, and the order of W be a power of p, then W belongs to P.*

§ 6. The substitutions of H will transform J into n/n' expressions that do not differ by constant factors merely. The product π of these factors will be a relative or absolute invariant for H. Evidently, $A(\pi) = \zeta\pi$, A being any substitution of H. If

$$S(\pi) = \phi\pi,$$

then H contains an invariant subgroup H_1 of index p, consisting of all the substitutions C for which

$$C(\pi) = \pi.$$

To this subgroup would belong every substitution T of H. Accordingly, *unless the group H_1 of Theorem IV exists, we must have*

$$S(\pi) = \pi. \tag{3}$$

§ 7. To examine the nature of the factors of π, let us write H symbolically in the form

$$H = P' + PV_1 + PV_2 + \cdots + PV_{n-n'}. \tag{4}$$

The group P', operating upon J, will furnish just one of the factors of π, namely J. The p^λ substitutions represented by the symbol PV_i will furnish, say, p^{k_i} factors of π, whose product we shall indicate by π_i. Now, for $j \neq i$, one of the factors of π_j may be equal, apart from a constant multiplier, to one of the factors of π_i. In such a case we find $\pi_j = \pi_i \times$ (a constant). We shall then agree to say that $\pi_j = 1$, if $j > i$. With this understanding we may write

$$\pi = J\pi_1\pi_2 \cdots \pi_{n-n'}.$$

Since $S(J) = \theta J$, $S(\pi_i) = \zeta\pi_i$, it follows that, if (3) is to hold, there is at least one index ρ such that

$$S(\pi_\rho) = \phi\pi_\rho. \tag{5}$$

§ 8. First, let π_ρ consist of a single factor. Then the alternative case 1°) of Theorem I is proved in the following manner. Let V be one of the substitutions PV_ρ of (4). Then we may suppose that

$$\pi_\rho = V(J).$$

The function π_ρ is here a relative invariant for P. Hence, J is a relative invariant for $V^{-1}PV$. Hence, by (B), § 5,

$$V^{-1}PV = P.$$

Now, since V does not belong to P', the group generated by V and P', though containing P invariantively, does not fulfill the other conditions imposed upon P' in § 3. Case 1°) of Theorem I thus follows.

§ 9. Next, let π_ρ consist of more than one factor,

$$\pi_\rho = y_1 y_2 \cdots y_{p^k}. \tag{6}$$

C) Then we shall prove that *V, one of the substitutions of the symbol PV_ρ, transforms a certain subgroup Q_1 of P, which contains S, into itself or into another subgroup of P. Moreover, V does not transform that subgroup of Q_1 which is also a subgroup of P'_1 into itself or into another subgroup of P'_1.*

Substituting (6) in (5), it might happen that, for some index j, we have $S(y_j) \neq \zeta y_j$. Then we may suppose that

$$S(y_j) = y_{j+1},\ S(y_{j+1}) = y_{j+2},\ \cdots,\ S(y_{j-1+p^l}) = \theta' y_j \qquad (\theta'^p = 1).$$

Thus, if $y_j = W(J)$, we get

$$W^{-1} S^{p^l} W(J) = \theta' J. \tag{7}$$

Hence (§ 5, *B*) $W^{-1} S^{p^l} W < P$.

This substitution, being of lower order than S, must belong to P'_1 (§ 3, *A*). The function J is, however, an absolute invariant for P'' and therefore for P'_1 (cf. §§ 4, 5). Accordingly $\theta' = 1$.

§ 10. The p^k factors of π_ρ (6), permuted transitively by P, fall into p sets of p^{k-1} factors each, forming imprimitive sets. Let the p products of the factors of these sets be indicated by

$$a_1,\ a_2,\ \cdots,\ a_p; \qquad a_1 a_2 \cdots a_p = \pi_\rho.$$

Substituting in (5), either we have

$$S(a_i) = \zeta a_i \qquad (i = 1, 2, \cdots, p); \tag{8}$$

or we may put

$$S(a_1) = a_2,\ S(a_2) = a_3,\ \cdots,\ S(a_p) = \phi a_1.$$

The latter relations would, however, lead to an equation like (7), $\theta' \neq 1$, contrary to what was proved in § 9. Accordingly, (8) is true, and $\zeta = \phi$ for some index i, say $i = 1$:

$$S(a_1) = \phi a_1. \tag{9}$$

That subgroup of P for which $a_1, a_2, \cdots, a_p$ are relative invariants shall be designated by $Q_{\lambda-\nu}$, where $\nu = \lambda - k$. Its order is $p^{\lambda-1}$.

The p^{k-1} factors of a_1 fall into p sets of imprimitivity for the group $Q_{\lambda-\nu}$. The respective products shall be indicated by

$$b_1,\ b_2,\ \cdots,\ b_p; \qquad b_1 b_2 \cdots b_p = a_1.$$

We may assume that $S(b_1) = \phi b_1$. The factors $b_1, b_2, \cdots, b_p$ are relative invariants for a group $Q_{\lambda-\nu-1} < Q_{\lambda-\nu}$, of order $p^{\lambda-2}$. Proceeding thus, we finally arrive at groups Q_3, Q_2, Q_1, of orders $p^{\nu+2}, p^{\nu+1}, p^{\nu}$ respectively, having sets of relative invariants respectively designated by

$$w_1, w_2, \cdots, w_p; \qquad v_1, v_2, \cdots, v_p; \qquad y_1, y_2, \cdots, y_p.$$

These invariants satisfy the relations

$$w_1 = v_1 v_2 \cdots v_p; \qquad v_1 = y_1 y_2 \cdots y_p.$$

It will be noticed that S belongs to all of these groups, and we have, corresponding to (9),

$$S(w_1) = \phi w_1, \ S(v_1) = \phi v_1, \ S(y_1) = \phi y_1.$$

§ 11. Let $y_1 = V(J)$, V being one of the p^{λ} substitutions $P V_\rho$ [(4), § 7]. Then J is a relative invariant for the group $V^{-1} Q_1 V$, which must therefore be a subgroup of P (§ 5). The first statement in (C), § 9, is thus true.

Let A be a substitution of Q_2, but not of Q_1. Then we may suppose that

$$y_i = A^{-i+1}(y_1) \qquad (i = 2, \cdots, p);$$

and we get

$$\phi = \frac{S(v_1)}{v_1} = \frac{S(y_1)}{y_1} \cdot \frac{S(y_2)}{y_2} \cdots = \frac{S \cdot ASA^{-1} \cdot A^2SA^{-2} \cdot \cdots \cdot A^{p-1}SA^{-p+1}(y_1)}{y_1}$$

$$(10) \qquad = \frac{(SA)^p A^{-p}(y_1)}{y_1} = \frac{V^{-1}(SA)^p A^{-p} V(J)}{J}.$$

Now $(SA)^p A^{-p} < Q_1$, and is also contained in every subgroup of P of order $p^{\lambda-1}$. It is therefore contained in P_1'. For this group J is an absolute invariant. Therefore, if the second part of (C), § 9, were not true, then (10) could not be true.

§ 12. We shall now prove case 2°) of Theorem I. Let $Q^{(0)}$ be any subgroup of P such that $Q^{(0)} \geqq Q_1$, and let

$$(11) \qquad Q^{(0)}, Q^{(1)}, Q^{(2)}, \cdots, Q^{(n)} = P$$

be that series of groups in which each member is the largest subgroup of P containing the one immediately to the left of it invariantively. If H contains a substitution T which is commutative with one of these groups, say $Q^{(m)}$, $m > 0$, without being commutative with $Q^{(m-1)}$, then 2°) of Theorem I is true; the group Q in the theorem representing a certain group $> Q^{(m-1)}$ and $\leqq Q^{(m)}$. We shall therefore assume that every substitution P which is commutative with $Q^{(m)}$ of (11), or of any similar series, is commutative also with Q^{m-1}, $m > 0$. Under

this assumption the following theorem by Frobenius holds: If K represents the greatest subgroup of H in which $Q^{(k)}$ is invariant $(n > k \geqq 0)$, then $Q^{(k+1)}$ is a Sylow subgroup of K.*

D) By means of this theorem we can prove that, *if V be the substitution considered in* §§ 9, 11, *we may write*

$$V = W_1 W_2 W_3 \cdots W_r,$$

where $W_1, W_2, \cdots, W_r$ are substitutions of H, respectively commutative with the following members of a certain series of groups:

$$L_1 = Q_1,\ L_2,\ L_3,\ \cdots,\ L_r = P, \tag{12}$$

each a subgroup of those that follow.

§ 13. To prove (D), let $VPV^{-1} = M$. Then M and P have in common the group Q_1, but no greater group. Let Q_1' be the greatest subgroup of P containing Q_1 invariantively, and Q_1'' the corresponding subgroup of M. The groups Q_1' and Q_1'' will generate a group N in which Q_1' is a Sylow subgroup by Frobenius's theorem, § 12. There is therefore in N a substitution W_1 such that $Q_{1,1} = W_1^{-1} Q_1'' W_1 \leqq Q_1'$, $Q_{1,1} > Q_1$. We have also $W_1^{-1} Q_1 W_1 = Q_1$. Let now $(W_1^{-1} V) P (W_1^{-1} V)^{-1} = M_1$. Then P and M_1 have in common a subgroup, say L_2, $\geqq Q_{1,1}$, and therefore $L_2 > Q_1$. If $L_2 = P$, then (D) is proved, the series (12) consisting of the two terms Q_1, P; in this case we find

$$V = W_1 W_2,$$

where W_2 is a certain substitution of H commutative with P.

If $L_2 < P$, let L_2' be the largest subgroup of P containing L_2 invariantively, L_2'' the corresponding subgroup of M_1. The groups L_2' and L_2'' generate a group in which L_2' is a Sylow subgroup, etc. Proceeding as above, the proof of (D) may be completed without difficulty.

§ 14. To complete the proof of case 2°) of Theorem I, let us consider the substitutions $W_1, W_2, \cdots, W_r$, respectively commutative with the groups of (12). If they were at the same time commutative with the corresponding subgroups of P_1', i. e., if W_i, which is commutative with L_i, is also commutative with that subgroup (L_i^0) of L_i which is a subgroup of P_1', then (C), § 9, would not be true. For, the group

$$V^{-1} L_1^0 V = W_r^{-1} \{ \cdots [W_2^{-1} (W_1^{-1} L_1^0 W_1) W_2] \cdots \} W_r$$

* Paper referred to in § 1, p. 1325, Lemma I. The above theorem is apparently more general than that given by FROBENIUS, account being taken of the underlying assumptions as stated in both places. The method of proof by FROBENIUS is valid for the case considered here.

would then be a subgroup of P_1'. Accordingly, for some subscript i, there is a substitution, W_i, commutative with L_i, but not with L_i^0.

Now if G be the largest subgroup of H in which L_i is invariant, then L_i' is a Sylow subgroup of G (§§ 12, 13). Evidently, L_i^0 is invariant in L_i'. Since L_i^0 is not invariant in G (W_i belongs to G), it follows that there is a substitution T in G which is not commutative with L_i^0. The group L_i takes the place of Q in Theorem I, 2°).

§ 15. To prove Theorem II, let

$$Q_1,\ Q_2,\ \cdots,\ Q_{\lambda-\nu},\ P$$

be the series of groups arrived at in § 10. Let $A_{\lambda-\nu+1}$ be a substitution belonging to P, but not to $Q_{\lambda-\nu}$. Then we may suppose that

$$a_i = A_{\lambda-\nu+1}^{-i+1}(a_1) \qquad (i=2,\ 3,\ \cdots,\ p).$$

We obtain

$$\phi = \frac{S(\pi_\rho)}{\pi_\rho} = \frac{1}{a_1}\left[S\cdot A_{\lambda-\nu+1}SA_{\lambda-\nu+1}^{-1}\cdot A_{\lambda-\nu+1}^{2}SA_{\lambda-\nu+1}^{-2}\cdot\cdots\cdot(a_1)\right]$$

$$= \frac{1}{a_1}(SA_{\lambda-\nu+1})^p A_{\lambda-\nu+1}^{-p}(a_1).$$

The function a_1, being a relative invariant for $Q_{\lambda-\nu}$, is plainly an absolute invariant for the group $R_{\lambda-\nu}$, whose substitutions are common to all the subgroups of $Q_{\lambda-\nu}$ of order $p^{\lambda-2}$. It follows that $(SA_{\lambda-\nu+1})^p A_{\lambda-\nu+1}^{-p}$ does not belong to $R_{\lambda-\nu}$.

Starting now with a substitution $A_{\lambda-\nu}$ belonging to $Q_{\lambda-\nu}$ but not to $Q_{\lambda-\nu+1}$, we may suppose

$$b_i = A_{\lambda-\nu}^{-i+1}(b_1) \qquad (i=2,\ 3,\ \cdots,\ p).$$

Substituting in $\phi = S(a_1)/a_1$, we find that $(SA_{\lambda-\nu})^p A_{\lambda-\nu}^{-p}$ does not belong to $R_{\lambda-\nu-1}$, etc.

The group Q_1 is $\leqq Q$ of Theorem I, 2°), by §§ 12–14.

§ 16. We proceed to prove Theorem III. Considering the groups $Q_1, Q_2, \cdots, P$ of § 10, let A designate a substitution belonging to Q_2, but not to Q_1. We will assume that A permutes the letters $y_1, y_2, \cdots, y_p$ in the order

$$A: \qquad (y_1 y_2 \cdots y_p),$$

and that

$$S(y_i) = \theta^{\alpha_i}(y_i) \qquad (i=1,\ 2,\ \cdots,\ p).$$

Then, since $S(v_1) = \phi v_1$,

$$\alpha_1 + \alpha_2 + \cdots + \alpha_p \not\equiv 0 \pmod{p}.$$

We shall say that *the constants* $\theta^{\alpha_1}, \theta^{\alpha_2}, \cdots, \theta^{\alpha_p}$ *are the multipliers of* S.

The substitution

$$S' = S^{x_1} \cdot A^{-1} S^{x_2} A \cdot A^{-2} S^{x_3} A^2 \cdot \cdots \cdot A^{-p+1} S^{x_p} A^{p-1},$$

belonging to Q_1, will have the multipliers θ, 1, 1, $\cdots$, 1, provided the congruences

$$\begin{aligned} \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_p x_p &\equiv 1, \\ \alpha_2 x_1 + \alpha_3 x_2 + \cdots + \alpha_1 x_p &\equiv 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \alpha_p x_1 + \alpha_1 x_2 + \cdots + \alpha_{p-1} x_p &\equiv 0 \pmod{p} \end{aligned}$$

are satisfied. The necessary condition is that the determinant Δ of the coefficients does not vanish (mod p). We find

$$\begin{aligned} \Delta &= (\alpha_1 + \theta\alpha_2 + \theta^2\alpha_3 + \cdots)(\alpha_1 + \theta^2\alpha_2 + \theta^4\alpha_3 + \cdots)\cdots \\ &\equiv \Delta^p \equiv (\alpha_1^p + \alpha_2^p + \alpha_3^p + \cdots)^p \equiv \alpha_1 + \alpha_2 + \alpha_3 + \cdots \not\equiv 0 \pmod{p}. \end{aligned}$$

The substitutions S', $A^{-1}S'A$, $A^{-2}S'A^2$, $\cdots$, all belonging to Q_1, are readily seen to generate a group of order p^p at least. The group Q_2 is accordingly of order p^{p+1} at least. Theorem III, α), is thus true when $Q_2 = P$.

§ 17. Let $Q_2 < P$. The functions v_1, v_2, $\cdots$, v_p (§ 10) contain p^2 factors y_1, y_2, $\cdots$. We may assume that

$$v_i = y_{(i-1)p+1} y_{(i-1)p+2} \cdots y_{ip} \qquad (i = 1, 2, \cdots, p).$$

We shall say that *the p factors of v_i constitute the set V_i*. The letter B shall designate a substitution belonging to Q_3 but not to Q_2. It permutes the letters v_1, v_2, $\cdots$, v_p in some such order as

$$B: \qquad (v_1 v_2 \cdots v_p).$$

Let R represent a substitution of the group Q_2. It has v_i for a relative invariant. We may assume that it permutes the members of the set V_i in the order $(y_{(i-1)p+1} y_{(i-1)p+2} \cdots y_{ip})$, or some power of this order,

$$R: \qquad (y_{(i-1)p+1} y_{(i-1)p+2} \cdots y_{ip})^n.$$

If $n = p$, i. e., if the functions $y_{(i-1)p+1}$, $\cdots$ are relative invariants for R, we shall say that *R is linear in V_i*. Otherwise we shall say that *R is circular in V_i*. Thus, S is linear in V_1, and A is circular in V_1.

§ 18. Let us now suppose that R is linear in V_i and write

$$R(y_{(i-1)p+j}) = \theta^{\beta_j} y_{(i-1)p+j} \qquad (j = 1, 2, \cdots, p).$$

We shall say that R *is of rank* m *in* V_i *if* $\beta_1 + \theta\beta_2 + \theta^2\beta_3 + \cdots + \theta^{p-1}\beta_p$ *is divisible by* $(1-\theta)^m$, *but by no higher power of* $1-\theta$; $\beta_1, \beta_2, \cdots, \beta_p$ *not being all equal* $(mod\ p)$. *If* $\beta_1 \equiv \beta_2 \equiv \cdots \equiv \beta_p \not\equiv 0\ (mod\ p)$, *we shall say that* R *is of rank* $p-1$; *if* $\beta_1 \equiv \beta_2 \equiv \cdots \equiv \beta_p \equiv 0\ (mod\ p)$, *we shall say that* R *is of rank* p.

Let A_1 be any substitution contained in Q_2. Then $A_1 R A_1^{-1} R^{-1}$ is linear in V_i. It is of rank p if A_1 is linear in V_i; if A_1 is circular, the commutator in question is of rank $m+1$ if $m < p$, and of rank p if $m = p$.

§ 19. We can construct by the method of § 16, using S and B, a substitution S_1, $< Q_2$, such that

$$S_1(v_1) = \theta v_1, \qquad S_1(v_i) = v_i \qquad (i = 2, 3, \cdots, p).$$

The following three cases may arise:

(1) S_1 is linear in $V_1, V_2, \cdots, V_p$;
(2) S_1 is circular in $V_1, V_2, \cdots, V_p$;
(3) S_1 is partly linear and partly circular.

Consider case (1). The substitution S_1 is of rank 0 in V_1 and of rank $\geqq 1$ in $V_2, V_3, \cdots, V_p$. The substitution $S_2 = A S_1 A^{-1} S_1^{-1}$ is of rank 1 in V_1 and of rank $\geqq 2$ in $V_2, V_3, \cdots$. The substitution $S_3 = A S_2 A^{-1} S_2^{-1}$ is of rank 2 in V_1 and of rank $\geqq 3$ in $V_2, V_3, \cdots$. Proceeding thus, we get a series of substitutions of which the last, $S_p = A S_{p-1} A^{-1} S_{p-1}^{-1}$, is of rank $p-1$ in V_1 and of rank p in $V_2, V_3, \cdots$.

The substitutions

$$S_p,\ B S_p B^{-1},\ B^2 S_p B^{-2},\ \cdots,\ B^{p-1} S_p B^{-p+1}$$

will generate a group Q' of order p^p at least. This group does not contain S_{p-1}. The substitutions

$$S_{p-1},\ B S_{p-1} B^{-1},\ B^2 S_{p-1} B^{-2},\ \cdots,\ B^{p-1} S_{p-1} B^{-p+1}$$

and the group Q' will generate a group Q'' of order p^{2p} at least, etc. Finally, we have a group $Q^{(p)}$, of order $\geqq p^{p^2}$, linear in $V_1, V_2, \cdots, V_p$, which group with A and B generate a group of order $\geqq p^{p^2+2}$.

§ 20. Consider case (2). Constructing the substitutions

$$R_1 = B S_1 B^{-1}, \qquad R_2 = S_1 R_1 S_1^{-1} R_1^{-1},$$

we find that R_2 is linear in $V_1, V_2, \cdots, V_p$, of rank 1 in V_1 and V_2 and of rank $\geqq 2$ in $V_3, V_4, \cdots, V_p$.

Now, S is linear in V_1 (§ 17). It cannot be linear in all the sets

$V_2, V_3, \cdots$, or S_1 would be linear throughout. Hence there is a substitution, $W = B^n S B^{-n}$, which is circular in V_1 and linear in V_2. Then the substitution $R_3 = W R_2 W^{-1} R_2^{-1}$ is linear in all the sets $V_1, V_2, \cdots$. It is of rank 2 in V_1 and of rank $\geqq 3$ in the remaining sets. Proceeding as in § 19, we construct a group $Q^{(p-2)}$ of order $\geqq p^{(p-2)p}$, linear throughout, a subgroup in the group generated by A, B and R_3. This group $Q^{(p-2)}$ and the substitutions R_2, BR_2B^{-1}, $B^2R_2B^{-2}, \cdots$ generate a group $Q^{(p-1)}$ of order $\geqq p^{p^2-p-1}$, linear throughout. This group and the substitutions S_1, BS_1B^{-1}, $B^2S_1B^{-2}, \cdots$ generate a group $\leqq Q_2$, of order $\geqq p^{p^2-1}$. Accordingly, the order of Q_3 is $\geqq p^{p^2}$.

The same processes, with slight variations, will dispose of case (3). Theorem III, α), will be found true.

§ 21. In proving Theorem III, β), we may assume that $Q_2 = P$.

There is a substitution $S' < Q_2$, but not $< Q_1$, which can take the place of S in Theorem I, Q_1 being substituted for P_1 in the theorem, and a corresponding group Q' taking the place of Q_1 in Theorems II and III, α). If $P = Q'$, Theorem III, β) is true. If $P > Q'$, then there exists a set of functions

$$V_1' : \qquad y_1', y_2', \cdots, y_p',$$

corresponding to the set V_1 defined in § 17. These functions are permuted transitively by P; otherwise the order of this group would be $\geqq p^{p^2}$ [Theorem III, α)]. We will look upon P as transforming simultaneously the two sets V_1 and V_1', making use of the results of § 16.

We may evidently assume $S' = A$ of § 16. *Then A is linear and of rank 0 in V_1', and is circular in V_1.*

There must be a substitution A' which is circular in V_1'. We can evidently find an integer n such that the substitution $A'A^n$ is linear in V_1. Again, such numbers m and k can be found that the substitution

$$R = (A'A^n)^k S^m$$

is linear and of rank 0 in V_1 and is circular in V_1'.

The substitutions A and R will now generate a group of order $\geqq p^{2p-1}$. To prove this, we construct first $R_1 = ARA^{-1}R^{-1}$, which is linear and of rank 1 in both V_1 and V_1'; then the substitutions

$$R_2 = AR_1A^{-1}R_1^{-1}, \; R_3 = AR_2A^{-1}R_2^{-1}, \; \cdots,$$

which are all linear, of rank p in V_1', and of ranks 2, 3, $\cdots$, respectively, in V_1; finally, we construct the substitutions

$$W_2 = RR_1R^{-1}R_1^{-1}, \; W_3 = RW_2R^{-1}W_2^{-1}, \; W_4 = RW_3R^{-1}W_3^{-1}, \; \cdots,$$

all linear, of rank p in V_1, and of ranks 2, 3, 4, $\cdots$, respectively, in V_1'.

The substitutions $R_1, R_2, R_3, \cdots; W_2, W_3, \cdots$ will generate a group of order $\geqq p^{2p-3}$, linear in V_1 and V_1'. Again this group together with A and R will generate a group of order $\geqq p^{2p-1}$.

§ 22. Theorem IV is proved as follows. The invariant subgroup H_1 exists unless the condition $S(\pi) = \pi$ is satisfied (cf. § 6). But Theorems I, II and III were true in consequence of this condition.

INFINITE DISCONTINUOUS GROUPS OF BIRATIONAL TRANSFORMATIONS WHICH LEAVE CERTAIN SURFACES INVARIANT*

BY

VIRGIL SNYDER

It was shown by SCHWARZ † that no algebraic curve of genus greater than unity can remain invariant under a continuous group of birational transformations. Later HURWITZ ‡ showed that no such curve could belong to any birational group of infinite order.

The corresponding theory for surfaces is by no means complete. While those belonging to continuous groups have been determined, only a few isolated examples are known of surfaces having an infinite discontinuous group. §

All the groups which have been discussed are illustrations of two principles, the first of which refers to quartic surfaces and will be considered in two parts, the latter including the former as a particular case; the second principle is applicable to a much wider category. I propose to discuss these principles and apply the second to the determination of an extended family of new surfaces having an infinite group.

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† *Ueber diejenigen algebraischen Gleichungen zwischen zwei veränderlichen Grössen, welche eine Schaar rationaler eindeutig umkehrbarer Transformationen in sich zulassen*, Crelle's Journal, vol. 87 (1879), pp. 139–146.

‡ *Ueber diejenigen algebraischen Gebilde, welche eindeutige Transformationen in sich zulassen*, Mathematische Annalen, vol. 32 (1888), pp. 290–308, reprinted from the Göttinger Nachrichten of February 7, 1887. See also NOETHER, Mathematische Annalen, vol. 21 (1883).

§ HUMBERT: *Sur la décomposition des fonctions θ en facteurs*, Comptes Rendus, vol. 126, (1898), pp. 394–396; *Sur les fonctions abéliennes singulières*, ibid., pp. 508–510, and Liouville's Journal, ser. 5, vol. 6 (1900), pp. 279–386, see page 372; PAINLEVÉ: *Sur les surfaces qui admettent un groupe infini discontinu de transformations birationnelles*, Comptes Rendus, vol. 126, (1898), pp. 512–514; HUTCHINSON: *The Hessian of the cubic surface II*, Bulletin of the American Mathematical Society, vol. 6 (1900), pp. 328–337, and *On some birational transformations of the Kummer surface into itself*, ibid., vol. 7 (1901), pp. 211–217; FANO: *Sopra alcune superficie del 4° ordine rappresentabili sul piano doppio*, Rendiconti del Istituto Lombardo, vol. 39 (1906), pp. 1071–1086. The first three of these articles are concerned with special forms of the Kummer surface; the treatment is entirely transcendental. The next two treat the general Kummer surface and two others into which it can be transformed; the treatment is partly transcendental and partly algebraic. The last one gives an outline of the theory of those quartic surfaces having a net of hyperelliptic curves; the treatment is purely geometric.

§ 1. *Quartic surfaces that possess a net of hyperelliptic curves of genus two.* *a. The nodal inversion* (N).

1. The line joining any point A_1 on a quartic surface F_4 to a conical point N of the surface will meet it again in a point B_1. A (1, 1) correspondence exists between A_1, B_1, defining a non-linear birational transformation of order 2 which will be indicated by (N). If F_4 has two conical points N_1, N_2, these define two such transformations (N_1), (N_2). Every plane section through N_1N_2 will remain invariant under the group generated by (N_1), (N_2). By taking N_1, N_2 as the vertices $(0, 0, 0, 1)$, $(0, 0, 1, 0)$ of the tetrahedron of reference the equation of a plane section may be written in the form

$$x_1^2(ax_2^2+bx_2x_3+cx_3^2)+x_1x_4(a'x_2^2+b'x_2x_3+c'x_3^2)+x_4^2(a''x_2^2+b''x_2x_3+c''x_3^2)=0.$$

The necessary and sufficient condition that the operations (N_1), (N_2) are commutative is the vanishing of the determinant *

$$(a) \qquad \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

If now the binodal quartic be transformed into a cubic, the operations (N_1), (N_2) become ordinary quadric inversions with regard to the polar conics of two ordinary points of the curve. If we still call these points N_1, N_2, the geometric condition that (N_1), (N_2) be commutative is that the tangents at N_1, N_2 meet on the curve, i. e., that N_1, N_2 be conjugate points. In general, the question whether (N_1), (N_2) define a finite group is thus reduced to STEINER's celebrated "Schliessungsproblem." †

In general, if $(N_1N_2)^k=1$, k must satisfy a certain relation which can be most easily expressed in terms of elliptic functions. Since any binodal quartic or non-singular cubic can be birationally derived from a space quartic curve of the first kind c_4, the (2, 2) correspondence between the points in which the lines through N_1, N_2 meet the curve again can be defined by means of the generators of a quadric surface passing through the quartic curve. The quartic curve may be defined by the equations

$$x_1=\rho\wp''(u), \qquad x_2=\rho\wp'(u), \qquad x_3=\rho\wp(u), \qquad x_4=\rho.$$

Four points corresponding to the values u_1, u_2, u_3, u_4 of the parameter u will lie in the same plane if, and only if,

$$u_1+u_2+u_3+u_4\equiv 0 \pmod{2\omega_1, 2\omega_2}.$$

* STURM: *Die Lehre von den geometrischen Verwandtschaften*, vol. 1 (1908), p. 267.

† J. STEINER: *Geometrische Lehrsätze*, Crelle's Journal, vol. 32 (1845), pp. 182–184. If the cycle is closed for one point, it will be for every point.

Given any two points u and v on c_4. Through the line joining them can be passed a quadric surface of the pencil having c_4 for basis curve. If $u + v \equiv c$, then the parameters of the points on any generator of the second system must satisfy the relation $u + v \equiv -c$. Starting with the point u, we can obtain v by the equation $u + v \equiv c$, then from v we can find u_1 on the other generator by $v + u_1 \equiv -c$. Similarly $u_1 + v_1 \equiv c$, $v_1 + u_2 \equiv -c$, etc. If $u_k = u$, by eliminating the intermediate terms we obtain the condition

$$2kc \equiv 0 .$$

For an arbitrary F_4 with two or more double points this condition will not be satisfied for any finite integer k, hence:

Quartic surfaces having m $(1 < m \leqq 16)$ double points exist which are invariant under a noncontinuous group of birational transformations of infinite order.

2. This result is interesting in view of KLEIN's* question regarding operations which leave the 16-nodal Kummer surface invariant. By means of properly chosen operations of the linear G_{16}, any (N_i) can be transformed into any other (N_k). This gives rise to at least thirty-two operations. By duality we have at once 32 more. The question is whether the surface is invariant under any other than these 64 operations. This question was answered in the affirmative by HUMBERT† by means of hyperelliptic theta functions.

On applying condition (a) to the equation of the surface referred to a Göpel tetrad, this question can be answered in the affirmative immediately. Similarly, by making use of the invariants whose vanishing expresses that $(N_i N_k)$ is finite it is seen that the surface has an infinite group, as the condition for finiteness would impose a relation among the three essential constants of the surface. By means of the known G_{32} the generators of this group can now be easily determined.

3. Another interesting case is furnished by the Weddle surface, the locus of the vertex of a quadric cone through six given points. Since this surface can be birationally transformed into the Kummer surface, it must have a G_{32}. This group was shown by BAKER to be defined by the six operations (N_i).‡

* *Ueber Konfigurationen, welche der Kummerschen Fläche zugleich ein- und umgeschrieben sind*, Mathematische Annalen, vol. 27 (1885), pp. 106–142. See p. 142.

† *Théorie générale des surfaces hyperelliptiques*, Liouville's Journal, ser. 4, vol. 9 (1893), pp. 29–170 and pp. 361–475. See page 466.

‡ This same result can be obtained from the parameters belonging to two points collinear with N_i as given in HUMBERT's third paper cited above, p. 470, but this fact is not there mentioned. BAKER: *Elementary note on the Weddle quartic surface*, Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1903), pp. 247–261, gives an algebraic proof. A much more extensive discussion of the transformations of both surfaces is given by BAKER: *An introduction to the theory of multiply periodic functions* (1907); see pp. 69–82.

By the preceding principles this theorem can be obtained at once as follows. The equation of the surface may be written in the form

$$\begin{vmatrix} \frac{a_1}{x_1} & x_1 & a_1 & 1 \\ \frac{a_2}{x_2} & x_2 & a_2 & 1 \\ \frac{a_3}{x_3} & x_3 & a_3 & 1 \\ \frac{a_4}{x_4} & x_4 & a_4 & 1 \end{vmatrix} = 0,$$

the six nodes being at the four vertices of the tetrahedron of reference and the two points $(1, 1, 1, 1)$, (a_1, a_2, a_3, a_4). In a plane containing any pair of nodes, as $k_4x_3 = k_3x_4$, the two points on $x_1 = 0$ are $(0, 0, k_3, k_4)$, $[0, k_3(a_2-a_4) + k_4(a_3 - a_2), k_3(a_3 - a_4), k_4(a_3 - a_4)]$. Similarly the two points on $x_2 = 0$ are $(0, 0, k_3, k_4)$, $[k_3(a_1 - a_4) + k_4(a_3 - a_1), 0, k_3(a_3 - a_4), k_4(a_3 - a_4)]$. The necessary and sufficient condition that (N_1), (N_2) be commutative is that the point $[k_3(a_1 - a_4) + k_4(a_3 - a_1), k_3(a_2 - a_4) + k_4(a_3 - a_2), k_3(a_3 - a_4), k_4(a_3 - a_4)]$ lie on the surface. By direct substitution we find that it does lie on the surface; similarly for the other pairs of nodes.

4. Moreover the result can be obtained geometrically. Let c_3 be the space cubic curve passing through the six points N_i. It lies on the Weddle surface defined by these points. The quadric cone having its vertex at N_i and passing through the five other points N_k contains c_3 and is the tangent cone at the conical point N_i of the Weddle surface.*

Any plane section through N_iN_k will contain the line N_iN_k and a cubic curve passing through N_iN_k. The tangent to the plane cubic at N_i is the generator of the tangent cone at N_i lying in the given plane and not passing through N_k. Similarly for N_k. Since these generators meet on c_3, it follows that N_i, N_k are conjugate points with regard to every plane cubic section passing through them, hence (N_i), (N_k) are commutative.

It is known that through any given plane quartic curve a Kummer surface can be passed, while all the plane sections of a Weddle surface are restricted, the relation among the constants being expressed by the equation

$$A^2 + 144B = 0,$$

wherein A is the cubic and B the sextic invariant of a quartic curve.†

* HIERHOLZER: *Ueber Kegelschnitte im Raume*, Mathematische Annalen, vol. 2 (1870), pp. 563–586. See page 582.

† F. MORLEY and J. R. CONNER: *Plane sections of a Weddle surface*, American Journal of Mathematics, vol. 31 (1909), pp. 263–270.

The preceding results furnish a new geometric interpretation of this important theorem for the case in which the plane passes through two conical points of the surface.

§ 1*b*. *The Involutions* (I).

5. A quartic surface F_4 of genus one ($p_n = p_g = p = 1$) which contains one curve of genus 2 will contain a net of such curves, any two of which will meet in two points, forming a rational involution I. (FANO, l. c.) The curves of order 4 are plane sections, and the surface has a double point. These curves define the inversions (N). In case the curves are of order 5 they are cut from F_4 by a net of quadrics, all having for residual the same twisted cubic.

Given a curve c_6 of order six and genus 2 on F_4. It is then one of a net σ_1, defining an involution I_1. Each c_6 of σ_1 will be cut in ∞^1 pairs of points of I_1, forming the canonical g_2^1. Through each c_6 pass ∞^2 cubic surfaces F_3. The residual intersection will be another sextic c_6', also of genus 2, and belonging to a net σ_2, distinct from σ_1 provided no F_3 of the net can be found which touches F_4 in every point of c_6. The net σ_2 will define an involution I_2 on F_4.

The ∞^1 lines determined by the pairs of points in g_2^1 on c_6 define a ruled cubic surface R_3 whose double directrix is a quadrisecant of c_6. The other points of intersection of each generator with F_4 will belong to the canonical g_2^1 on the residual c_6'. The curves c_6, c_6' have four points of intersection on the double directrix. Every line joining a pair of points in I_1 will also join a pair of points in I_2. The lines joining any point to its conjugate will therefore define a congruence of order 2, one of the lines joining a given point to its conjugate in I_1 and the other joining the same point to its conjugate in I_2. Starting with any point on the surface we can first find its conjugate in I_1 by a birational transformation of order 2, then the conjugate of the latter as to I_2, also a transformation of order 2. If a line of I_1 be given, there are two lines, one through each of the points of this line belonging to I_1, which connect it with its conjugate in I_2. Hence between the lines of I_1, I_2 there is a (2, 2) correspondence, and the condition for periodicity is reduced to that of the preceding case.*

The determination of these hyperelliptic curves by algebraic processes is in general a very difficult problem.

§ 2. *Systems of bitangents* (T).

6. Let $a \equiv \Sigma a_i x_i = 0$, $b = 0$ be the equations of a straight line, the coefficients a_i, b_i being rational functions of two non-homogeneous parameters κ, τ. If the coördinates of a fixed point y be substituted for x_i, the number of roots in κ, τ defines the order of the congruence T, that is, the number of lines

* FANO, l. c., showed by a different method that the operation ($I_1 I_2$) is in general of infinite order.

belonging to the system which pass through the point. The locus of the point y for which two of the lines coincide is the focal surface of the congruence. Thus, except for particular cases, the line of a given congruence of any order which touches the focal surface F at a given point P can be rationally separated from the other lines passing through P by the process of partial elimination. We shall be concerned only with such congruences as have a single line (counted twice) passing through P on F, and lying in its tangent plane. The other lines of T passing through P have one point of intersection with F at P.*

Congruences of the first order can have no focal surface; for those of the second order there is no residual line, while in the case of cubic congruences points on the focal surface are characterized by one double and one single root in κ, τ.

Every line of T is a bitangent to F, the points of tangency being P_1, P_2. The operation of interchanging P_1, P_2 is a birational transformation of order 2 which will be denoted by (T). If F is the complete focal surface for more than one congruence, we have two or more operations of order 2, and their product will be in general of infinite order. The necessary and sufficient condition that the transformations (T_1), (T_2) be birational is that the two congruences T_1, T_2 be rationally separable.

7. A number of important surfaces having this property can be obtained as the apparent contour of SEGRE's cubic variety Γ in four dimensional space S_4,† quartics by projecting from a point on Γ, and sextics by projecting from any point in S_4. In particular, if Γ contains a plane and can be generated by trilinear systems, the contour will be the focal surface of at least two congruences, and the equations of the transformations (T_1), (T_2) can be determined.

Let

$$\Gamma \equiv \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & 0 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0$$

be the equation of a cubic variety containing the plane $x_4 = 0$, $x_5 = 0$. A space $\kappa_2 x_4 - \kappa_1 x_5 = 0$ passing through this plane will cut from Γ a quadric surface, one such quadric passing through any point of Γ.

The two systems of generators are rationally separable, being

$$\frac{u_3}{u_1 k_2 - u_2 k_1} = \frac{x_3}{x_1 k_2 - x_2 k_1} = \tau_1, \qquad \frac{u_3}{x_3} = \frac{u_1 k_2 - u_2 k_1}{x_1 k_2 - x_2 k_1} = \tau_2,$$

* An example of congruences of another kind is furnished by a family of quadrics having one variable parameter, when the equations of the two systems of generators cannot be rationally separated.

† *Sulle varietà cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario.* Memorie di Torino, ser. 2, vol. 39 (1889), pp. 1–48.

the second one being defined by the pencil of spaces passing through the other plane $u_3 = 0$, $x_3 = 0$ on Γ.

If now a plane be passed through any line τ_1 and a fixed point K, the section of Γ made by this plane will consist of τ_1 and a conic cutting τ_1 in two points P', P''. The locus of the points P_1, P_2 in which the lines KP', KP'' are cut by an S_3 not passing through K is the focal surface of the congruence T_1, the points P_1, P_2 being the points of contact. If $k \equiv (c_1, c_2, \cdots)$, and S_3 be defined by $\Sigma e_i x_i = 0$, the equations take the form

$$\frac{u_3 - \tau_1(u_1 k_2 - u_2 k_1)}{u_3(c) - \tau_1[u_1(c) k_2 - u_2(c) k_1]} = \frac{x_3 - \tau_1(x_1 k_2 - x_2 k_1)}{c_3 - \tau_1(c_1 k_2 - c_2 k_1)} = \frac{k_2 x_4 - k_1 x_5}{k_2 c_4 - k_1 c_5}$$

from which x_5, for example, can be eliminated by means of $\Sigma e_i x_i = 0$. If τ_1 be eliminated, a cubic in $k_1 : k_2$ results, whose envelope is the focal surface.

If then we start with a point $(x_1', \cdots, x_5')$ or x' on Γ, the associated S_3 and the line τ_1 are known and the points $P' \equiv (x')$, $P'' \equiv (x'')$ are at once defined as the roots of a quadratic equation, one of which is known. The other root is expressed as a rational function of the given one, and the equations of (T_1) are determined. In the same way we determine (T_2).

If the system T_1 belongs to a linear complex, the points P_1, P_2 are poles of the tangent planes at P_2, P_1 respectively, so that (T_1) may be regarded as the product of the two commutative operations, duality as to the surface and duality as to the complex. When T_1, T_2 both belong to linear complexes, the group generated by (T_1), (T_2) may be finite. In particular, the necessary and sufficient condition that (T_1), (T_2) are commutative is that the complexes to which T_1, T_2 belong are in involution.

The lines of the congruence T_1 can be arranged on a system of ∞^1 quadric surfaces, the congruence T_2 being composed of the other system of generators of these quadrics. Any line l of T_1 will therefore determine two lines m_1, m_2 of T_2, so that a (2, 2) correspondence exists between l, m. Thus the locus of P_1, P_2 corresponding to the quadric defined by l, m is a space curve c_4 of order 4 of the first kind. By the operations (T_1), (T_2) the curve c_4 remains invariant, so that each quadric of the system goes into itself. The points P_1, P_2 define a (2, 2) correspondence upon c_4, hence the discussion of the periodicity of the operation $(T_1 T_2)$ can be reduced to that of the preceding case.*

When the focal surface is of order 6 the residual section made by any quadric of bitangents is another c_4 which is also the curve of contact of another quadric. If the line $P_1 P_2$ cuts F_6 in Q_1, Q_2 the operation of interchanging Q_1, Q_2 is also birational, since only a single line of each congruence passes through Q_1

* For the literature concerning the (2, 2) correspondence on c_4, see O. STAUDE: *Flächen zweiten Grades und ihre Systeme und Durchdringungskurven*, Encyklopädie der mathematischen Wissenschaften, III C2., no. 123; in particular, footnote 513.

apart from the one (counted twice) which touches F_6 at Q_1. All the theorems regarding periodicity can be applied immediately to this case also.

8. An interesting illustration is furnished by the variety Γ having ten nodes. The contour from any point on Γ is the general Kummer surface. The six systems of bitangents belong to six linear complexes mutually in involution, hence the operations (T_i) generate the well-known G_{32} first found by Klein.*

The product of any two operations (T_i), (T_k) is a linear transformation, and thus the linear group of order 16 is defined. Many of the properties of the Kummer surface and of its systems of bitangents can be deduced readily from this starting point.

9. An illustration of a different kind, wherein the group generated by the $(T_i)(T_k)$ is finite, is obtained from the variety

$$(1) \qquad \Gamma \equiv x_1x_2x_3 + \lambda x_4x_5x_6 = 0, \qquad \Sigma x_i = 1, \qquad \lambda \neq 1,$$

the center of projection being $P \equiv (1, 1, 1, -1, -1, -1)$. The details of this case will sufficiently illustrate the general procedure. The line of the system† I which passes through the point x' may be expressed by the equations

$$(2) \qquad \frac{x_1'x_4 - x_4'x_1}{x_1' + x_4'} = \frac{x_2'x_5 - x_5'x_2}{x_2' + x_5'} = \frac{x_3'x_6 - x_6'x_3}{x_3' + x_6'} = l.$$

Solve these equations for x_4, x_5, x_6 in terms of l and substitute the results in (1). The equations of the conic and of the line may be written in the form

$$(3) \qquad x_5'x_6'(x_1' + x_4')x_2x_3 + x_4'x_6'(x_2' + x_5')x_1x_3 + x_4'x_5'(x_3' + x_6')x_1x_2 = 0,$$

$$(4) \qquad x_2'x_3'(x_1' + x_4')x_1 + x_1'x_3'(x_2' + x_5')x_2 + x_1'x_2'(x_3' + x_6')x_3 = 0.$$

If now x' be taken on the first polar of Γ as to P,

$$(5) \qquad H \equiv x_1x_2 + x_1x_3 + x_2x_3 - \lambda(x_4x_5 + x_5x_6 + x_6x_4) = 0,$$

it will be one of the points of intersection of the line and the conic. Making (3), (4) simultaneous and making use of (2), we obtain the coördinates of x'', the second point of intersection. The results are

$$(6) \qquad x_1'' = \frac{1}{x_4'}, \qquad x_2'' = \frac{1}{x_5'}, \qquad x_3'' = \frac{1}{x_6'}, \qquad x_4'' = \frac{1}{x_1'}, \qquad x_5'' = \frac{1}{x_2'}, \qquad x_6'' = \frac{1}{x_3'}.$$

* *Zur Theorie der Liniencomplexe des ersten und zweiten Grades*, Mathematische Annalen, vol. 2 (1870), pp. 201–226.

† For this notation and the discussion of this variety, see my paper: *Surfaces derived from the cubic variety having nine double points in four dimensional space*, these Transactions, vol. 10 (1909), pp. 71–78.

Let the lines joining x' to P be cut by the polar S_3 of P as to Γ

$$x_1 + x_2 + x_3 + \lambda(x_4 + x_5 + x_6) = 0. \tag{7}$$

To distinguish between points x on Γ and points on the S_3 defined by (7), coördinates in the latter will be denoted by y_i. It is defined by

$$y_1 + y_2 + y_3 = 0, \qquad y_4 + y_5 + y_6 = 0. \tag{8}$$

From a point x on $\Gamma(x)$ we obtain the corresponding point in S_3 by means of the equations

$$\begin{aligned} y_1 &= x_2 + x_3 - 2x_1, & y_4 &= x_5 + x_6 - 2x_4, \\ y_2 &= x_1 + x_3 - 2x_2, & y_5 &= x_4 + x_6 - 2x_5, \\ y_3 &= x_1 + x_2 - 2x_3, & y_6 &= x_4 + x_5 - 2x_6, \end{aligned} \tag{9}$$

and the reciprocal relations, giving x when y is known, are

$$\begin{aligned} x_1 &= 2y_1 H(y) - 3\Gamma(y), & x_4 &= 2y_4 H(y) + 3\Gamma(y), \\ x_2 &= 2y_2 H(y) - 3\Gamma(y), & x_5 &= 2y_5 H(y) + 3\Gamma(y), \\ x_3 &= 2y_3 H(y) - 3\Gamma(y), & x_6 &= 2y_6 H(y) + 3\Gamma(y). \end{aligned} \tag{10}$$

Now by means of equations (6), (9), (10) we can obtain the equations of the birational transformation in S_3. The results are

$$\begin{aligned} \rho y_1' &= -\lambda[2H(y)(y_4y_5 + y_5y_6 - 2y_4y_6) + 3\Gamma(y)(y_5 - 2y_4 - 2y_6)], \\ \rho y_2' &= -\lambda[2H(y)(y_4y_6 + y_5y_6 - 2y_4y_5) + 3\Gamma(y)(y_6 - 2y_4 - 2y_5)], \\ \rho y_3' &= -\lambda[2H(y)(y_4y_6 + y_4y_5 - 2y_5y_6) + 3\Gamma(y)(y_4 - 2y_5 - 2y_6)], \\ \rho y_4' &= 2H(y)(y_1y_2 + y_2y_3 - 2y_1y_3) - 3\Gamma(y)(y_2 - 2y_1 - 2y_3), \\ \rho y_5' &= 2H(y)(y_1y_3 + y_2y_3 - 2y_1y_2) - 3\Gamma(y)(y_3 - 2y_1 - 2y_2), \\ \rho y_6' &= 2H(y)(y_1y_2 + y_1y_3 - 2y_2y_3) - 3\Gamma(y)(y_1 - 2y_2 - 2y_3), \end{aligned} \tag{11}$$

the equations of the surface being

$$y_1 + y_2 + y_3 = 0, \qquad y_4 + y_5 + y_6 = 0, \qquad 27(1-\lambda)\Gamma^2(y) + 4H^3(y) = 0.$$

The transformation defined by (11) is not a Cremona transformation, being birational for points of the surface only.

In the same manner we may obtain the systems (T_2), (T_3), defined by II, III, $\cdots$. If we use the notation $(x_i x_k x_l)$ to indicate the cyclic substitution

$x_i = \rho x'_k$, etc., the results may be written in the form

$$T_2 = T_1 l_2,\ T_3 = T_1 l_2^2,\ T_4 = T_1 l_4,\ T_5 = T_1 l_5,\ T_6 = T_1 l_4 l_5 l_4,$$

wherein

$$l_2 = (y_1 y_2 y_3)(y_4 y_6 y_5);\ l_4 = (y_2 y_3)(y_5 y_6);\ l_5 = (y_1 y_2)(y_4 y_5).$$

Moreover

$$T_1 l_2 = l_2 T_1,\ T_1 l_4 = l_4 T_1,\ T_1 l_5 = l_5 T_1,$$

from which it follows that $T_1, \cdots, T_6$ generate a group of order 36 which contains a linear subgroup of order 18. The surface is also invariant under the odd substitutions $t = (y_2 y_3)$ etc., making another linear group of order 18. The operations (T_i) combine with these in the same manner as with the preceding, making a total group of order 72.

10. From any point P_1 on F_6 can be drawn just one line of I touching F_6 at P_1. Let P_2 be the second point of contact of this line. Similarly, the line $P_2 P_3$ belongs to IV and touches F_6 at P_2 and at P_3. The line $P_3 P_4$ belongs to I. Since $(T_1 T_4)^2 = 1$ it follows that $P_4 P_1$ belongs to IV and has its points of contact at P_4 and at P_1. The vertices P_1, P_2, P_3, P_4 define a tetrahedron inscribed in F_6 and the planes $P_1 P_2 P_3$, etc. are all tangent planes. The transformation $(T_1 T_4)$ transforms P_1 into P_3 and P_2 into P_4; it is the axial involution l_4. We have the following theorem:

There are nine systems of ∞^2 tetrahedra which are inscribed in and circumscribed about F_6. Two opposite edges of the tetrahedra of each system always meet two fixed lines.

11. When the line σ touches the residual conic, section of Γ, $P_1 = P_2$ and the corresponding line in S_3 has four coincident points in common with F_6. The locus of the point of contact in I is a curve defined as the intersection of $\Gamma(x)$ with $H(x)$ and the variety defined by the equation

$$x_1 x_2 x_3 x_4 x_5 x_6 \left[\frac{(x_1 + x_4)^4}{x_1^2 x_4^2} + \frac{(x_2 + x_5)^4}{x_2^2 x_5^2} + \frac{(x_3 + x_6)^4}{x_3^2 x_6^2} \right]$$
$$-2(x_1+x_4)^2(x_2+x_5)^2(x_3+x_6)^2 \left[\frac{x_1 x_4}{(x_1 + x_4)^2} + \frac{x_2 x_5}{(x_2 + x_5)^2} + \frac{x_3 x_6}{(x_3 + x_6)^2} \right] = 0.$$

By means of equations (6), (9), (10) the equations of the curve in S_3 can be obtained. Let C_1 denote the curve at the points of which the lines of I have four point contact. From any point P_1 of this curve draw the line of IV. From the preceding theorem the second point of contact must also lie on C_1. The ruled surfaces belonging to IV, V, VI which have C_1 for directrix curve are such that every generator of each is a bisecant of the curve. If the line of II touching F_6 at P_1 also touches it at P', and the line $P'P''$ belongs to I, then P'' is a point of C_2. *The six curves C_i are birationally equivalent.*

CORNELL UNIVERSITY,
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PROPER MULTIPLE INTEGRALS OVER ITERABLE FIELDS

BY

ERNEST B. LYTLE *

Introduction.

The theory of multiple integrals has generally been confined to functions having metric (defined later in § 1) domains of definition. In his book † and in an article on *Improper Multiple Integrals* ‡ PIERPONT has given a definition of integrals applicable to non-metric domains.

Using this PIERPONT definition of an integral I have found a new class of fields for which the fundamental relation

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f$$

holds. This general class of fields, which I have called iterable, includes all metric fields and a large class of non-metric fields.

For simplicity the results are stated throughout the paper for functions of two independent variables; but the method of treatment is general and the extension to the case of any number of independent variables is thought to be obvious. The notation used by PIERPONT has been followed, and the reader is referred for further explanation to his *Lectures*.

§ 1. *Preliminary definitions.*

Let $\mathfrak{A} = \{x, y\}$ be a two-dimensional point aggregate lying within a square. Effect a rectangular division of the plane of norm δ. Let

$$\delta_1', \ \delta_2', \ \delta_3', \ \cdots$$

be those rectangles containing at least one point of $\mathfrak{A}$. Let

$$\delta_1'', \ \delta_2'', \ \delta_3'', \ \cdots$$

denote those rectangles all of whose points belong to $\mathfrak{A}$. Then

$$\lim_{\delta=0} \Sigma \delta_i' = \bar{\mathfrak{A}}$$

* Presented to the Society (New York), April 25, 1908.

† *Lectures on the Theory of Functions of Real Variables*, vol. 1 (1905). I shall refer to this as *Lectures*.

‡ These Transactions, vol. 7 (1906). Referred to later under the title *Improper Integrals*.

is defined to be the *outer* (upper) *content* of $\mathfrak{A}$, and

$$\lim_{\delta=0} \Sigma\delta_i'' = \underline{\mathfrak{A}}$$

is the *inner* (lower) *content* of $\mathfrak{A}$. It has been shown (*Lectures*, p. 513) that these limits always exist and are finite. When

$$\bar{\mathfrak{A}} = \underline{\mathfrak{A}}$$

their common value is called the *content* of $\mathfrak{A}$, and $\mathfrak{A}$ is said to be a *metric aggregate.*

Let $\mathfrak{A} = \{x, y\}$ be a limited point aggregate. Let $f(x, y)$ be a limited single-valued function defined over $\mathfrak{A}$. Effect a rectangular division D of the plane of norm δ. Let

$$\delta_1', \ \delta_2', \ \delta_3', \ \cdots,$$

be those rectangles of D which contain points of $\mathfrak{A}$. Let M_i, m_i be the maximum and minimum respectively of $f(x, y)$ in the rectangle δ_i'. Put

$$\bar{S}_D = \Sigma M_i \delta_i', \qquad \underline{S}_D = \Sigma m_i \delta_i'. \tag{1}$$

Then

$$\lim_{\delta=0} \bar{S}_D = \bar{\int_{\mathfrak{A}}} f \tag{2}$$

is defined to be the *outer upper integral* of $f(x, y)$ over the field $\mathfrak{A}$, and

$$\lim_{\delta=0} \underline{S}_D = \underline{\int_{\mathfrak{A}}} f \tag{3}$$

is the *outer lower integral* of $f(x, y)$ over $\mathfrak{A}$. These limits always exist and are finite. (See *Lectures*, p. 510.)

When the integrals (2) and (3) are equal we denote their common value by

$$\int_{\mathfrak{A}} f \tag{4}$$

and say that f is *integrable* over $\mathfrak{A}$.

It is important to note that the relations

$$\bar{\int_{\mathfrak{A}}} f = \min \bar{S}_D, \qquad \underline{\int_{\mathfrak{A}}} f = \max \underline{S}_D \tag{5}$$

which hold for Riemann integrals are not always true for outer integrals. (See *Lectures*, p. 508.)

Let $\mathfrak{A}$ be a limited plane point set. Let $\mathfrak{C}$ be the points of $\mathfrak{A}$ lying on a line parallel to the y-axis. Let $\mathfrak{B}$ be the projection of $\mathfrak{A}$ on the x-axis. Then when

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \bar{\mathfrak{A}} \tag{6}$$

we define $\mathfrak{A}$ to be *iterable relatively to* x.

If C is a linear section of $\mathfrak{A}$ parallel to the x-axis and B is the projection of $\mathfrak{A}$ on the y-axis, then when

$$\int_B \bar{C} = \bar{\mathfrak{A}} \tag{7}$$

$\mathfrak{A}$ is *iterable relatively to* y.

The following examples show that both iterable and non-iterable aggregates exist.

EXAMPLE 1. Let $\mathfrak{A} = \{x, y\}$ where x takes the rational values in a unit interval and y takes all real values in a unit interval. Then $\mathfrak{A}$ is non-metric, for

$$\underline{\mathfrak{A}} = 0, \qquad \bar{\mathfrak{A}} = 1.$$

But $\mathfrak{A}$ is iterable relative to both x and y, for

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \int_B \bar{C} = 1 = \bar{\mathfrak{A}}.$$

EXAMPLE 2. Let $\mathfrak{A} = \{x, y\}$ be the points of a unit square, specified for rational x by $0 \leqq y \leqq \frac{1}{2}$, and for irrational x by $\frac{1}{2} \leqq y \leqq 1$. Here $\mathfrak{A}$ is non-metric, for

$$\underline{\mathfrak{A}} = 0, \qquad \bar{\mathfrak{A}} = 1.$$

Also $\mathfrak{A}$ is iterable relatively to y, for

$$\int_B \bar{C} = 1 = \bar{\mathfrak{A}},$$

but not iterable relatively to x, for

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \tfrac{1}{2} \neq \bar{\mathfrak{A}}.$$

By similar methods fields may readily be constructed which are iterable neither relatively to x nor relatively to y.

Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2,$$

and for each x let

$$\bar{\mathfrak{C}} = \bar{\mathfrak{C}}_1 = \bar{\mathfrak{C}}_2.$$

Then we say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *thoroughly mixed* partials of $\mathfrak{A}$.

The notation $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is used to specify that $\mathfrak{B}$ is the projection of $\mathfrak{A}$ on the x-axis and $\mathfrak{C}$ is a linear section of $\mathfrak{A}$ parallel to the y-axis.

§ 2. *Properties of iterable fields.*

THEOREM 1. *Every metric aggregate is iterable relatively to both x and y.*

It has been shown (see *Lectures*, p. 538) that, for a function $f(x, y)$ limited

in the metric field $\mathfrak{A}$,

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int} dx \overline{\int} f dy \leqq \overline{\int} dx \overline{\int} f dy \leqq \overline{\int}_{\mathfrak{A}} f,$$

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int} dy \overline{\int} f dx \leqq \overline{\int} dy \overline{\int} f dx \leqq \overline{\int}_{\mathfrak{A}} f.$$

In these relations put $f(x, y) = 1$ and they become

$$\bar{\mathfrak{A}} \leqq \underline{\int}_B \bar{C} \leqq \overline{\int}_B \bar{C} \leqq \bar{\mathfrak{A}}, \qquad \bar{\mathfrak{A}} \leqq \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}} \leqq \overline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}} \leqq \bar{\mathfrak{A}}.$$

Hence

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \int_B \bar{C} = \bar{\mathfrak{A}},$$

that is, $\mathfrak{A}$ is iterable relatively to x and relatively to y.

LEMMA. *Let $\mathfrak{A}$ be iterable relatively to x. Then if $\delta = \alpha \cdot \beta$ is a cell of any rectangular division D, we have*

$$\lim_{\delta=0} \Sigma(\alpha \Sigma \beta) = \lim_{\alpha=0} \Sigma(\alpha \lim_{\beta=0} \Sigma \beta).$$

For each x the inequality

$$\lim_{\beta=0} \Sigma \beta \leqq \Sigma_\beta$$

will hold. Hence

$$\Sigma(\alpha \lim_{\beta=0} \Sigma \beta) \leqq \Sigma(\alpha \Sigma \beta) \leqq \Sigma \delta.$$

But

$$\lim_{\delta=0} \Sigma(\alpha \lim_{\beta=0} \Sigma \beta) = \lim_{\alpha=0} \Sigma(\alpha \lim_{\beta=0} \Sigma \beta) = \int_{\mathfrak{B}} \bar{\mathfrak{C}},$$

$$\lim_{\delta=0} \Sigma \delta = \bar{\mathfrak{A}}.$$

Since $\mathfrak{A}$ is iterable relatively to x

$$\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \bar{\mathfrak{A}},$$

and our theorem follows.

This lemma enables us to evaluate an iterated limit by a simpler single limit. A similar lemma is true for a field iterable relatively to y.

THEOREM 2. *Every partial unmixed aggregate of an iterable aggregate is iterable.*

Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2,$$

where $\mathfrak{A}$ is iterable and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed.*

* See *Lectures*, § 711, p. 519.

Then for each x

$$\bar{\mathfrak{C}} \leqq \bar{\mathfrak{C}}_1 + \bar{\mathfrak{C}}_2,$$

and for every x-interval

$$\min \bar{\mathfrak{C}} \leqq \min \bar{\mathfrak{C}}_1 + \max \bar{\mathfrak{C}}_2.$$

Hence by integration over $\mathfrak{B}$,

$$\underline{\int} \bar{\mathfrak{C}} \leqq \underline{\int} \bar{\mathfrak{C}}_1 + \overline{\int} \bar{\mathfrak{C}}_2.$$

But

$$\underline{\int} \bar{\mathfrak{C}}_1 \leqq \bar{\mathfrak{A}}_1, \qquad \overline{\int} \bar{\mathfrak{C}}_2 \leqq \bar{\mathfrak{A}}_2.$$

Hence

$$(1) \qquad \underline{\int} \bar{\mathfrak{C}} \leqq \underline{\int} \bar{\mathfrak{C}}_1 + \overline{\int} \bar{\mathfrak{C}}_2 \leqq \bar{\mathfrak{A}}_1 + \bar{\mathfrak{A}}_2.$$

Since $\mathfrak{A}$ is iterable by hypothesis

$$\int \bar{\mathfrak{C}} = \bar{\mathfrak{A}},$$

and since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed

$$\bar{\mathfrak{A}} = \bar{\mathfrak{A}}_1 + \bar{\mathfrak{A}}_2.$$

Therefore by (1)

$$\underline{\int} \bar{\mathfrak{C}}_1 + \overline{\int} \bar{\mathfrak{C}}_2 = \bar{\mathfrak{A}}_1 + \bar{\mathfrak{A}}_2,$$

or

$$\left(\bar{\mathfrak{A}}_1 - \underline{\int} \mathfrak{C}_1\right) + \left(\bar{\mathfrak{A}}_2 - \overline{\int} \mathfrak{C}_2\right) = 0.$$

But

$$\bar{\mathfrak{A}}_1 - \underline{\int} \bar{\mathfrak{C}}_1 \geqq 0, \qquad \bar{\mathfrak{A}}_2 - \overline{\int} \bar{\mathfrak{C}}_2 \geqq 0,$$

hence

$$(2) \qquad \bar{\mathfrak{A}}_1 = \underline{\int} \bar{\mathfrak{C}}_1, \qquad \bar{\mathfrak{A}}_2 = \overline{\int} \bar{\mathfrak{C}}_2.$$

Similarly

$$(3) \qquad \underline{\int} \bar{\mathfrak{C}} \leqq \overline{\int} \bar{\mathfrak{C}}_1 + \underline{\int} \bar{\mathfrak{C}}_2 \leqq \bar{\mathfrak{A}}_1 + \bar{\mathfrak{A}}_2 = \bar{\mathfrak{A}},$$

from which follows

$$(4) \qquad \bar{\mathfrak{A}}_1 = \overline{\int} \bar{\mathfrak{C}}_1, \qquad \underline{\int} \bar{\mathfrak{A}}_2 = \bar{\mathfrak{C}}_2.$$

By (2) and (4), $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are each iterable.

COROLLARY. *If a discrete aggregate be subtracted from an iterable aggregate, then the remaining aggregate is iterable.*

Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2,$$

where $\mathfrak{A}$ is iterable and $\mathfrak{A}_2$ is discrete. Since $\mathfrak{A}_2$ is discrete $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed. Therefore $\mathfrak{A}_1$ is iterable by the preceding theorem.

§ 3. *Proper Integrals.*

A *proper* integral is one in which both the function and its field of integration are limited.

THEOREM 3.* *Let $f(x, y) \geqq 0$ be limited over the limited field $\mathfrak{A}$. Then*

$$\overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} f \leqq \overline{\int_{\mathfrak{A}}} f.$$

If the sign of f is changed, the theorem is converted into the following:

THEOREM 4. *Let $f(x, y) \leqq 0$ be limited over the limited field $\mathfrak{A}$. Then*

$$\underline{\int_{\mathfrak{A}}} f \leqq \underline{\int_{\mathfrak{B}}} \underline{\int_{\mathfrak{C}}} f.$$

THEOREM 5. *Let $f(x, y)$ and $g(x, y)$ each be limited over the limited field $\mathfrak{A}$. Let them be equal except at a discrete set of points. Then*

$$(a) \qquad \overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{A}}} g,$$

$$(b) \qquad \overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} f = \overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} g.$$

Let $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$, where $\mathfrak{A}_2$ is discrete. Let $f = g$ over $\mathfrak{A}_1$. Let D be a rectangular division of the plane of norm $\delta = \alpha \cdot \beta$. Let

$$\delta_{11}, \ \delta_{12}, \ \delta_{13}, \ \cdots$$

be those rectangles which contain no points of $\mathfrak{A}_2$. Let

$$\delta_{21}, \ \delta_{22}, \ \delta_{23}, \ \cdots$$

contain points of $\mathfrak{A}_2$. Then

$$\Sigma M_i \delta_i = \Sigma M_{1i} \delta_{1i} + \Sigma M_{2i} \delta_{2i},$$

$$\Sigma N_i \delta_i = \Sigma N_{1i} \delta_{1i} + \Sigma N_{2i} \delta_{2i},$$

* Restated for later use. For proof see RICHARDSON, these Transactions, vol. 9 (1908), p. 347.

where M_i and N_i are the maxima of f and g respectively in δ_i. Subtracting we get

$$\Sigma M_i \delta_i - \Sigma N_i \delta_i = \Sigma M_{2i} \delta_{2i} - \Sigma N_{2i} \delta_{2i},$$

since $M_{1i} = N_{1i}$. On passing to the limit as $\delta \doteq 0$, this becomes

$$\overline{\int_{\mathfrak{A}}} f - \overline{\int_{\mathfrak{A}}} g = 0,$$

for each limit in the second member is zero because $\mathfrak{A}_2$ is discrete and both f and g are limited. This gives our relation (a).

From (a) and theorem 10 below it follows that

$$\overline{\int_{\mathfrak{C}}} g = \overline{\int_{\mathfrak{C}}} f$$

at all points of $\mathfrak{B}$ except a discrete set. Relation (b) then follows by application of relation (a) to the set $\mathfrak{B}$.

THEOREM 6. *Let $f(x, y)$ be limited over a limited field $\mathfrak{A}$ and negative only at a discrete set of points Δ. Then*

$$\overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} f \leqq \overline{\int_{\mathfrak{A}}} f.$$

Let $g(x, y) = f(x, y)$ at all points of $\mathfrak{A}$ except at Δ where $g = 0$. Then $g \geqq 0$, and by theorem 3

$$\overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} g \leqq \overline{\int_{\mathfrak{A}}} g.$$

But by the preceding theorem

$$\overline{\int_{\mathfrak{A}}} g = \overline{\int_{\mathfrak{A}}} f, \qquad \overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} g = \overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} f,$$

and substituting these in the above relation we have our theorem.

THEOREM 7. *Let $f(x, y)$ be limited over a limited field $\mathfrak{A}$ and positive only at a discrete set of points. Then*

$$\underline{\int_{\mathfrak{A}}} f \leqq \underline{\int_{\mathfrak{B}}} \underline{\int_{\mathfrak{C}}} f.$$

THEOREM 8. *Let $f(x, y)$ be limited over the limited field $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$. For each x let $\overline{\mathfrak{C}} = \overline{\mathfrak{C}}_1 = \overline{\mathfrak{C}}_2$. Let $f \geqq 0$ over $\mathfrak{A}_1$ and $f \leqq 0$ over $\mathfrak{A}_2$. Then*

$$(A) \qquad \underline{\int_{\mathfrak{A}}} f \leqq \underline{\int_{\mathfrak{B}}} \underline{\int_{\mathfrak{C}}} f \leqq \overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} f \leqq \overline{\int_{\mathfrak{A}}} f.$$

This inequality will appear frequently in the succeeding pages and we shall refer to it as *relation* (A).

Since $\bar{\mathfrak{C}} = \bar{\mathfrak{C}}_1 = \bar{\mathfrak{C}}_2$, the positive and negative values of the function are each everywhere dense in $\mathfrak{A}$. Hence every rectangle containing a point of $\mathfrak{A}$ also contains a point of $\mathfrak{A}_1$, and

$$\overline{\int_{\mathfrak{A}}} f = \lim \Sigma_{\mathfrak{A}} M_i \delta_i = \lim \Sigma_{\mathfrak{A}_1} M_i \delta_i = \overline{\int_{\mathfrak{A}_1}} f.$$

Similarly for each x we have

$$\overline{\int_{\mathfrak{C}}} f = \overline{\int_{\mathfrak{C}_1}} f.$$

But the projections of $\mathfrak{A}$ and $\mathfrak{A}_1$ on the x-axis are equal, hence

$$\overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} f = \overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}_1}} f.$$

Since $f \geqq 0$ over $\mathfrak{A}_1$, we have by theorem 3,

$$\overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}_1}} \quad \overline{\int_{\mathfrak{A}_1}} f$$

and the equals of both members give

$$\overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} f \leqq \overline{\int_{\mathfrak{A}}} f. \tag{1}$$

In the same way, using theorem 4 and $\mathfrak{A}_2$, we get

$$\underline{\int_{\mathfrak{A}}} f \leqq \underline{\int_{\mathfrak{B}}} \underline{\int_{\mathfrak{C}}} f. \tag{2}$$

But we always have

$$\underline{\int_{\mathfrak{B}}} \underline{\int_{\mathfrak{C}}} f \leqq \overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} f. \tag{3}$$

Combining (1), (2) and (3) we get relation (A), which was to be proved.

Examples of thoroughly mixed sets $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are well known; for example, in $\mathfrak{A}_1$ let x be real and y be rational, and in $\mathfrak{A}_2$ let x be real and y be irrational.

THEOREM 9. *Let $f(x, y)$ be limited over the limited iterable field $\mathfrak{A}$. Then the existence of the multiple integral $\int_{\mathfrak{A}} f$ and the iterated integral $\int_{\mathfrak{B}} \int_{\mathfrak{C}} f$ is sufficient for their equality.*

Let $|f| \leqq F$. Then $f + F \geqq 0$, and by theorem 3

$$\overline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} (f + F) \leqq \overline{\int_{\mathfrak{A}}} (f + F).$$

But

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} f + F\int_{\mathfrak{B}} \bar{\mathfrak{C}} = \int_{\mathfrak{B}}\int_{\mathfrak{C}}(f+F) \leqq \overline{\int_{\mathfrak{B}}}\overline{\int_{\mathfrak{C}}}(f+F) \leqq \overline{\int_{\mathfrak{A}}}(f+F)$$

$$= \overline{\int_{\mathfrak{A}}} f + F\bar{\mathfrak{A}} = \int_{\mathfrak{A}} f + F\bar{\mathfrak{A}}.$$

Hence

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} f \leqq \int_{\mathfrak{A}} f + F\left(\bar{\mathfrak{A}} - \int_{\mathfrak{B}} \bar{\mathfrak{C}}\right). \tag{1}$$

Since $f - F \leqq 0$, we get in a similar way by theorem 4,

$$\int_{\mathfrak{A}} f - F\bar{\mathfrak{A}} = \underline{\int_{\mathfrak{A}}}(f-F) \leqq \int_{\mathfrak{B}}\int_{\mathfrak{C}}(f-F) = \int_{\mathfrak{B}}\int_{\mathfrak{C}} f - F\int_{\mathfrak{B}} \bar{\mathfrak{C}},$$

and therefore

$$-F\left(\bar{\mathfrak{A}} - \int_{\mathfrak{B}} \bar{\mathfrak{C}}\right) + \int_{\mathfrak{A}} f \leqq \int_{\mathfrak{B}}\int_{\mathfrak{C}} f. \tag{2}$$

But since $\mathfrak{A}$ is iterable

$$\bar{\mathfrak{A}} - \int_{\mathfrak{B}} \bar{\mathfrak{C}} = 0,$$

and this with (1) and (2) gives

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}}\int_{\mathfrak{C}} f.$$

We observe in this connection that the mere existence of the multiple and the iterated integrals is not sufficient in general for their equality.* The theorem just cited may not be true when the field is not iterable and the function limited, as is shown by the following example.

EXAMPLE 3. Let $f(x, y) = 1$ over the field $\mathfrak{A} = \{x, y\}$ where for rational x, $0 \leqq y \leqq \frac{1}{2}$ and for irrational x, $\frac{1}{2} \leqq y \leqq 1$. Then

$$\int_{\mathfrak{A}} f = 1, \qquad \int_{\mathfrak{B}}\int_{\mathfrak{C}} f = \tfrac{1}{2}.$$

That is, both integrals exist but are unequal.

THEOREM 10. *Let $\mathfrak{A} = \mathfrak{B}\cdot\mathfrak{C}$ be discrete. Then the points $\mathfrak{B}_\sigma$ at which $\bar{\mathfrak{C}} \geqq \sigma$, where σ denotes an arbitrary small positive number, form a discrete set.*†

THEOREM 11. *Let $f(x, y) \geqq 0$ be limited over the limited field $\mathfrak{A}$.*

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*Bulletin of the American Mathematical Society, November, 1906, p. 61.

†Cf. SCHÖNFLIES: *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 8 (1900), p. 96.

Let $\underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}} = \bar{\mathfrak{A}}$. *Then*

$$(A) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f.$$

By theorem 3 we have

$$(1) \qquad \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f.$$

But $f - F \leqq 0$, hence by theorem 4 and *Lectures*, p. 535, we have

$$\underline{\int}_{\mathfrak{A}} f - F\bar{\mathfrak{A}} \leqq \underline{\int}_{\mathfrak{A}} (f - F) \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} (f - F) \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f - F \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}}.$$

And since by hypothesis $\bar{\mathfrak{A}} = \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}}$, this gives

$$(2) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f.$$

Relation (A) then follows from (1) and (2).

THEOREM 12. Let $f(x, y) \leqq 0$ *be limited over the limited field* $\mathfrak{A}$. *Let* $\underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}} = \bar{\mathfrak{A}}$. *Then*

$$(d) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f.$$

THEOREM 13. *Let* $f(x, y)$ *be limited over the limited field* $\mathfrak{A}$. *Let* $\mathfrak{C}$ *be an integrable function of* x *and let* C *be any constant. Then*

$$(e) \qquad \overline{\int}_{\mathfrak{A}} (f + C) = \overline{\int}_{\mathfrak{A}} f + C\bar{\mathfrak{A}}.$$

$$(f) \qquad \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} (f + C) = \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f + C \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}}.$$

Relation (e) follows as an application of a corollary of RICHARDSON'S.*

By an application of relation (e) we get relation (f), thus

$$\overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} (f + C) = \overline{\int}_{\mathfrak{B}} \left[\overline{\int}_{\mathfrak{C}} f + C\bar{\mathfrak{C}} \right] \qquad \text{by } (e)$$

$$= \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f + C \underline{\int}_{\mathfrak{B}} \bar{\mathfrak{C}}$$

by hypothesis and Richardson's corollary.

* These Transactions, vol. 9 (1908), p. 350.

We may obtain similar relations for lower integrals.

THEOREM 14. *Let $f(x, y)$ be limited over the limited field $\mathfrak{A}$ which is iterable relatively to x. Then*

$$(A) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{C}} \overline{\int}_{\mathfrak{B}} f \leqq \overline{\int}_{\mathfrak{A}} f.$$

Since $f + F \geqq 0$, by theorem 3

$$\overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} (f + F) \leqq \overline{\int}_{\mathfrak{A}} (f + F).$$

By theorem 13 we have

$$\overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} (f + F) = \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f + F \int_{\mathfrak{B}} \overline{\mathfrak{C}}$$

$$\overline{\int}_{\mathfrak{A}} (f + F) = \overline{\int}_{\mathfrak{A}} f + F\overline{\mathfrak{A}}.$$

Therefore

$$(1) \qquad \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f + F\left(\overline{\mathfrak{A}} - \int_{\mathfrak{B}} \overline{\mathfrak{C}}\right).$$

Again, since $f - F \leqq 0$, by theorem 4

$$\underline{\int}_{\mathfrak{A}} (f - F) \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} (f - F).$$

But by relations for lower integrals similar to those of theorem 13

$$\underline{\int}_{\mathfrak{A}} (f - F) = \underline{\int}_{\mathfrak{A}} f - F\overline{\mathfrak{A}}, \qquad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} (f - F) = \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f - F \underline{\int}_{\mathfrak{C}} \overline{\mathfrak{C}}.$$

Therefore

$$(2) \qquad -F\left(\overline{\mathfrak{A}} - \underline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}}\right) + \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f.$$

But $\mathfrak{A}$ being iterable, we have

$$(3) \qquad \overline{\mathfrak{A}} - \int_{\mathfrak{B}} \overline{\mathfrak{C}} = 0.$$

Therefore relation (A) follows at once from (1), (2) and (3).

Relation (A) may be written

$$(g) \qquad \underline{\int} f d\mathfrak{A} \leqq \underline{\int} dx \underline{\int} f dy \leqq \overline{\int} dx \overline{\int} f dy \leqq \overline{\int} f d\mathfrak{A}.$$

If the field $\mathfrak{A}$ is iterable relatively to y, we obtain in a similar way

$$(h) \qquad \underline{\int} f d\mathfrak{A} \leqq \underline{\int} dy \underline{\int} f dx \leqq \overline{\int} dy \overline{\int} f dx \leqq \overline{\int} f d\mathfrak{A}.$$

COROLLARY 1. *Let $f(x, y)$ be integrable over any limited field which is iterable relatively to both x and y. Let f also be integrable over any section of $\mathfrak{A}$ parallel to either axis. Then*

$$\int f d\mathfrak{A} = \int dx \int f dy = \int dy \int f dx.$$

This follows at once by putting

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} = \overline{\int} f d\mathfrak{A} = \int f d\mathfrak{A}$$

in relations (g) and (h) above. It gives *a very general condition for inversion* in iterated integrals.

In conclusion, it may be shown that *the equality*

$$\underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f$$

is not sufficient for the existence of

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$

For consider the following example. Let $\mathfrak{A} = \{x, y\}$, where $0 \leqq x = p/q \leqq 1$, $0 \leqq y \leqq 1/q$. For rational y let $f(x, y) = +1$ and for irrational y let $f(x, y) = -1$. Then $\mathfrak{A}$ is discrete and f is limited, hence

$$\int_{\mathfrak{A}} f = \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f = 0$$

by theorem 14. But

$$\underline{\int}_{\mathfrak{C}} f = -\frac{1}{q}, \qquad \overline{\int}_{\mathfrak{C}} f = +\frac{1}{q}$$

for each x in $\mathfrak{B}$. Therefore

$$\int_{\mathfrak{C}} f$$

does not exist at any point in $\mathfrak{B}$ and

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f$$

cannot then exist.

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ON A CLASS OF HYPERFUCHSIAN FUNCTIONS*

BY

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Introduction.

As a generalization of the fuchsian functions, PICARD introduced the hyperfuchsian. These he defined † as uniform functions of two independent complex variables which are absolutely invariant for an infinite discontinuous group of linear fractional transformations on the variables. Moreover the group, when expressed in its homogeneous form, is such as to leave an indefinite Hermitian form invariant.

In a series of memoirs ‡ PICARD has established the existence and general theory of such functions. In a particular example he uses as variables the ratios of the values of the integral

$$\int [z(z-1)(z-x)(z-y)]^{-\frac{1}{3}} dz$$

taken along three independent contours in the associated Riemann surface, and generates the group by the motions of the two branch-points x and y. Following and extending PICARD's study of this particular group, ALEZAIS § has considered the Riemann surface

$$w^3 = (z-\alpha)(z-\beta)(z-\gamma)(z-\delta)$$

and the values of the moduli of periodicity of a set of associated linearly independent integrals of the first kind along a properly chosen system of cross-cuts on the surface. The group is generated by the monodromy of the four branch-points α, β, γ, δ, and contains PICARD's group as a subgroup. Considering the PICARD group to be of special interest, ALEZAIS studies it in some detail and

* Presented to the Society, December 31, 1908.

† Acta Mathematica, vol. 5 (1885), p. 121. See also R. ALEZAIS, *Sur une classe de fonctions hyperfuchsiennes*, etc., Paris (1901), p. 1. This thesis contains (p. 3) a complete bibliography of Picard's papers on the problem. An abridgment of the thesis under the same title is in the Annales de l'École Normale Supérieure, (3) vol. 19 (1902), p. 261.

‡ Acta Mathematica, vol. 2 (1882), p. 1, and vol. 5 (1885), p. 21; also several notes in Comptes Rendus, 1882–1885.

§ Loc. cit.

constructs quotients of the products of theta-functions of half-integer characteristics and zero arguments as hyperfuchsian functions belonging to the group.

The only other binomial Riemann surface of genus 3 having five branch-points may be written

$$w^4 = (z-a)(z-b)(z-c)^2(z-d)^2(z-e)^2.$$

The present paper determines the monodromy group of this surface and a set of hyperfuchsian functions belonging to its group. The fourth section contains an extension of a known result on the number of linearly independent moduli of an abelian integral of the first kind connected with a binomial Riemann surface. In the final section an upper limit is fixed for the number of generators of monodromy groups connected with binomial Riemann surfaces, and a method is given for picking out a set of transformations from which the generators may be determined.

§ 1. *The monodromy group.*

Consider the Riemann surface of genus 3,

$$w^4 = (z-a)(z-b)(z-c)^2(z-d)^2(z-e)^2, \tag{1}$$

where a, b, c, d, e may be assumed finite. We will denote the different values of w corresponding to the same value of z by w_1, w_2, w_3, w_4, and will assign the branch w_m to sheet m of the surface. The several values of w will be connected by the relation $w_m = i^{m-1} w_1$.

A set of linearly independent integrals of the first kind is selected as follows:

$$\begin{gathered} u_1 = \int \frac{dz}{w}; \qquad u_2 = \int (z-c)(z-d)(z-e)\frac{dz}{w^3}; \\ u_3 = \int z(z-c)(z-d)(z-e)\frac{dz}{w^3}. \end{gathered} \tag{2}$$

The branch-lines and cross-cuts for the surface are chosen as indicated in Fig. 1.

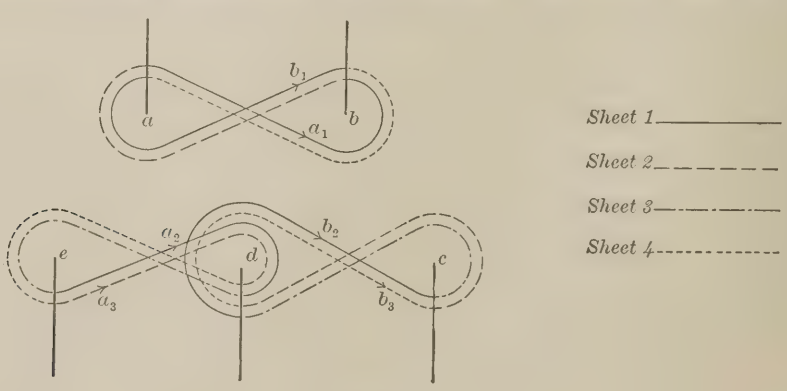

FIG. 1.

Denoting the results of integrating u_i along the paths a_j, b_j by $A_j^{(i)}$, $B_j^{(i)}$ respectively, the table of moduli of periodicity of the integrals (2) is

$$
(3)\qquad
\begin{array}{c|ccc|ccc}
 & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\
\hline
u_1 & iA_1^{(1)} & -B_2^{(1)} & iB_2^{(1)} & A_1^{(1)} & A_2^{(1)} & -iA_2^{(1)} \\
u_2 & -iA_1^{(2)} & -B_2^{(2)} & -iB_2^{(2)} & A_1^{(2)} & A_2^{(2)} & iA_2^{(2)} \\
u_3 & -iA_1^{(3)} & -B_2^{(3)} & -iB_2^{(3)} & A_1^{(3)} & A_2^{(3)} & iA_2^{(3)}
\end{array}
$$

subject to the conditions

$$iA_1^{(1)}A_1^{(2)} - A_2^{(2)}B_2^{(1)} + A_2^{(1)}B_1^{(2)} = 0,$$

$$iA_1^{(1)}A_1^{(3)} - A_2^{(3)}B_2^{(1)} + A_2^{(1)}B_1^{(3)} = 0.$$

The normal table of moduli assumes the form

$$
(3')\qquad
\begin{array}{c|ccc|ccc}
 & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\
\hline
v_1 & \pi i & 0 & 0 & \pi i(i) & \pi i(-u) & \pi i(-iu) \\
v_2 & 0 & \pi i & 0 & \pi i(-u) & \pi i\left(\dfrac{-iu^2}{2} - v\right) & \pi i\left(\dfrac{u^2}{2}\right) \\
v_3 & 0 & 0 & \pi i & \pi i(-iu) & \pi i\left(\dfrac{u^2}{2}\right) & \pi i\left(\dfrac{iu^2}{2} - v\right)
\end{array}
$$

where

$$u = \frac{A_1^{(1)}}{B_2^{(1)}}, \qquad v = \frac{A_2^{(1)}}{B_2^{(1)}}.$$

Denote by a'_{ik} the real part of the element of this table which represents the modulus of periodicity of the integral v_k at the cut b_i. If r_i, r_k are integers, then * $\sum_{ik} a'_{ik} r_i r_k$ must be negative; this condition will be satisfied if †

$$(4)\qquad u\bar{u} - i(v - \bar{v}) < 0,$$

wherein $\bar{u}$, $\bar{v}$, are the conjugates of u, v.

We now define the group G which we desire to investigate as the totality of linear fractional transformations on the variables u, v due to the monodromy of the branch-points of the given surface (1). The branch-points are supposed to move continuously over the plane in such a way that at the end of the movement they occupy as a whole their original positions and the Riemann surface has returned to its original form. Since $u = A_1^{(1)}/B_2^{(1)}$, $v = A_2^{(1)}/B_2^{(1)}$, and $A_1^{(1)}$, $A_2^{(1)}$, $B_2^{(1)}$ are the values of the integral u_1 along the paths a_1, a_2, b_2 respectively, in determining the transformation on u, v due to any movement of the

* STAHL, *Abelsche Funktionen*, p. 122 (XV).

† KRAZER, *Thetafunktionen*, p. 17 (XVII).

branch-points, it will be necessary merely to investigate the value of u_1 along the paths a_1', a_2', b_2', which will denote the paths into which a_1, a_2, b_2 have been changed. For brevity write

$$(5) \qquad x = A_1^{(1)}, \qquad y = A_2^{(1)}, \qquad z = B_2^{(1)}.$$

The values of u_1 along the new paths will be denoted by x', y', z' respectively, and the variables u, v will have been transformed into the variables u', v', where

$$u' = \frac{x'}{z'}, \qquad v' = \frac{y'}{z'}.$$

By a positive interchange of two branch-points we shall mean that the positions are interchanged in such a manner that the line joining the two shall have been rotated counter-clockwise about its middle point. When necessary the actual paths of the points interchanged are supposed to vary from the circle described by the end points of the joining line sufficiently to prevent any third point being enclosed in the completed path of the two points. A point will be said to move around another point positively when the enclosed area is to the left of the moving point; this area will be assumed to enclose but a single branch-point unless the contrary is stated. Denote by $(\kappa\lambda)$ the transformation on the variables arising from a positive interchange of the two points κ and λ. This same notation will be used to denote the motion of κ positively around λ if κ and λ cannot be interchanged under the above restrictions on the movement of the branch-points.

By interchanging the branch-points a and b positively, it is readily found that x, y, z undergo the transformation

$$x' = ix, \qquad y' = y, \qquad z' = z;$$

whence

$$(ab) \qquad u' = -\frac{u}{i}, \qquad v' = \frac{iv}{i},$$

the coefficients being so written that their determinant is unity. Similarly we obtain

$$(cd) \qquad u' = u, \qquad v' = v - 1,$$

$$(de) \qquad u' = \frac{u}{v+1}, \qquad v' = \frac{v}{v+1},$$

$$(ae) \qquad u' = \frac{-u+1+i}{i}, \qquad v' = \frac{(-1+i)u + iv + 1 - i}{i}.$$

Every other monodromy of the branch-points which leaves the surface unaltered may be considered as a combination of the above four taken in proper

order, and consequently the transformation arising from such a monodromy may be expressed in terms of the above four. To show this we may proceed as follows, distinguishing two cases, namely, when a single pair of branch-points is moved, and when more than two are moved.

Under the first case we have operations such as interchanging c and e, or moving b around d. To produce the motion of b around d, for instance, we may perform in succession $(ab)^{-1}$, (de), (ae), $(de)^{-1}$, (ab); or (de), $(ab)^{-1}$, (ae), (ab), $(de)^{-1}$. Thus for our transformation on the variables we have

$$(bd) = (ab)^{-1}(de)(ae)(de)^{-1}(ab) = (de)(ab)^{-1}(ae)(ab)(de)^{-1},$$

where the transformations on the variables are applied from *right to left* as written. The last written equality may be easily verified; for (ab) and (de) are commutative, as are their powers, since the corresponding motions on the surface may be performed simultaneously without interference. Suppose now that one of the branch-points, say a, performs any circuit whatsoever and either returns to its original position or makes the possible interchange with the branch-point b. We may resolve its path (by proper deformation) into a series of circuits around the remaining branch-points singly, together with the possible interchange at the end. Since each of these circuits about the single branch-points and the interchange are severally expressible in terms of the four fundamental motions, it follows that the transformation due to the motion of a is expressible in terms of the four fundamental transformations. So with each of the other points b, c, d, e.

In case two or more branch-points move at the same time, no difficulty will arise if the paths do not intersect. For the paths are traversed independently and the movement giving rise to one path may be considered as having been completed before any other branch-point begins to move. The above argument shows that the transformation arising from these several motions can be expressed in terms of the four fundamental ones. If the paths of the moving points do intersect, we must specify the order in which two or more branch-points pass through a given point of intersection. For two branch-points must not coincide, and moreover the order specified must be compatible with the geometric possibilities of the surface. When this order is established the portions of the paths between the points of intersection may be considered as new paths of the form treated above. At the points of intersection we have simply an interchange of two points or one point moving around another. Since any such motion can now be resolved into the four fundamental ones, each taken the proper number of times and in the proper order, it follows that the transformation arising from any monodromy of the branch-points can be expressed in terms of (ab), (cd), (de), (ae).

These four fundamental transformations are expressible in terms of the two,

$$(6)\qquad A \equiv \begin{vmatrix} 1 & 1-i & 0 \\ 1-i & -1-i & -1 \\ 0 & 1 & 0 \end{vmatrix}, \qquad B \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix},$$

the determinant of the coefficients only being written. The expressions are

$$\begin{aligned} (ab) &= BA^5B^{-1}A^5BA, \\ (cd) &= B, \\ (de) &= A^{-1}BA, \\ (ae) &= A^{-2}B^{-1}A^{-2}B^{-1}. \end{aligned}$$

Moreover A is a transformation of the given group, since

$$A = (ae)(cd)(de)(ab)^{-1}(cd).$$

Since B is non-periodic, the theorem follows:

The monodromy group G connected with the Riemann surface (1) *is of infinite order and is generated by the two operators, A and B, defined by* (6). *The quadratic form $u\bar{u} - i(v - \bar{v})$ is a relative invariant for the group G.*

§ 2. *The arithmetic form of the group.*

Since the coefficients of the generating transformations of G are complex integers, that is, are of the form $m + ni$, where m and n are integers, it follows that every transformation of G has complex integers for coefficients. We wish to characterize further these coefficients.

For this purpose write the general transformation of G in the form

$$(7)\quad u' = \frac{a_1u + b_1v + c_1}{\Delta}, \qquad v' = \frac{a_2u + b_2v + c_2}{\Delta}, \qquad \Delta = a_3u + b_3v + c_3,$$

where the coefficients are complex integers with determinant unity. Since the monodromy group leaves $\phi = i(v - \bar{v}) - u\bar{u}$ relatively invariant, the transformed form ϕ' will be connected with the original by the relation

$$\phi' = \frac{\phi}{\Delta\bar{\Delta}},$$

where $\bar{\Delta}$ is the conjugate of Δ. In order that this relation may subsist the coefficients in (7) must satisfy the conditions

$$(8)\qquad \begin{aligned} &a_1\bar{a}_1 - i(a_2\bar{a}_3 - \bar{a}_2a_3) = 1, && a_1\bar{b}_1 - i(a_2\bar{b}_3 - a_3\bar{b}_2) = 0, \\ &b_1\bar{b}_1 - i(b_2\bar{b}_3 - \bar{b}_2b_3) = 0, && a_1\bar{c}_1 - i(a_2\bar{c}_3 - a_3\bar{c}_2) = 0, \\ &c_1\bar{c}_1 - i(c_2\bar{c}_3 - \bar{c}_3c_2) = 0, && b_1\bar{c}_1 - i(b_2\bar{c}_3 - b_3\bar{c}_2) = -i. \end{aligned}$$

Using the relations (8) to solve (7), we obtain

$$
(7') \qquad
\begin{aligned}
\frac{u}{\Delta} &= \bar{a}_1 u' - i\bar{a}_3 v' + i\bar{a}_2, \\
\frac{v}{\Delta} &= i\bar{c}_1 u' + \bar{c}_3 v' - \bar{c}_2, \\
\frac{1}{\Delta} &= -i\bar{b}_1 u' - \bar{b}_3 u' + \bar{b}_2 = \Delta',
\end{aligned}
$$

which give the inverse of transformation (7). On account of the relative invariance of ϕ', the coefficients of (7′) must satisfy the relations

$$
(8') \qquad
\begin{aligned}
&a_1\bar{a}_1 + i(b_1\bar{c}_1 - \bar{b}_1 c_1) = 1, && a_1\bar{a}_2 + i(b_1\bar{c}_2 - \bar{b}_2 c_1) = 0, \\
&a_2\bar{a}_2 + i(b_2\bar{c}_2 - \bar{b}_2 c_2) = 0, && a_1\bar{a}_3 + i(b_1\bar{c}_3 - \bar{b}_3 c_1) = 0, \\
&a_3\bar{a}_3 + i(b_3\bar{c}_3 - \bar{b}_3 c_3) = 0, && a_2\bar{a}_3 + i(b_2\bar{c}_3 - \bar{b}_3 c_2) = i.
\end{aligned}
$$

Denote by $A_1, \cdots$ the minors of $a_1, \cdots$ in the determinant of (7); solve (7) for u/Δ, v/Δ, $1/\Delta$, and we obtain

$$
\begin{aligned}
\frac{u}{\Delta} &= A_1 u' + A_2 v' + A_3, \\
\frac{v}{\Delta} &= B_1 u' + B_2 v' + B_3, \\
\frac{1}{\Delta} &= C_1 u' + C_2 v' + C_3.
\end{aligned}
$$

Comparing this form of the inverse transformation with (7′), we obtain the relations

$$
(9) \qquad
\begin{aligned}
\bar{a}_1 &= b_2 c_3 - b_3 c_2 = A_1, & \bar{b}_1 &= i(a_2 b_3 - a_3 b_2) = iC_1, \\
\bar{a}_2 &= -i(b_1 c_2 - b_2 c_1) = -iA_3, & \bar{b}_2 &= a_1 b_2 - a_2 b_1 = C_3, \\
\bar{a}_3 &= -i(b_1 c_3 - b_3 c_1) = iA_2, & \bar{b}_3 &= a_1 b_3 - a_3 b_1 = -C_2, \\
\bar{c}_1 &= i(a_2 c_3 - a_3 c_2) = -iB_1, \\
\bar{c}_2 &= a_1 c_2 - a_2 c_1 = -B_3, \\
\bar{c}_3 &= a_1 c_3 - a_3 c_1 = B_2.
\end{aligned}
$$

These relations may be solved for a_1, a_2, a_3 (or a_1, b_1, c_1) in terms of the other elements, which will be connected by three bilinear relations taken from (8′) [or (8)].

Since $(a_2\bar{a}_3 - \bar{a}_2 a_3)$ and similar expressions are divisible by 2, from (8) and (8′) it follows that

$$a_1\bar{a}_1 \equiv 1, \qquad a_2\bar{a}_2 \equiv 0, \qquad a_3\bar{a}_3 \equiv 0, \qquad b_1\bar{b}_1 \equiv 0, \qquad c_1\bar{c}_1 \equiv 0 \pmod 2,$$

from which

$$a_1 = 2m + 1 + 2ni \qquad \text{or} \qquad 2m + (2n+1)i,$$

$$a_2 \equiv 0, \qquad a_3 \equiv 0, \qquad b_1 \equiv 0, \qquad c_1 \equiv 0 \quad [\bmod (1+i)].$$

From the relation $\bar{b}_2 = a_1 b_2 - a_2 b_1$ it follows that

$$\bar{b}_2 - a_1 b_2 \equiv 0 \pmod 2.$$

If $a_1 = 2m + 1 + 2ni$, this congruence gives $\bar{b}_2 - b_2 \equiv 0 \pmod 2$. If $a_1 = 2m + (2n+1)i$, the above congruence becomes

$$\bar{b}_2 - ib_2 \equiv 0 \pmod 2.$$

If we write $b_2 = b_2' + ib_2''$, this congruence becomes

$$(1-i)(b_2' + b_2'') \equiv 0 \pmod 2,$$

whence b_2' and b_2'' are of the same parity, and therefore $b_2 \equiv 0 \ [\bmod (1+i)]$. A similar argument applied to the other relations (9) yields the conditions

$$b_3 \equiv 0, \qquad c_2 \equiv 0, \qquad c_3 \equiv 0 \quad [\bmod (1+i)].$$

Consequently $b_2c_3 - b_3c_2 \equiv 0 \pmod 2$. But $\bar{a}_1 = b_2c_3 - b_3c_2 \equiv 0 \pmod 2$, which violates the assumption

$$a_1 = 2m + (2n+1)i,$$

and hence this value of a_1 is impossible. We accordingly have

$$a_1 \equiv 1 \pmod 2.$$

Therefore $a_1 = 1 + 2m + 2ni$, and by (8) and (9),

$$b_2c_3 - b_3c_2 \equiv 1, \qquad b_2\bar{c}_3 - \bar{b}_3c_2 \equiv 1 \pmod 2.$$

The residues of complex integers (mod 2) are $0, 1, i, 1+i$. The two above congruences reduce the possible combinations of residues of b_2, b_3, c_2, c_3 to 48. Using these 48 possible residues in connection with the equations (9), we are led to 192 transformations which are incongruent modulo 2.

By using the relations (9) the periods of the transformations of G may be readily determined. The characteristic equation for the general transformation of the group is

$$\begin{vmatrix} a_1 - \rho & b_1 & c_1 \\ a_2 & b_2 - \rho & c_2 \\ a_3 & b_3 & c_3 - \rho \end{vmatrix} = 0,$$

or, by (9),

$$\rho^3 - d\rho^2 + d\rho - 1 = 0, \qquad d = a_1 + b_2 + c_3 = a + bi,$$

a and b being integers. If the transformation is periodic, all the roots of the characteristic equation must be roots of unity, and conversely. Hence $|d| \leqq 3$. Let $\rho = (1 + it)/(1 - it)$, and the characteristic equation becomes

$$(1 + a)t^3 + bt^2 + (a - 3)t + b = 0.$$

If the roots of this equation are real, those of the characteristic equation are roots of unity, and conversely. The roots of this equation will be real if

$$G^2 + 4H^3 \leqq 0,$$

where

$$G = \tfrac{2}{27}(9a^2b + 36ab + 27b + b^3),$$

$$H = \frac{3(a+1)(a-3) - b^2}{9}.$$

The possible values of d for which this equation is satisfied are $-1, 0, 1, -1 \pm 2i, 2, \pm i, -1 \pm i, 3$. Construct the corresponding equation in ρ and factor; the periods follow immediately. With the notation

$$\varepsilon = e^{\frac{\pi i}{3}}, \qquad i = e^{\frac{\pi i}{2}}, \qquad j = e^{\frac{\pi i}{4}}$$

the tabulated results are:

d	-1	0	1	$-1 \pm 2i$	2	$\pm i$	$-1 \pm i$	3
ρ	$1, -1, -1$	$1, -\varepsilon, \varepsilon^2$	$1, i, -i$	$-1, \pm i, \pm i$	$1, \varepsilon, -\varepsilon^2$	$i, j, -j, -i, ij, -ij$	$-1, \pm i\varepsilon, \mp i\varepsilon^2,$	$1, 1, 1$
Period	$2, \infty$	3	4	$4, \infty$	6	8	12	∞

For $d = -1, -1 \pm 2i$ all second order minors of the determinant of the characteristic equation must vanish for the double root if the period is finite. For all values of d other than the above the substitution is non-periodic.*

* The group G, considered as abstract, is apparently characterized by only the conditions of this section. The problem arises: Is G the maximum group which satisfies these conditions? The question is as yet unanswered; but there is good reason to believe that G is the maximum group. If we transform G into a group G' by means of

$$\begin{vmatrix} -i & 1 & 1 \\ -1 & 0 & -i \\ 0 & 1 & 1 \end{vmatrix},$$

then the coefficients of each linear transformation in G' will be complex integers with determinant 1. Moreover the invariant locus for the new group is the hypersphere $u\bar{u} + v\bar{v} = 1$. The group G' is apparently the maximum group of substitutions having these restrictions, and would accordingly be the simplest analogue, in the case of two variables, to the modular group of linear transformations on a single complex variable.

§ 3. *The hyperfuchsian functions.*

Using the general transformation of the group G in its homogeneous form

$$
(7'')\qquad
\begin{aligned}
x' &= a_1 x + b_1 y + c_1 z,\\
y' &= a_2 x + b_2 y + c_2 z,\\
z' &= a_3 x + b_3 y + c_3 z,
\end{aligned}
$$

we see that corresponding to this linear transformation on the variables, the elements of the table of moduli (3) undergo a linear transformation. The corresponding transformation on the allied theta series requires the determination of a set of integral coefficients $\alpha_{\iota\kappa}$, $\beta_{\iota\kappa}$, $\gamma_{\iota\kappa}$, $\delta_{\iota\kappa}$ which satisfy the identities

$$
-B'^{(i)}_{\kappa} = \sum_{\mu=1}^{3} (-\alpha_{\kappa\mu} B^{(i)}_{\mu} + \beta_{\kappa\mu} A^{(i)}_{\mu}),
$$

$$
A'^{(i)}_{\kappa} = \sum_{=}^{3} (-\gamma_{\kappa\mu} B^{(i)}_{\mu} + \delta_{\kappa\mu} A^{(i)}_{\mu}),
$$

wherein $A^{(i)}_{\kappa}$, etc., have the meanings assigned in § 1, and $A'^{(i)}_{\kappa}$ is the value of $A^{(i)}_{\kappa}$ after transformation. Replacing $A^{(i)}_{\kappa}$, $A'^{(i)}_{\kappa}$, etc., in the above identities by their values given by (5) and (7″), the theta transformation is given by

$$
\begin{array}{ccc|ccc}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \beta_{11} & \beta_{12} & \beta_{13}\\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_{21} & \beta_{22} & \beta_{23}\\
-\beta_{21} & -\alpha_{23} & \alpha_{22} & \alpha_{21} & -\beta_{23} & \beta_{22}\\
\hline
-\beta_{11} & -\alpha_{13} & \alpha_{12} & \alpha_{11} & -\beta_{13} & \beta_{12}\\
\gamma_{21} & \gamma_{22} & \gamma_{23} & \delta_{21} & \delta_{22} & \delta_{23}\\
-\delta_{21} & -\gamma_{23} & \gamma_{22} & \gamma_{21} & -\delta_{23} & \delta_{22}
\end{array}
$$

wherein the coefficients are subject to certain well-known bilinear relations, and are connected with the coefficients of (7″) by means of the relations

$$
(10)\qquad
\begin{aligned}
a_1 &= \alpha_{11} - \beta i_{11}, & b_1 &= -\beta_{13} - i\beta_{12}, & c_1 &= \alpha_{13} + i\alpha_{12},\\
a_2 &= \delta_{21} + i\gamma_{21}, & b_2 &= \delta_{22} - i\delta_{23}, & c_2 &= -\gamma_{22} + i\gamma_{23},\\
a_3 &= -\beta_{21} - i\alpha_{21}, & b_3 &= -\beta_{22} - i\beta_{23}, & c_3 &= \alpha_{22} - i\alpha_{23}.
\end{aligned}
$$

The bilinear relations mentioned above, when taken by columns, reduce to the following:

$$
\alpha_{11}^2 + 2\alpha_{21}\delta_{21} + \beta_{11}^2 - 2\beta_{21}\gamma_{21} = 1,
$$

$$
-\alpha_{12}\beta_{13} + \alpha_{13}\beta_{12} + \alpha_{22}\delta_{22} + \alpha_{23}\delta_{23} - \beta_{22}\gamma_{22} - \beta_{23}\gamma_{23} = 1,
$$

$$\alpha_{11}\alpha_{13} - \alpha_{12}\beta_{11} - \alpha_{21}\gamma_{22} + \alpha_{22}\gamma_{21} + \alpha_{23}\delta_{21} - \beta_{21}\gamma_{23} = 0,$$

$$\alpha_{11}\alpha_{12} + \alpha_{13}\beta_{11} + \alpha_{21}\gamma_{23} + \alpha_{22}\delta_{21} - \alpha_{23}\gamma_{21} - \beta_{21}\gamma_{22} = 0,$$

$$\alpha_{11}\beta_{13} - \alpha_{21}\delta_{22} - \beta_{11}\beta_{12} - \beta_{21}\delta_{23} + \beta_{22}\gamma_{21} + \beta_{23}\delta_{21} = 0,$$

$$\alpha_{11}\beta_{12} + \alpha_{21}\delta_{23} + \beta_{11}\beta_{13} - \beta_{21}\delta_{22} + \beta_{22}\delta_{21} - \beta_{23}\gamma_{21} = 0,$$

$$\alpha_{12}\beta_{12} + \alpha_{13}\beta_{13} + \alpha_{22}\delta_{23} - \alpha_{23}\delta_{22} + \beta_{22}\gamma_{23} - \beta_{23}\gamma_{22} = 0,$$

$$\alpha_{12}^2 + \alpha_{13}^2 + 2\alpha_{22}\gamma_{23} - 2\alpha_{23}\gamma_{22} = 0,$$

$$\beta_{12}^2 + \beta_{13}^2 + 2\beta_{22}\delta_{23} - 2\beta_{23}\delta_{22} = 0.$$

Rewrite the first, ninth, and eighth of these equations as indicated below. Multiply the sixth equation by i, and subtract the fifth from the result. Multiply the fourth by $-i$, and add the third to the result. From the second multiplied by $-i$ subtract the seventh. The results of these operations in the order indicated are

$$(\alpha_{11} - i\beta_{11})(\alpha_{11} + i\beta_{11}) - i(\delta_{21} + i\gamma_{21})(-\beta_{21} + i\alpha_{21}) + i(\delta_{21} - i\gamma_{21})(-\beta_{21} - i\alpha_{21}) = 1,$$

$$(-\beta_{13} - i\beta_{12})(-\beta_{13} + i\beta_{12}) - i(\delta_{22} - i\delta_{23})(-\beta_{22} - i\beta_{23}) + i(\delta_{22} + i\delta_{23})(-\beta_{22} + i\beta_{23}) = 0,$$

$$(\alpha_{13} + i\alpha_{12})(\alpha_{13} - i\alpha_{12}) - i(-\gamma_{22} + i\gamma_{23})(\alpha_{22} + i\alpha_{23}) + i(-\gamma_{22} - i\gamma_{23})(\alpha_{22} - i\alpha_{23}) = 0,$$

$$(\alpha_{11} - i\beta_{11})(-\beta_{13} + i\beta_{12}) - i(\delta_{21} + i\gamma_{21})(-\beta_{22} - i\beta_{23}) + i(-\beta_{21} - i\alpha_{21})(\delta_{22} + i\delta_{23}) = 0,$$

$$(\alpha_{11} - i\beta_{11})(\alpha_{13} - i\alpha_{12}) - i(\delta_{21} + i\gamma_{21})(\alpha_{22} + i\alpha_{23}) + i(-\beta_{21} - i\alpha_{21})(-\gamma_{22} - i\gamma_{23}) = 0,$$

$$(-\beta_{13} - i\beta_{12})(\alpha_{13} - i\alpha_{12}) - i(\delta_{22} - i\delta_{23})(\alpha_{22} + i\alpha_{23}) + i(-\beta_{22} + i\beta_{23})(-\gamma_{22} - i\gamma_{23}) = -i.$$

In these equations substitute the values of a_i, b_i, c_i from (10), and we obtain the fundamental equations of condition (8). In like manner the bilinear relations obtained by taking the elements of the theta-transformation by rows may be combined to produce a set of equations similar to the above, out of which the conditions of (8′) can be formed.

Since the equations (9) are consequences of the conditions (8) and (8′), we may substitute in (9) the values of a_i, b_i, c_i given by (10). Separating the real and imaginary parts of the resulting equations, we obtain a solution of the bilinear relations connecting the coefficients of the theta transformation. This solution gives six coefficients explicitly in terms of the remaining twelve, which are connected by four bilinear relations.

Because of the particular moduli of the problem at hand certain linear relations exist among the theta functions. To determine what these are we follow the values of any theta function under the transformation which corresponds to identity in G. For this latter $a_1 = b_2 = c_3$, $a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 0$.

Using (10), we find the theta transformation to be

$$\begin{array}{ccc|ccc} m & 0 & 0 & n & 0 & 0 \\ 0 & m & n & 0 & 0 & 0 \\ 0 & -n & m & 0 & 0 & 0 \\ \hline -n & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & 0 & -n & m \end{array}$$

with $m^2 + n^2 = 1$, and m, n integers. This transformation yields four cases, viz.: $m = 1,\ n = 0;\ m = -1,\ n = 0;\ m = 0,\ n = +1;\ m = 0,\ n = -1$. The corresponding relations between the theta functions with zero arguments are*

$$(11)\qquad \begin{aligned} \vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)) &= \vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)), \\ \vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)) &= \vartheta\begin{bmatrix} -g \\ -h \end{bmatrix}((0)), \\ \vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)) &= e^{2\pi i g_1 h_1}\vartheta\begin{bmatrix} -h_1 & g_3 & -g_2 \\ g_1 & h_3 & -h_2 \end{bmatrix}((0)), \\ \vartheta\begin{bmatrix} g \\ h \end{bmatrix}((0)) &= e^{2\pi i g_1 h_1}\vartheta\begin{bmatrix} h_1 & -g_3 & g_2 \\ -g_1 & -h_3 & h_2 \end{bmatrix}((0)). \end{aligned}$$

By using the relations (10) the theta-transformations corresponding to the generators of G are determined to be

$$A:\ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 & -1 & -1 \end{array},\qquad B^n:\ \begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & n & 0 & 0 & 1 & 0 \\ 0 & 0 & n & 0 & 0 & 1 \end{array},$$

and the corresponding transformations on the theta-functions with zero arguments are†:

$$(12)\qquad \begin{aligned} A:\ \vartheta\left[\begin{smallmatrix} g_1-h_2+h_3 & h_2 & h_3 \\ h_1+h_2+h_3 & g_1-g_2+h_1-h_2+h_3 & g_1-g_3-h_1-h_2-h_3 \end{smallmatrix}\right]((0))_b \\ = ive^{-2\pi i[h_2(h_2-g_1+g_2)+h_3(g_1-g_3)]}\vartheta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]((0))_a, \\ B^n:\ \vartheta\left[\begin{smallmatrix} g_1 & g_2 & g_3 \\ h_1 & \frac{1}{2}n-ng_2+h_2 & \frac{1}{2}n-ng_3+h_3 \end{smallmatrix}\right]((0))_b = e^{n\pi i[g_2+g_3-g_2^2-g_3^2]}\vartheta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]((0))_a. \end{aligned}$$

* KRAZER u. PRYM, *Neue Grundlagen einer Theorie der allgemeinen Thetafunktionen*, etc., p. 121.
† Loc. cit.

We now wish to determine the simplest possible products of theta functions with zero arguments which are relatively invariant under the group G. If a product of two functions ϑ_1 and ϑ_2 is relatively invariant, then it follows that ϑ_1 is a relative invariant under A^2 and B^2. For this to be true, the transformed characteristic must be reducible to one of the four given in (11). In addition to this, the characteristic of ϑ_1 must be transformed into that of ϑ_2 by A or B. This requires further conditions similar to the above. The characteristic of ϑ_1 will then be the simultaneous solution of these sets of conditions, if such a solution exists. A similar procedure may be used to determine whether the product of several theta-functions can be relatively invariant under G or not. The following products are believed to be the simplest:

$$
(13)\qquad
\begin{aligned}
\phi_1 &\equiv \begin{bmatrix}000\\000\end{bmatrix}\begin{bmatrix}000\\011\end{bmatrix}\begin{bmatrix}011\\000\end{bmatrix},\\
\phi_2 &\equiv \begin{bmatrix}000\\100\end{bmatrix}\begin{bmatrix}000\\111\end{bmatrix}\begin{bmatrix}011\\100\end{bmatrix}\begin{bmatrix}011\\111\end{bmatrix},\\
\phi_3 &\equiv \begin{bmatrix}000\\110\end{bmatrix}\begin{bmatrix}000\\101\end{bmatrix}\begin{bmatrix}001\\100\end{bmatrix}\begin{bmatrix}001\\110\end{bmatrix}\begin{bmatrix}010\\100\end{bmatrix}\begin{bmatrix}010\\101\end{bmatrix},\\
\phi_4 &\equiv \begin{bmatrix}000\\010\end{bmatrix}\begin{bmatrix}010\\000\end{bmatrix}\begin{bmatrix}010\\001\end{bmatrix}\begin{bmatrix}101\\101\end{bmatrix}\begin{bmatrix}101\\111\end{bmatrix}\begin{bmatrix}111\\101\end{bmatrix},
\end{aligned}
$$

in which for the sake of brevity we write

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix} \equiv \vartheta\begin{bmatrix}\frac{1}{2}\lambda\\ \frac{1}{2}\mu\end{bmatrix}((0))_a$$

(λ, μ integers). The function $\phi \equiv \left[\begin{smallmatrix}011\\011\end{smallmatrix}\right]$ is also relatively invariant, but from (11) we find that $\phi = -\phi$, and hence that $\phi \equiv 0$. Denote by ϕ_i' the transformed form of ϕ_i. By using (11) and (12) the following table is deduced for the generating transformations:

	ϕ_1'	ϕ_2'	ϕ_3'	ϕ_4'
A	$-v^3\phi_1$	$-v^4\phi_2$	$v^6\phi_3$	$v^6\phi_4$
B	$i\phi_1$	$-\phi_2$	$-\phi_3$	$-\phi_4$

None of the functions ϕ_i vanish identically. To show this it will be sufficient to find values of u, v for which some one theta function of each ϕ_i is different from zero, since the factors of each product are permuted by the transformations of the group. Take the particular values $u = 0$, $v = -i$, which satisfy the condition (4), and substitute in the functions $\left[\begin{smallmatrix}000\\000\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}000\\100\end{smallmatrix}\right] = \left[\begin{smallmatrix}100\\000\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}010\\100\end{smallmatrix}\right] = \left[\begin{smallmatrix}101\\000\end{smallmatrix}\right]$,

and $\left[\begin{smallmatrix}010\\000\end{smallmatrix}\right]$. This gives

$$\begin{bmatrix}000\\000\end{bmatrix}=\sum_{m_i=-\infty}^{\infty} e^{-\pi[m_1^2+m_2^2+m_3^2]},$$

$$\begin{bmatrix}100\\000\end{bmatrix}=\sum_{m_i=-\infty}^{\infty} e^{-\pi[(m_1+\frac{1}{2})^2+m_2^2+m_3^2]},$$

$$\begin{bmatrix}101\\000\end{bmatrix}=\sum_{m_i=-\infty}^{\infty} e^{-\pi[(m_1+\frac{1}{2})^2+m_2^2+(m_3+\frac{1}{2})^2]},$$

$$\begin{bmatrix}010\\000\end{bmatrix}=\sum_{m_i=-\infty}^{\infty} e^{-\pi[m_1^2+(m_2+\frac{1}{2})^2+m_3^2]},$$

which evidently do not vanish.

None of the functions ϕ_1, ϕ_2, ϕ_3 vanishes identically, and the functions

$$F_1=\frac{\phi_3}{\phi_1^2}, \qquad F_2=\frac{\phi_4}{\phi_1^2} \tag{15}$$

are absolutely invariant under the group G. These functions are believed to be as simple as any of the kind under discussion and belonging to the group.*

§ 4. *Linear independence of the moduli.*

It will be observed that in Table (3) the six moduli of a given integral u_i are expressed linearly in terms of three such. This is a particular case of a general theorem which we proceed to deduce.

Denote by

$$u_i=\int\frac{R_i(z)\,dz}{w^{\eta_i}} \qquad (i=1,2,\cdots,p),$$

where $R_i(z)$ is a rational integral function of z, and η_i is an integer, a system of p linearly independent integrals of the first kind connected with the Riemann surface

$$w^\nu=(z-a)^\alpha(z-b)^\beta\cdots(z-n)^\lambda$$

having μ branch-points which may be assumed finite, and with

$$0<\alpha<\nu,\ 0<\beta<\nu,\ \cdots,\ 0<\lambda<\nu,\ \alpha+\beta+\cdots+\lambda\equiv 0 \pmod{\nu}.$$

Denote the different values of w corresponding to the same value of z by $w_1, w_2, \cdots, w_\nu$, and assign the branch w_m to the mth sheet of the surface.

* PICARD has shown (Acta Mathematica, vol. 5 (1884), p. 175) that we can choose two hyperfuchsian functions, relative to a group G, such that every other function belonging to the same group is an algebraic function of the two chosen. Whether or not the two functions (15) are suitable for that purpose cannot be determined without a knowledge of the fundamental region for G.

The several values of w will be connected by the relation

$$w_m = \theta^{(m-1)} w_1, \qquad \theta = e^{\frac{2\pi i}{\nu}}.$$

Denote by u any integral of the first kind, and by $u^{(m)}$ the value of u taken over any path starting from a point in sheet m. Then

$$(17) \qquad u^{(m)} = \int \frac{R(z)\,dz}{w_m^\eta} = \int \frac{R(z)\,dz}{\theta^{(m-1)\eta} w_1^\eta} = \theta^{\rho_m} \int \frac{R(z)\,dz}{w_1^\eta} = \theta^{\rho_m} u^{(1)},$$

where $(m-1)\eta + \rho_m \equiv 0 \pmod{\nu}$, with $0 \leqq \rho_m < \nu$.

By a simple circuit about a branch-point is meant a loop which starts and ends at z_0 in two different sheets which are successive in the order of the winding of the sheets for that branch-point. Denote by $\alpha^{(m)}, \beta^{(m)}, \cdots$, the values of $u^{(m)}$ along simple circuits about the branch-points $a, b, \cdots$, respectively. The values of $\alpha^{(m)}, \beta^{(m)}, \cdots$, for the different values of m are connected by the relation (17) as follows: $\alpha^{(m)} = \theta^{\sigma_m}\alpha^{(1)}$, $\beta^{(m)} = \theta^{\sigma'_m}\beta^{(1)}, \cdots$, where σ_m, σ'_m, etc., are defined similarly to ρ_m. Since any closed path on the surface can be resolved by continuous deformation into a set of simple circuits about the branch-points, it follows that the value of $u^{(m)}$ along any closed path may be expressed in the form

$$c'_1\alpha^{(1)} + c'_2\beta^{(1)} + \cdots + c'_\mu\lambda^{(1)},$$

where c'_e is a sum of roots of unity. For the integrals along the several simple circuits about a may each be expressed in the form $\pm\,\theta^{\sigma_m}\alpha^{(1)}$, depending on the direction in which the integral is taken; consequently the portion of the value of $u^{(m)}$ due to paths around a is of the form $c'_1\alpha^{(1)}$ where c'_1 is a sum of roots of unity. Similarly for the other branch-points. In particular, take u positively over a path around all the branch-points, and we obtain a relation of the form

$$(18) \qquad c_1\alpha^{(1)} + c_2\beta^{(1)} + \cdots + c_\mu\lambda^{(1)} = 0,$$

since the given path may be deformed into a point path.

Assume an appropriate set of branch-lines and cross-cuts, and define A_j, B_j, as in § 1. Since the moduli of periodicity of u are the results of integrating u along these cross-cuts of the surface, we have as the values of A_j, B_j

$$(19) \qquad \begin{aligned} A_j &= c_{1j}\alpha^{(1)} + c_{2j}\beta^{(1)} + \cdots + c_{\mu j}\lambda^{(1)}, \\ B_j &= d_{1j}\alpha^{(1)} + d_{2j}\beta^{(1)} + \cdots + d_{\mu j}\lambda^{(1)}, \end{aligned} \qquad (j = 1, 2, \cdots, p),$$

where c_{ij}, d_{ij} are constants. Substitute the value of $\lambda^{(1)}$ from (18) in the equations (19) and we have $2p$ equations expressing the moduli of periodicity A_j, B_j in terms of $\mu - 1$ quantities $\alpha^{(1)}, \beta^{(1)}, \cdots$. Not more than $\mu - 2$ of the $2p$ moduli $A_j, B_j (j = 1, 2, \cdots p)$ are linearly independent. To prove this, assume that $\mu - 1$ of them are independent. Solve these $\mu - 1$ linearly independent

equations for each of the $\mu - 1$ quantities $\alpha^{(1)}, \beta^{(1)}, \cdots$. Thus we find that each of the quantities $\alpha^{(1)}, \beta^{(1)}, \cdots$ is expressed as a linear function of $\mu - 1$ constants A_j, B_j. In particular, then,

$$\alpha^{(1)} = \text{const.}$$

But since $\alpha^{(1)}$ is the value of $u^{(1)}$ along an open path on the Riemann surface, it is a function of the end points and is not constant. Consequently the assumption made is false, and we have the theorem: *Not more than $\mu - 2$ of the moduli of periodicity of an abelian integral of the first kind associated with a binomial Riemann surface having μ branch-points are linearly independent.* *

§ 5. *The fundamental transformations for the group of the general binomial surface.*

In § 1 certain fundamental transformations arising from the monodromy of the branch-points of a Riemann surface were determined, and from them the generating tranformations of the group were found. The method employed there will be generalized in the present section; by analyzing separate cases we are finally led to the general theorem. We assume the conventions and definitions of § 1.

Consider the binomial Riemann surface

$$w^\nu = (z - a_1)^\alpha (z - a_2)^\alpha \cdots (z - a_m)^\alpha,$$

where $m\alpha \equiv 0 \pmod{\nu}$, $0 < \alpha < \nu$, α and ν integers and relative primes. Let z_0 be any arbitrarily chosen point of the surface, and with it as center swing a radius vector counter-clockwise; let the branch-points be numbered $1, 2, \cdots, m$, in the order in which the radius vector meets them. Denote by $(kl) = (lk)$ the transformation produced on any set of variables, functionally connected with the branch points a_k, a_l of the surface, due to the interchange of these branch-points. In the following discussion the symbol stands for the interchange itself. The interchange of the points is to be made in the manner indicated on pp. 40, 41.

If a_i, a_k, a_l are three branch-points such that $i < k < l$, then (il) is equivalent to either $t_1 = (ik)^{-1}(kl)(ik)$ or $t_2 = (kl)(ik)(kl)^{-1}$. For in t_1, $(ik)^{-1}$ means the interchange of a_i and a_k clockwise, then (kl) the interchange of a_l and a_k $(\equiv a_i)$ counter-clockwise, thus putting a_i in the place of a_l, then (ik) the interchange of a_k $(\equiv a_l)$ and a_i $(\equiv a_k)$ putting a_l in place of a_i and the path of a_k may be deformed into a point path without passing over a branch-point: the paths of a_i and a_k are between z_0 and a_l. A similar analysis verifies t_2. An analogous procedure applies if four or more branch-points are permuted to obtain an interchange. In particular, if $k < l$, we may express (kl) in terms

* See FRICKE-KLEIN, *Automorphe Functionen*, II, p. 135, where the essential statement of this theorem for $\mu = 4$ is credited to J. WELLSTEIN.

of successive interchanges of branch-points, and obtain

$$(kl) = (k, k+1)^{-1}(k+1, k+2)^{-1} \cdots (l-2, l-1)^{-1}(l-1, l)(l-1, l-2)\cdots(k+1, k).$$

It follows that the transformations due to every simple interchange can be expressed in terms of $m-1$ transformations $(i, i+1)$, $i = 1, 2, \cdots, m-1$.

We will next discuss all other admissible types of motions of the branch-points and the corresponding transformations of the variables. Let a_k perform a complete loop about a_l; the resultant transformation is $(kl)^2$, for the complete loop is the succession of two interchanges of the points a_k and a_l. In such a case the path of neither point can be deformed to a point-path. Next assume that a_k and a_l $(k < l)$ are interchanged so that a branch-point a_i is enclosed in the total path of the other points. The resultant transformation is

$$(ki)(il)(ki) = (il)(ki)(il),$$

for in either case the path of a_i is a point-path, that of a_l is between z_0 and a_i while a_i is between z_0 and that of a_k. If a_j is situated similarly to a_i then the resulting transformation is

$$(ki)(ij)(jl)(ij)(ik) = (lj)(ji)(ik)(ji)(jl).$$

In case a_i lies between z_0 and the paths of a_k, a_l, the corresponding transformation is

$$(ki)(il)(ik)^{-1} = (il)^{-1}(ik)(il),$$

and similarly for more points situated like a_i.

By an argument similar to that on p. 41 we may resolve any motion of any number of branch-points into a sequence of motions of the types discussed above, and may consequently express the resultant transformations in terms of $m-1$ successive interchanges designated above.

Next consider the binomial Riemann surface

$$w^\nu = (z-a_1)^\alpha \cdots (z-a_m)^\alpha (z-b_1)^\beta \cdots (z-b_n)^\beta,$$

where $0 < \alpha \neq \beta < \nu$, $m\alpha + n\beta \equiv 0 \pmod{\nu}$, α, β, and ν are integers, and either α or β is relatively prime to ν. In a manner similar to that above we may select two sets of $m-1$, $n-1$ transformations in terms of which all the transformations due to all motions of the points a among themselves, and to the points b among themselves may be expressed. Denote by T_{12} the transformation arising from the motion of some point a around some point b once. Evidently the points can be so chosen that the last point of the set a moves around the first point of the set b. Then every transformation arising from proper motions of the points a, b can be expressed in terms of the $m+n-1$ indicated transformations. To prove this it will evidently be sufficient to express the

transformation arising from the motion of an arbitrary point a around an arbitrary point b. Let a_i and b_j be the points. Interchange a_j and a_m, b_j and b_1; then use T_{12}, which puts a_m around b_1; then interchange a_i and a_m, b_j and b_1, care being taken as to the direction of interchange. This completes the motions, and all paths except those of a_i, b_j are point-paths, thus furnishing the required representation of the transformation. In general the number $m+n-1$ of fundamental transformations cannot be reduced. If, however, $n=1$, then $T_{12}=(m, m-1)^{-1}\cdots(12)^{-2}\cdots(m, m-1)^{-1}$, and the number of fundamental transformations is not greater than $m-1$.

In the case of several sets of branch-points $m_1, m_2, \cdots, m_n$ in number, which cannot be interchanged, a fundamental transformation will be required corresponding to the motion of some point of a set around a point of every other set. The corresponding transformations will be designated by T_{kl}, where k and l are different and refer to the above n sets of branch-points. The number of these will be $\frac{1}{2}n(n-1)$, and this number will diminish by unity if some m_i is unity. The number of transformations of each set is known from the above discussion to be m_l-1. To sum up, we have the following theorem:

The number of generators of the group arising from all possible motions of the branch-points of the binomial Riemann surface

$$w^\nu=\prod_{m_1}(z-a_i)^{\alpha_1}\prod_{m_2}(z-b_i)^{\alpha_2}\cdots\prod_{m_n}(z-l_i)^{\alpha_n},$$

where

$$0<\alpha_i\neq\alpha_j<\nu, \qquad \sum_{i=1}^{n}m_i\alpha_i\equiv 0 \pmod{\nu},$$

α_i and ν are integers, and some α_i is relatively prime to ν, does not exceed

$$\sum_{i=1}^{n}m_i-n+\frac{n(n-1)}{2}=\sum_{i=1}^{n}m_i+\frac{n(n-3)}{2}.$$

If a detailed knowledge of an individual case is available, this number may be reduced. The group of § 1 is an instance in point.

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PERIODIC ORBITS ABOUT AN OBLATE SPHEROID

BY

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§ 1. *Introduction.*

The orbit of a particle about an oblate spheroid is not in general closed geometrically. The motion of the particle is not, therefore, in general, periodic from a geometric point of view. But if we consider the orbits as described by the particle in a revolving meridian plane which passes constantly through the particle several classes of closed orbits can be found in which the motion is periodic. The failure of these orbits to close in space arises from the incommensurability of the period of rotation of the line of nodes with the period of motion in the revolving plane. When these periods happen to be commensurable the orbits are closed in space and the motion is therefore periodic, though the period may be very great. Indeed, it seems that most of the difficulty in giving mathematical expressions for the orbits about an oblate spheroid rests upon the question of incommensurability of periods. The difficulty arising from the motion of the node can be overcome by the use of the revolving plane, but other incommensurabilities, such as that introduced by the eccentricity, can not be eliminated in this manner.

Orbits closed in the revolving plane are considered most conveniently in two general classes: (1) Those which reënter after one revolution, (2) those which reënter after many revolutions. The existence of both classes is established in this paper and convenient methods for constructing the solutions are given. Orbits which reënter after the first revolution are naturally the simpler and will be considered in the first part of the paper. Those lying in the equatorial plane of the spheroid become straight lines in the revolving plane, and it is shown that within the realm of convergence of the series employed all orbits in the equatorial plane are periodic. When the orbits do not lie in the equatorial plane there exists one, and only one, orbit for any arbitrarily assigned values of the inclination and the mean distance. These orbits reduce to circles with the vanishing of the oblateness of the spheroid.

In considering orbits which reënter only after many revolutions the differential equations are found to be very complex, and one would despair of proving the existence of these orbits by the ordinary direct computation of the necessary coefficients. However, a proof of their existence and a method for the construc-

tion of the solutions are given by the aid of certain theorems, which are here established, on the character of the solutions of non-homogeneous linear differential equations with periodic coefficients.

These periodic orbits of many revolutions involve five constants, four of which are entirely arbitrary, and the fifth subject only to the limitation that it shall satisfy certain commensurability conditions. One constant, only, is missing for a complete integration of the differential equations. These orbits are all symmetric with respect to the equatorial plane.

§ 2. *The differential equations.*

The differential equations of motion of a particle about an oblate spheroid are *

$$
(1)\qquad
\begin{aligned}
\frac{d^2x}{dt^2} &= -\frac{k^2Mx}{R^3}\left[1+\tfrac{3}{10}b^2\,\frac{x^2+y^2-4z^2}{R^4}\,\mu^2+\cdots\right]=\frac{\partial V}{\partial x},\\
\frac{d^2y}{dt^2} &= -\frac{k^2My}{R^3}\left[1+\tfrac{3}{10}b^2\,\frac{x^2+y^2-4z^2}{R^4}\,\mu^2+\cdots\right]=\frac{\partial V}{\partial y},\\
\frac{d^2z}{dt^2} &= -\frac{k^2Mz}{R^3}\left[1+\tfrac{3}{10}b^2\,\frac{3(x^2+y^2)-2z^2}{R^4}\,\mu^2+\cdots\right]=\frac{\partial V}{\partial z}.
\end{aligned}
$$

The symbols employed are defined as follows: x, y, z are rectangular coördinates, the origin being at the center of the spheroid and the xy-plane the plane of the equator, k is the Gaussian constant, M is the mass of the spheroid, b is the polar radius of the spheroid, μ is the eccentricity of the spheroid,

$$
R=\sqrt{x^2+y^2+z^2},\qquad V=\frac{k^2M}{R}\left[1+\frac{b^2}{10}\,\frac{x^2+y^2-2z^2}{R^4}\,\mu^2+\cdots\right].
$$

Since

$$
\frac{1}{x}\frac{\partial V}{\partial x}=\frac{1}{y}\frac{\partial V}{\partial y},
$$

we obtain one integral of areas, namely,

$$
(2)\qquad x\frac{dy}{dt}-y\frac{dx}{dt}=c_1.
$$

That is, the projection of the area described by the radius vector upon the equatorial plane is proportional to the time. We have also the vis viva integral

$$
(3)\qquad \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=2V+c_2.
$$

For further integration we are compelled to resort to series. Poincaré has shown in his *Les méthodes nouvelles de la mécanique céleste* that if certain

* MOULTON, *Celestial Mechanics*, p. 113.

conditions are fulfilled it is possible to obtain periodic orbits represented by power series in a parameter when periodic orbits are known for zero values of this parameter. These conditions are at least partially fulfilled in the present problem, for the right members of the differential equations are analytic in x, y, z and the parameter μ. Furthermore, for $\mu = 0$, the equations reduce to the ordinary two-body problem for which periodic solutions are known. It is our purpose to show that the remaining necessary conditions also are fulfilled and that periodic solutions persist for values of $\mu \neq 0$. These periodic series are very satisfactory, for the general character of the orbits represented by them is easily obtained. The solutions thus found are rigorous, but they are not general, their existence depending upon special initial conditions.

It will be advantageous to transform the differential equations to cylindrical coördinates by the substitutions

$$
(4)\qquad
\begin{aligned}
x &= ar\cos v, & k^2 M &= n^2 a^3,\\
y &= ar\sin v, & c_1 &= ck\sqrt{Ma},\\
z &= aq, & nt &= \tau,\\
R &= a\sqrt{r^2+q^2}, & \frac{3}{10}\frac{b^2}{a^2} &= \theta_1^2.
\end{aligned}
$$

After these substitutions equations (1) become

$$
(5)\qquad
\begin{aligned}
&(a) & r'' - rv'^2 &= \frac{-r}{(r^2+q^2)^{\frac{3}{2}}}\left[1 + \frac{r^2-4q^2}{(r^2+q^2)^2}\theta_1^2\mu^2 + \cdots\right],\\
&(b) & rv'' + 2r'v' &= 0,\\
&(c) & q'' &= \frac{-q}{(r^2+q^2)^{\frac{3}{2}}}\left[1 + \frac{3r^2-2q^2}{(r^2+q^2)^2}\theta_1^2\mu^2 + \cdots\right],
\end{aligned}
$$

the accents denoting the derivatives with respect to τ.

The integral of (b) is $r^2 v' = c$, by means of which v' can be eliminated from the first of these equations. After the elimination the equations take the form

$$
(6)\qquad
\begin{aligned}
&(a) & r'' &= \frac{c^2}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} - \frac{r^3-4rq^2}{(r^2+q^2)^{\frac{7}{2}}}\theta_1^2\mu^2 + \cdots,\\
&(b) & q'' &= -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} - \frac{3r^2q-q^3}{(r^2+q^2)^{\frac{7}{2}}}\theta_1^2\mu^2 + \cdots,\\
&(c) & v' &= \frac{c}{r^2}.
\end{aligned}
$$

The first two of these equations are independent of the third so that r and q may be considered as rectangular coördinates in a revolving plane which passes

always through the z-axis and through the particle itself. The problem is thus reduced to the consideration of the motion in this plane, for, when r is known, v is obtained by a simple quadrature from (6c).

§ 3. *Surfaces of zero velocity.*

The velocity integral in the revolving plane is

$$r'^2 + q'^2 = \frac{2}{(r^2+q^2)^{\frac{1}{2}}} + \frac{2}{3}\frac{r^2-2q^2}{(r^2+q^2)^{\frac{5}{2}}}\theta_1^2\mu^2 + \cdots - \frac{c^2}{r^2} + c_2.$$

If we put the velocity equal to zero the resulting equation represents a two-parameter family of curves. For assigned values of the parameters c and c_2 there is defined a certain curve in the revolving plane. On one side of this curve the motion is real while on the other side it is imaginary. For values of $c_2 < 0$ this curve is closed, and the motion is real on the inside. As the plane revolves this curve generates a surface of the general form of an anchor ring. For $\mu^2 = 0$ this curve belongs to the ordinary two-body problem, and its equation is

$$\frac{2}{(r^2+q^2)^{\frac{1}{2}}} - \frac{c^2}{r^2} + c_2 = 0.$$

The motion is elliptic, parabolic, or hyperbolic according as c_2 is negative, zero, or positive. Putting

$$r = \rho\cos\phi, \qquad q = \rho\sin\phi,$$

we find, on solving for ρ,

$$\rho = \frac{1}{c_2}\left[-1 \pm \sqrt{1 + \frac{c_2 c^2}{\cos^2\phi}}\right].$$

For negative values of c_2 this equation represents two closed ovals which do not enclose the origin. If $c_2 c^2 = -1$ the ovals shrink upon the points $\rho = -1/c_2$, $\phi = 0$ and π. The corresponding orbit is therefore a circle in the equatorial plane. As c_2 approaches zero the ovals open out rapidly and approach the limiting curves

$$\rho = \frac{c^2}{2\cos^2\phi}.$$

For values of $c_2 > 0$ there is but one positive value for ρ, which is

$$\rho = \frac{1}{c_2}\left[-1 + \sqrt{1 + \frac{c_2 c^2}{\cos^2\phi}}\right].$$

If $c^2 \neq 0$ none of these curves crosses the axis $\phi = 90°$. But if $c = 0$ we have the circle $\rho = -2/c_2$ inside of which the motion is real when c_2 is negative.

For values of $\mu^2 \neq 0$, but sufficiently small, we can put

$$r = (\rho + \bar{\rho}) \cos \phi, \qquad q = (\rho + \bar{\rho}) \sin \phi,$$

and solve for $\bar{\rho}$ as a power series in μ^2. We find

$$\bar{\rho} = \frac{1}{3} \frac{2 - 3 \cos^2 \phi}{\rho(1 + c_2 \rho)} \theta_1^2 \mu^2 + \cdots,$$

which is the correction to be applied to the corresponding curves in the two-body problem.

Part I.

ORBITS WHICH REËNTER AFTER ONE REVOLUTION.

§ 4. *Symmetry.*

Returning to the differential equations (6*a*) and (6*b*) we observe that if we change

$$r \text{ into } + r, \qquad q \text{ into } - q, \qquad \tau \text{ into } - \tau,$$

the differential equations remain unchanged. Hence, if at some epoch, $\tau = \tau_0$,

$$r = \alpha, \qquad r' = 0,$$

$$q = 0, \qquad q' = \beta,$$

that is, if at the epoch $\tau = \tau_0$, the particle crosses the r-axis perpendicularly, it follows from the form of the differential equations that the orbit is symmetrical with respect to the r-axis and with respect to the epoch $\tau = \tau_0$. In other words r will be an even series in $(\tau - \tau_0)$ and q will be an odd series in $(\tau - \tau_0)$. If now at some other epoch, $\tau = \tau_0 + T$, the particle again crosses the r-axis perpendicularly the orbit will be symmetrical with respect to this epoch also. It is clear therefore that the orbit is a closed one, and that the motion in it is periodic, for, by hypothesis, at $\tau = \tau_0 - T$ it must have been at the same point and moving with the same velocity in the same direction. The motion is therefore periodic with the period $2T$. Hence with these initial values sufficient conditions for periodicity are that at $\tau = \tau_0 + T$

$$r' = q = 0.$$

From the integral of area, $v' = c/r^2$, it follows that if r is periodic v will have the form

$$v = \text{constant} \times \tau + \text{periodic terms}.$$

§ 5. *Existence of periodic orbits in the equatorial plane.*

If $q \equiv 0$ equations (6) reduce to

$$(8)\qquad \begin{aligned} &(a)\quad r'' = \frac{c^2}{r^3} - \frac{1}{r^2} - \frac{\theta_1^2\mu^2}{r^4} - \frac{\theta_2^2\mu^4}{r^6} - \cdots, \\ &(b)\quad v' = \frac{c}{r^2}. \end{aligned}$$

The first of these equations is independent of the second and can be integrated separately. It represents motion in a straight line in the revolving plane. It admits the constant solution

$$r_0 = 1, \qquad c_0^2 = 1 + \theta_1^2\mu^2 + \theta_2^2\mu^4 + \cdots,$$

which represents a point in the revolving plane, or a circle in the equatorial plane. In order to investigate the oscillations about this point let us put

$$r = 1 + e\rho, \qquad c^2 = c_0^2 + e\epsilon,$$

where ρ is a variable whose initial value is arbitrarily assigned, e is an arbitrary parameter corresponding to the eccentricity in the two-body problem, and ϵ is a parameter to be determined so that ρ shall be periodic.

On substituting these values (8a) and expanding as power series in e, the terms independent of e cancel out and it is possible to divide through by e. The equation then becomes

$$(9)\qquad \begin{aligned} \rho'' + [1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots]\,\rho = \epsilon\,[1 - 3\rho e + 6\rho^2 e^2 - 10\rho^3 e^3 + \cdots] \\ + \rho^2 e[3 - 4\theta_1^2\mu^2 - 15\theta_2^2\mu^4 + \cdots] \\ + \rho^3 e^2[-6 + 10\theta_1^2\mu^2 + 46\theta_2^2\mu^4 + \cdots] \\ + \rho^4 e^3[10 - 20\theta_1^2\mu^2 - 111\theta_2^2\mu^4 + \cdots] \\ \cdots\cdots\cdots\cdots\cdots \end{aligned}$$

We can simplify this equation somewhat by dividing through by the coefficient of ρ in the left member and then substituting

$$\mathrm{T} = \tau\sqrt{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots}, \qquad \delta = \frac{\epsilon}{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots}.$$

The equation then becomes

$$(10)\qquad \begin{aligned} \frac{d^2\rho}{d\mathrm{T}^2} + \rho = \delta\,[1 - 3\rho e + 6\rho^2 e^2 - 10\rho^3 e^3 + \cdots] + [3 + a_1]\,\rho^2 e \\ + [-6 + a_2]\,\rho^3 e^2 + [10 + a_3]\,\rho^4 e^3 + \cdots, \end{aligned}$$

where

$$a_1 = -3\theta_1^2\mu^2 - (\theta_1^4 + 6\theta_2^2)\mu^4 + \cdots,$$

$$a_2 = +4\theta_1^2\mu^2 + (4\theta_1^4 + 28\theta_2^2)\mu^4 + \cdots,$$

$$a_3 = -10\theta_1^2\mu^2 - (10\theta_1^4 + 81\theta_2^2)\mu^4 + \cdots.$$

Equation (10) can be integrated for ρ as a power series in δ and e. Let us take the initial values

$$\rho = -1, \qquad \rho' = 0.$$

By Poincaré's extension of Cauchy's theorem we know that the solution of equation (10) having the prescribed initial values exists and converges provided δ and e are sufficiently small, for all values of $\mathbf{T}$ in the interval $0 \leqq \mathbf{T} \leqq T$, where T is finite, but otherwise arbitrary. The condition for periodicity is simply

$$\rho' = 0 \text{ at } \mathbf{T} = T. \tag{11}$$

If we choose $T = \pi$ an inspection of equation (10) shows that for $e = 0$ the solution for ρ is periodic with the period 2π whatever may be the value of δ, so that equation (11) must carry e as a factor. After integrating equation (10) we find that the condition (11) is

$$0 = -\left[\tfrac{3}{2} + a_1\right]\pi\delta e - \left[\tfrac{3}{2} + \tfrac{5}{2}a_1 + \tfrac{5}{12}a_1^2 + \tfrac{3}{8}a_2\right]\pi e^2 + \text{higher degree terms.} \tag{12}$$

Aside from the factor e there remains an equation in which the linear terms in e and δ are present, and which may be solved. We find

$$\delta = (-1 + \cdots)e + \cdots. \tag{13}$$

If this value of δ be substituted in equation (10) it will then admit periodic solutions for ρ as power series in e having the period 2π for all values of e sufficiently small. Furthermore the solution as a power series in e with the prescribed initial conditions is unique.

§ 6. *Existence of periodic orbits not lying in the equatorial plane.*

For $\mu^2 = 0$ the differential equations (6) admit the circular solution

$$\begin{matrix} r = 1, & q = 0, & v = \tau, \\ r' = 0, & q' = 0, & v' = 1, \end{matrix} \qquad c^2 = 1. \tag{14}$$

In order to investigate the existence of orbits not lying in the equatorial plane but having the period 2π for $\mu^2 \neq 0$, let us put

$$r = 1 + \rho, \qquad q = 0 + \sigma, \qquad c^2 = 1 + \epsilon, \tag{15}$$

and take the initial conditions

$$\rho = \alpha, \qquad \rho' = 0, \qquad \sigma = 0, \qquad \sigma' = \beta\mu.$$

The conditions for periodicity (§ 4) are then

$$\rho' = \sigma = 0 \text{ at } \tau = \pi.$$

We have three arbitrary constants at our disposal, α, β, and ϵ, and two conditions to be satisfied. Hence one constant remains undetermined. We will therefore let β remain arbitrary and determine α and ϵ so as to satisfy the two conditions. After making the substitutions (15) and expanding, equations (6) become

$$\begin{aligned} \rho'' + \rho &= \epsilon - 3\rho\epsilon + 3\rho^2 + \tfrac{3}{2}\sigma^2 - \theta_1^2\mu^2 + 6\rho^2\epsilon - 6\rho^3 - 6\rho\sigma^2 \\ &\qquad + 4\rho\theta_1^2\mu^2 + \text{higher degree terms}, \\ \sigma'' + \sigma &= 3\rho\sigma - 6\rho^2\sigma + \tfrac{3}{2}\rho^3 - \sigma\theta_1^2\mu^2 + \text{higher degree terms}. \end{aligned} \tag{16}$$

In order to integrate these equations let us put

$$\begin{aligned} \rho &= \sum_{i,j,k=0}^{\infty} \rho_{ijk}\epsilon^i\alpha^j\mu^k, \\ \sigma &= \sum_{i,j,k=1}^{\infty} \sigma_{ijk}\epsilon^i\alpha^j\mu^k. \end{aligned} \qquad (i+j+k>0), \tag{17}$$

The ρ_{ijk} and σ_{ijk} can be found by successive integrations, the constants of integration being determined so as to satisfy the initial conditions. In the series thus obtained take $\tau = \pi$. The two conditions for periodicity give us the two equations:

$$\begin{aligned} &(a)\ \rho'_{\tau=\pi} = 0 = a_1\epsilon^2 + a_2\epsilon\alpha + a_3\epsilon^3 + a_4\epsilon^2\alpha + a_5\epsilon\alpha^2 + a_6\alpha^3 + a_7\epsilon\mu^2 + a_8\alpha\mu^2 + a_9\mu^4 + \cdots, \\ &(b)\ \sigma_{\tau=\pi} = 0 = \beta\mu\,[b_1\epsilon + b_2\epsilon^2 + b_3\alpha^2 + b_4\epsilon\alpha + b_5\mu^2 + \cdots], \end{aligned} \tag{18}$$

where the a_i and b_i are constants which have been found from the actual integrations to be distinct from zero. Equation (18a) involves only the even powers of μ while (18b) involves only the odd powers. After dividing (18b) by $\beta\mu$ we can solve it for ϵ as a power series in α and μ^2 of the form

$$\epsilon = c_1\alpha^2 + c_2\mu^2 + c_3\alpha^3 + c_4\alpha\mu^2 + c_5\mu^4 + \cdots. \tag{19}$$

On substituting (19) in (18a) we obtain a series of the form

$$0 = d_1\alpha\mu^2 + d_2\alpha^3 + d_3\mu^4 + d_4\alpha^2\mu^2 + d_5\alpha^4 + \cdots. \tag{20 (a)}$$

If in this equation we make the substitution

$$\alpha = \left(\gamma - \frac{d_3}{d_1}\right)\mu^2, \tag{b}$$

we obtain

$$0 = \mu^4[f_1\gamma + f_2\mu^2 + f_3\gamma\mu^2 + \cdots], \tag{c}$$

which can be solved uniquely for γ as a power series in μ^2. This solution substituted in (20b) gives α as a power series in μ^2, and this value of α substituted in (19) gives ϵ as a power series in μ^2. We thus have a solution

$$\alpha = \mu^2 P_1(\mu^2), \qquad \epsilon = \mu^2 P_2(\mu^2), \qquad \beta \text{ arbitrary.}$$

Newton's parallelogram shows that equation (20a) has two additional solutions, but as they are imaginary we do not stop to develop them.

§ 7. *Existence of periodic orbits in a meridian plane.*

If in equations (6) we take the area constant $c^2 = 0$ the motion of the particle is in a meridian plane; that is, the revolving plane has ceased to revolve and the orbit in this plane is the true orbit. After changing to polar coördinates by the substitution

$$r = p \cos \phi, \qquad q = p \sin \phi, \qquad c^2 = 0,$$

the differential equations become

$$(21) \qquad \begin{aligned} p'' - p\phi'^2 + \frac{1}{p^2} &= -\frac{-\frac{3}{4} + \frac{3}{2}\cos 2\phi + \frac{1}{4}\cos 4\phi}{p^4}\theta_1^2\mu^2 + \cdots, \\ p\phi'' + 2p'\phi' &= -\frac{\frac{1}{2}\sin 2\phi - \frac{1}{4}\sin 4\phi}{p^4}\theta_1^2\mu^2 + \cdots. \end{aligned}$$

For $\mu^2 = 0$ we have the periodic solution

$$p = 1, \qquad \phi = \tau,$$

that is, a circle. For $\mu^2 \neq 0$ let us introduce ρ and σ by

$$p = 1 + \rho, \qquad \phi = \tau + \sigma,$$

with the initial values

$$\rho = \alpha, \qquad \rho' = 0, \qquad \sigma = 0, \qquad \sigma' = \beta,$$

where α and β are two new arbitraries. By Poincaré's theorem ρ, ρ', σ and σ' are expansible as power series in α, β and μ^2 with τ entering the coefficients. The conditions for periodicity are that at $\tau = \pi$

$$\rho' = \sigma = 0.$$

If we perform the integration and then set $\tau = \pi$, we obtain the two following conditional equations:

$$(22) \qquad \begin{aligned} &(a) \quad \sigma_{\tau=\pi} = 0 = a_1\alpha + a_2\beta + a_3\alpha^2 + a_4\alpha\beta + a_5\beta^2 + a_6\mu^2 + \cdots, \\ &(b) \quad \rho'_{\tau=\pi} = 0 = \qquad\qquad b_3\alpha^2 + b_4\alpha\beta + b_5\beta^2 + 0\cdot\mu^2 + b_7\mu^4 + \cdots, \end{aligned}$$

where the a_i and b_i are constants which have been found from the actual integrations to be distinct from zero.

The first of these two equations can be solved for α as a power series in β and μ^2. This expression for α substituted in (b) gives rise to an equation of the form

$$(c) \qquad 0 = c_1\beta\mu^2 + c_2\beta^3 + c_3\beta^2\mu^2 + c_4\mu^4 + \cdots.$$

This equation has the same form as ($20a$) and can be solved in the same way, giving a unique real value for β as a power series in μ^2 vanishing with μ^2. This expression for β substituted in the equation for α gives α as a power series in μ^2, vanishing with μ^2. Therefore real periodic orbits exist for $\mu^2 \neq 0$ which are analytic continuations of circular orbits for $\mu^2 = 0$.

We have thus proved the existence of the three following classes of periodic orbits which have the period 2π, the generating orbits being circles:

I. Orbits lying in the equatorial plane;
II. Orbits not lying in the equatorial plane;
III. Orbits in a meridian plane.

§ 8. *Construction of periodic solutions in the equatorial plane.*

Let us first consider orbits in the equatorial plane. We retake the differential equations (8) and by means of the transformations there given we pass at once to equation (10), which is explicitly

$$(23) \qquad \begin{aligned} \frac{d^2\rho}{d\mathrm{T}^2} + \rho = \delta[1 - 3\rho e + 6\rho^2 e^2 + \cdots] &+ [3 - 3\theta_1^2\mu^2 - (\theta_1^4 + 6\theta_2^2)\mu^4 + \cdots]\rho^2 e \\ &+ [-6 + 4\theta_1^2\mu^2 + (4\theta_1^4 + 28\theta_2^2)\mu^4 + \cdots]\rho^3 e^2 \\ &+ [10 - 10\theta_1^2\mu^2 - (10\theta_1^4 + 81\theta_2^2)\mu^4 + \cdots]\rho^4 e^3 + \cdots. \end{aligned}$$

It was shown in equation (13) that δ can be expanded uniquely as a power series in e in such a manner that the solution for ρ as a power series in e will be periodic with the period 2π. Since the series is periodic with the same period for all values of e sufficiently small, it follows that the coefficient of each power of e is itself periodic. Since the solution exists and is unique, it must be possible to determine the δ uniquely by the condition that the solution is periodic. In the existence proof it was shown that δ vanishes with e. Therefore ρ and δ have the form

$$(24) \qquad \rho = \rho_0 + \rho_1 e + \rho_2 e^2 + \rho_3 e^3 + \cdots, \qquad \delta = \delta_1 e + \delta_2 e^2 + \delta_3 e^3 + \cdots.$$

The ρ_j are to be determined by the integration of equation (23) and by the initial values

$$\rho = -1, \qquad \frac{d\rho}{d\mathrm{T}} = 0 \text{ at } \mathrm{T} = 0.$$

The δ_j are to be determined in such a manner that the ρ_j shall be periodic.

Substituting (24) in (23) and equating the coefficients, we find:

$$(25)\quad \begin{aligned}
&(a)\quad \frac{d^2\rho_0}{d\mathbf{T}^2}+\rho_0=0,\\
&(b)\quad \frac{d^2\rho_1}{d\mathbf{T}^2}+\rho_1=\delta_1+[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]\rho_0^2,\\
&(c)\quad \frac{d^2\rho_2}{d\mathbf{T}^2}+\rho_2=\delta_2-3\rho_0\delta_1+[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]2\rho_0\rho_1\\
&\qquad\qquad +[-6+4\theta_1^2\mu^2+(4\theta_1^4+28\theta_2^2)\mu^4+\cdots]\rho_0^3,\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\
&(d)\quad \frac{d^2\rho_n}{d\mathbf{T}^2}+\rho_n=\delta_n-3\rho_0\delta_{n-1}+[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]2\rho_0\rho_{n-1}\\
&\qquad\qquad +f_n(\rho_0,\cdots,\rho_{n-2}),\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

These equations can be integrated in succession. The solution of (a) which satisfies the initial conditions is

$$(26)\qquad \rho_0=-\cos\mathbf{T}.$$

Since the initial conditions are independent of e, every ρ_j except ρ_0 must vanish at $\mathbf{T}=0$. Substituting (26) in (25b) and integrating, we have

$$(27)\qquad \rho_1=\delta_1(1-\cos\mathbf{T})+[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]\left[\tfrac{1}{2}-\tfrac{1}{3}\cos\mathbf{T}-\tfrac{1}{6}\cos 2\mathbf{T}\right].$$

The constants of integration in equation (27) have been determined so as to satisfy the initial conditions, but the constant δ_1 is as yet undetermined.

Substituting these values of ρ_0 and ρ_1 in (25c), we find

$$(28)\quad \begin{aligned}
\frac{d^2\rho_2}{d\mathbf{T}^2}+\rho_2=&[\delta_2+\delta_1(3-3\theta_1^2\mu^2+\cdots)+3-6\theta_1^2\mu^2+\cdots]\\
&+[\delta_1(-3+6\theta_1^2\mu^2+\{2\theta_1^4+12\theta_2^2\}\mu^4+\cdots)\\
&\qquad+(-3+12\theta_1^2\mu^2-\{\tfrac{11}{2}\theta_1^2-9\theta_2^2\}\mu^4+\cdots]\cos\mathbf{T}\\
&+[\delta_1(3-3\theta_1^2\mu^2+\cdots)+3-18\theta_1^2\mu^2+\cdots]\cos 2\mathbf{T}\\
&+[3-4\theta_1^2\mu^2+\cdots]\cos 3\mathbf{T}.
\end{aligned}$$

In order that the solution of this equation may be periodic the coefficient of $\cos\mathbf{T}$ must be zero. This is the condition which determines δ_1 and consequently

$$\delta_1=-1+2\theta_1^2\mu^2+(\tfrac{3}{2}\theta_1^4-\theta_2^2)\mu^4+\cdots.$$

With this value of δ_1 equation (28) becomes

$$\frac{d^2\rho_2}{d\mathbf{T}^2}+\rho_2=[\delta_2+3\theta_1^2\mu^2+\cdots]+[-9\theta_1^2\mu^2+\cdots]\cos 2\mathbf{T}+[3-4\theta_1^2\mu^2+\cdots]\cos 3\mathbf{T}.$$

The solution of this equation which satisfies the initial conditions is

$$\begin{aligned}\rho_2=\delta_2(1-\cos\mathbf{T})+[3\theta_1^2\mu^2+\cdots]+[\tfrac{3}{8}-\tfrac{13}{2}\theta_1^2\mu^2+\cdots]\cos\mathbf{T}\\ +[3\theta_1^2\mu^2+\cdots]\cos 2\mathbf{T}+[-\tfrac{3}{8}+\tfrac{1}{2}\theta_1^2\mu^2+\cdots]\cos 3\mathbf{T}.\end{aligned}\tag{29}$$

The constant δ_2 is as yet undetermined. It is determined by the periodicity condition for ρ_3 in the same manner that δ_1 was determined by the periodicity condition of ρ_2. Without giving the details of the computation its value is found to be

$$\delta_2=-6\theta_1^2\mu^2+\cdots.$$

This method of integration can be carried as far as may be desired. In orderto show this let us suppose that $\rho_0,\cdots,\rho_{n-1}$ have been computed and all the constants determined with the exception of δ_{n-1}. From (25d) we have

$$\frac{d^2\rho_n}{d\mathbf{T}^2}+\rho_n=\delta_n-3\rho_0\delta_{n-1}+[3-3\theta_1^2\mu^2+\cdots]2\rho_0\rho_{n-1}+f_n(\rho_0,\cdots,\rho_{n-2}).\tag{30}$$

Here $f_n(\rho_0,\cdots,\rho_{n-2})$ is a polynomial in the ρ_j and contains only known terms. ρ_{n-1} depends upon δ_{n-1} in the following way:

$$\rho_{n-1}=\delta_{n-1}(1-\cos\mathbf{T})+\text{known terms}.$$

Equation (30) can therefore be written

$$\begin{aligned}\frac{d^2\rho_n}{d\mathbf{T}^2}+\rho_n=\delta_n+[3-3\theta_1^2\mu^2+\cdots]\delta_{n-1}+[-3+6\theta_1^2\mu^2+\cdots]\delta_{n-1}\cos\mathbf{T}\\ +[3-3\theta_1^2\mu^2+\cdots]\cos 2\mathbf{T}+\text{known terms}.\end{aligned}$$

In order that the solution of this equation shall be periodic the coefficient of $\cos\mathbf{T}$ must be zero. This condition determines δ_{n-1}. The equation can then be integrated and the constants of integration will be determined by the conditions that

$$\rho_n=\frac{d\rho_n}{d\mathbf{T}}=0\ \text{ at }\ \mathbf{T}=0.$$

Everything is then determined with the exception of δ_n, and we have then

$$\rho_n=(1-\cos\mathbf{T})\delta_n+\text{known terms}.$$

Substituting the values of δ_1 and δ_2 in the solution as far as it has been com-

puted, we find

$$\rho_0 = -\cos \mathbf{T},$$

$$\rho_1 = [\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\theta_1^4 - 4\theta_2^2)\mu^4 \cdots] + [-\theta_1^2\mu^2 + (-\tfrac{7}{6}\theta_1^4 + 3\theta_2^2)\mu^4 + \cdots]\cos \mathbf{T}$$
$$+ [-\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\tfrac{1}{6}\theta_1^4 + \theta_2^2)\mu^4 \cdots]\cos 2\mathbf{T},$$

$$\rho_2 = [-3\theta_1^2\mu^2 + \cdots] + [\tfrac{3}{8} - \tfrac{1}{2}\theta_1^2\mu^2 + \cdots]\cos \mathbf{T} + [3\theta_1^2\mu^2 + \cdots]\cos 2\mathbf{T}$$
$$+ [-\tfrac{13}{8} + \tfrac{1}{2}\theta_1^2\mu^2 + \cdots]\cos 3\mathbf{T},$$

$$\delta_1 = -1 + 2\theta_1^2\mu^2 + (\tfrac{3}{2}\theta_1^4 - \theta_2^2)\mu^4 + \cdots,$$

$$\delta_2 = -6\theta_1^2\mu^2 + \cdots.$$

From these expressions we have the following series for $r = 1 + e\rho$:

$$(a) \quad r = 1 - e\cos \mathbf{T} + \{[\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\theta_1^4 - 4\theta_2^2)\mu^4 + \cdots]$$
$$+ [-\theta_1^2\mu^2 + (-\tfrac{7}{6}\theta_1^4 + 3\theta_2^2)\mu^4 + \cdots]\cos \mathbf{T}$$
$$(31) \qquad + [-\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\tfrac{1}{6}\theta_1^4 + \theta_2^2)\mu^4 + \cdots]\cos 2\mathbf{T}\}\, e^2$$
$$+ \{[-3\theta_1^2\mu^2 + \cdots] + [\tfrac{3}{8} - \tfrac{1}{2}\theta_1^2\mu^2 + \cdots]\cos \mathbf{T}$$
$$+ [3\theta_1^2\mu^2 + \cdots]\cos 2\mathbf{T} + [-\tfrac{3}{8} + \tfrac{1}{2}\theta_1^2\mu^2 + \cdots]\cos 3\mathbf{T}\}\, e^3 + \cdots.$$

Substituting this value of r in the equation (8b)

$$r^2 \frac{dv}{d\mathbf{T}} = \sqrt{\frac{c_0^2 + e\epsilon}{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 \cdots}},$$

and integrating, we find

$$(b) \quad v - v_0 = \{[1 + \theta_1^2\mu^2 + (\tfrac{1}{2}\theta_1^4 + 2\theta_2^2)\mu^4 + \cdots]$$
$$+ [\theta_1^2\mu^2 + (-\tfrac{9}{4}\theta_1^4 + \tfrac{19}{2}\theta_2^2)\mu^4 + \cdots]\, e^2 + \cdots\}\, \mathbf{T}$$
$$(31) \qquad + \{[2 + 2\theta_1^2\mu^2 + (\theta_1^4 + \theta_2^2)\mu^4 + \cdots]\sin \mathbf{T}\}\, e$$
$$+ \{[2\theta_1^2\mu^2 + (\tfrac{13}{3}\theta_1^4 - 6\theta_2^2)\mu^4 + \cdots]\sin \mathbf{T}$$
$$+ [\tfrac{5}{4} + \tfrac{3}{4}\theta_1^2\mu^2 + (-\tfrac{14}{24}\theta_1^4 + \tfrac{3}{2}\theta_2^2)\mu^4 + \cdots]\sin 2\mathbf{T}\}\, e^2 + \cdots.$$

Equations (31a) and (31b) are the periodic solutions sought. If we return to the symbols defined in the original differential equations by means of equations (4), with the additional notation

$$n\sqrt{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots} = \nu,$$

we have the following expressions for the polar coördinates at any time t:

$$
\begin{aligned}
R = a\{ & 1 - e\cos\nu t + [\tfrac{1}{2} - \tfrac{1}{2}\cos 2\nu t]\, e^2 + [\tfrac{3}{8}\cos\nu t - \tfrac{3}{8}\cos 3\nu t]\, e^3 + \cdots \\
& + \frac{b^2}{a^2}\mu^2[(\tfrac{3}{20} - \tfrac{3}{10}\cos\nu t + \tfrac{3}{20}\cos 2\nu t)e^2 + (-\tfrac{9}{10} - \tfrac{3}{20}\cos\nu t \\
& + \tfrac{9}{10}\cos 2\nu t + \tfrac{3}{20}\cos 3\nu t)e^3 + \cdots] + \frac{b^4}{a^4}\mu^4[\cdots] + \cdots\},
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
v - v_0 = \nu t + 2\sin\nu t\cdot e + \tfrac{5}{4}\sin 2\nu t\cdot e^2 + \cdots + \frac{b^2}{a^2}\mu^2[(\tfrac{3}{10} + \tfrac{3}{10}e^2 + \cdots)\nu t \\
+ (\tfrac{3}{5}\sin\nu t)e^2 + (\tfrac{3}{5}\sin\nu t + \tfrac{9}{40}\sin 2\nu t)e^2 + \cdots] + \frac{b^4}{a^4}\mu^4[\cdots] + \cdots.
\end{aligned}
\tag{33}
$$

These equations contain four arbitrary constants, a, e, v_0 and t_0. Since the differential equations of motion in the equatorial plane were of the fourth order these series, within the realm of their convergence, represent the general solution. The expression for the radius vector, R, is always periodic with the period $2\pi/\nu$. At the expiration of this period v has increased by the quantity

$$
2\pi\left[\frac{b^2}{a^2}\mu^2(\tfrac{3}{10} + \tfrac{3}{10}e^2 + \cdots) + \cdots\right] = 2\pi\Theta
$$

in excess of 2π; that is, the line of apsides has rotated forwarded by this amount. If Θ is commensurable with unity the orbit is eventually closed geometrically. If $\Theta = I/J$, where I and J are integers relatively prime, then, at $t = 2J\pi/\nu$, $v = 2(I+J)\pi$, and the particle is at its initial position with its initial components of velocity. The particle has completed $I+J$ revolutions and the line of apsides has completed I revolutions. The mean sidereal period is

$$
P = \frac{2\pi}{\nu(1+\Theta)}.
$$

This formula for the rotation of the line of apsides has an interesting application in the case of Jupiter's fifth satellite. On the hypothesis that Jupiter is homogeneous and taking its equatorial diameter as 90,190 miles and its polar diameter as 84,570 miles, the mean distance of the satellite as 112,500 miles, the eccentricity of its orbit as .006 and its mean sidereal period as $11^h 0^m 23^s$, the above formula gives for the rotation of the line of apsides 1440° per year. The values derived from observations are somewhat discordant but are in the neighborhood of 880° per year. If we still keep the hypothesis that Jupiter is homogeneous in density and of the same oblateness as before, we are compelled, in order to relieve the discrepancy between theory and observation, to suppose that the value adopted for its polar radius was about 9,000 miles too

great. From the large reduction required it is clear that the hypothesis of homogeneity is very much in error.

§ 9. *Construction of periodic solutions not lying in the equatorial plane.*

By means of the area integral the problem has been reduced to the two equations

$$
(35)\qquad
\begin{aligned}
r'' &= \frac{c^2}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} - \frac{r^3 - 4rq^2}{(r^2+q^2)^{\frac{7}{2}}}\theta_1^2\mu^2 + \cdots,\\
q'' &= -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} - \frac{3r^2q - 2q^3}{(r^2+q^2)^{\frac{7}{2}}}\theta_1^2\mu^2 + \cdots.
\end{aligned}
$$

After the solution of these equations has been obtained the third coördinate is found from the equation

$$v' = \frac{c}{r^2}.$$

We have already proved [equations (14) to (20)] the existence of periodic solutions of these equations of the following type:

$$
(36)\qquad
\begin{aligned}
r &= 1 + \rho_2\mu^2 + \rho_4\mu^4 + \cdots,\\
q &= q_1\mu + q_3\mu^3 + q_5\mu^5 + \cdots,\\
c^2 &= 1 + c_2\mu^2 + c_4\mu^4 + \cdots,
\end{aligned}
$$

with the initial conditions

$$r'(0) = q(0) = 0, \qquad q'(0) = \beta\mu,$$

β being arbitrary. We can therefore integrate equations (35) so as to satisfy these initial conditions and determine the c_j in such a manner as to render the solution periodic.

Substituting (36) in (35) gives the equations

$$
(37)\qquad
\begin{aligned}
0 = [\rho_2'' + \rho_2 - \tfrac{3}{2}q_1^2 - c_2 + \theta_1^2]\mu^2 + [\rho_4'' + \rho_4 - 3\rho_2^2 - 3q_1q_3 + 6\rho_2q_1^2\\
+ \tfrac{15}{8}q_1^4 + (3c_2 - 4\theta_1^2)\rho_2 - \tfrac{15}{2}\theta_1^2q_1^2 - c_4]\mu^4 + \cdots,
\end{aligned}
$$

$$
(38)\qquad
\begin{aligned}
0 = [q_1'' + q_1]\mu + [q_3'' + q_3 - 3\rho_2q_1 - \tfrac{3}{2}q_1^3 + 3\theta_1^2q_1]\mu^3 + [q_5'' + q_5 - 3\rho_2q_3 - \tfrac{9}{2}q_1^2q_3\\
+ 6\rho_2^2q_1 - 3\rho_4q_1 + \tfrac{15}{2}\rho_2q_1^3 + \tfrac{15}{8}q_1^5 + 3\theta_1^2q_3 - 15\theta_1^2\rho_2q_1 - \tfrac{25}{2}\theta_1^2q_1^3]\mu^5 + \cdots.
\end{aligned}
$$

Equation (37) contains only even powers of μ while equation (38) contains only odd powers. For the integration we have:

Coefficient of μ.

$$(39)\qquad q_1'' + q_1 = 0.$$

The solution of this equation satisfying the initial conditions is

$$q_1 = \beta \sin \tau .$$

Coefficient of μ^2.

$$\begin{aligned} \rho_2'' + \rho_2 &= \tfrac{3}{2} q_1^2 + c_2 - \theta_1^2, \\ &= (\tfrac{3}{4}\beta^2 + c_2 - \theta_1^2) - \tfrac{3}{4}\beta^2 \cos 2\tau . \end{aligned} \tag{40}$$

The solution of this equation which satisfies the assigned initial conditions is

$$\rho_2 = (\tfrac{3}{4}\beta^2 + c_2 - \theta_1^2) + \alpha_2 \cos \tau + \tfrac{1}{4}\beta^2 \cos 2\tau .$$

The constant c_2 will be determined by the periodicity condition on q_3 to be

$$c_2 = 2\theta_1^2 - \beta^2 ,$$

and α_2 will be determined by the periodicity condition on ρ_4 to be

$$\alpha_2 = 0 .$$

If we anticipate these determinations we have

$$\rho_2 = (\theta_1^2 - \tfrac{1}{4}\beta^2) + \tfrac{1}{4}\beta^2 \cos 2\tau .$$

Coefficient of μ^3.

$$\begin{aligned} q_3'' + q_3 &= q_1(3\rho_2 + \tfrac{3}{2}q_1^2 - 3\theta_1^2), \\ &= (3\beta^2 + 3c_2 - 6\theta_1^2)\beta \sin \tau + \tfrac{3}{4}\alpha_2\beta \sin 2\tau . \end{aligned} \tag{41}$$

In order that the solution shall be periodic it is necessary that the coefficient of $\sin \tau$ shall be zero. Therefore

$$c_2 = 2\theta_1^2 - \beta^2 .$$

Substituting this value and integrating, we find

$$q_3 = \beta_3 \sin \tau - \tfrac{1}{4}\alpha_2\beta \sin 2\tau .$$

From the initial conditions we must have $q_3'(0) = 0$ and therefore

$$\beta_3 = \tfrac{1}{2}\alpha_2\beta .$$

But it will be shown in the next step that $\alpha_2 = 0$, and consequently

$$q_3 \equiv 0 .$$

Coefficient of μ^4.

$$\rho_4'' + \rho_4 = 3\rho_2^2 + 3q_1q_3 - 6\rho_2q_1^2 - \tfrac{15}{8}q_1^4 + (4\theta_1^2 - 3c_2)\rho_2 + \tfrac{15}{2}\theta_1^2q_1^2 + c_4 . \tag{42}$$

Before expanding the right member of this equation we will avoid useless labor by first examining the coefficient of $\cos \tau$ which we know must be zero from the periodicity condition. We notice first that terms in the coefficient of $\cos \tau$ can

arise only through terms involving ρ_2 and q_3, and second that all such terms carry α_2 as a factor. Since no other arbitrary enters the coefficient, we must take $\alpha_2 = 0$. It can be shown by induction that the constant of integration α_{2i} (the coefficient of $\cos\tau$) which arises in the integration of ρ_{2i} is determined by the periodicity condition on $\rho_{2(i+1)}$, and further that its value is always zero. The proof is omitted for the sake of brevity.

Substituting the value $\alpha_2 = 0$ in ρ_2 and q_3 and expanding the right member of (42), we find

$$\rho_4'' + \rho_4 = [c_4 + \theta_1^2 + \tfrac{11}{4}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4] + [\tfrac{1}{4}\theta_1^2\beta^2 - \tfrac{3}{16}\beta^4]\cos 2\tau + \tfrac{15}{64}\beta^4\cos 4\tau.$$

The solution of this equation satisfying the initial conditions and the properties just mentioned is

$$\rho_4 = [c_4 + \theta_1^2 + \tfrac{11}{4}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4] + [-\tfrac{1}{12}\theta_1^2\beta^2 + \tfrac{1}{16}\beta^4]\cos 2\tau - \tfrac{1}{64}\beta^4\cos 4\tau.$$

Anticipating the value of c_4 found below, we have

$$\rho_4 = [-3\theta_1^4 - \tfrac{11}{12}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4] + [-\tfrac{1}{12}\theta_1^2\beta^2 + \tfrac{1}{16}\beta^2]\cos 2\tau - \tfrac{1}{64}\beta^4\cos 4\tau.$$

Coefficient of μ^5.

$$\begin{aligned} q_5'' + q_5 &= 5\rho_2 q_3 + \tfrac{9}{2}q_1^2 q_3 - 6\rho_2^2 q_1 + 3\rho_4 q_1 - \tfrac{15}{2}\rho_2 q_1^2 - \tfrac{15}{8}q_1^5 \\ &\qquad - 3\theta_1^2 q_3 + 15\theta_1^2\rho_2 q_1 + \tfrac{25}{2}\theta_1^2 q_1^3, \\ &= [3c_4 + 12\theta_1^4 + 11\theta_1^2\beta^2]\beta\sin\tau - \theta_1^2\beta^3\sin 3\tau. \end{aligned} \tag{43}$$

From the periodicity condition we have

$$c_4 = -4\theta_1^4 - \tfrac{11}{3}\theta_1^2\beta^2.$$

Integrating and imposing the initial conditions, we find

$$q_5 = -\tfrac{3}{8}\theta_1^2\beta^3\sin\tau + \tfrac{1}{8}\theta_1^2\beta^3\sin 3\tau.$$

This is sufficient to make evident the general character of the series. The r-equation contains only even multiples of τ, and the q-equation contains only odd multiples. The r-equation contains only even powers of μ and of τ. The q-equation is odd in both these respects. The series are therefore triply even and odd.

Collecting the various coefficients we have the following series:

$$\begin{aligned} &(a) \qquad r = 1 + [(\theta_1^2 - \tfrac{1}{4}\beta^2) + \tfrac{1}{4}\beta^2\cos 2\tau]\mu^2 + [(-3\theta_1^4 - \tfrac{11}{12}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4) \\ &\qquad\qquad + (-\tfrac{1}{12}\theta_1^2\beta^2 + \tfrac{1}{16}\beta^4)\cos 2\tau - \tfrac{1}{64}\beta^4\cos 4\tau]\mu^4 + \cdots, \\ &(b) \qquad q = [\beta\sin\tau]\mu + [0]\mu^3 + [-\tfrac{3}{8}\theta_1^2\beta^3\sin\tau + \tfrac{1}{8}\theta_1^2\beta^3\sin 3\tau]\mu^5 + \cdots, \\ &(c) \qquad v - v_0 = [1 - \theta_1^2\mu^2 - (\tfrac{3}{2}\theta_1^4 + \tfrac{1}{6}\theta_1^2\beta^2)\mu^4 + \cdots]\tau + [-\tfrac{1}{4}\beta^2\sin 2\tau]\mu^2 \\ &\qquad\qquad + [(\tfrac{11}{12}\theta_1^2\beta^2 - \tfrac{1}{8}\beta^4)\sin 2\tau + \tfrac{1}{64}\beta^4\sin 4\tau]\mu^4 + \cdots, \\ &(d) \qquad c^2 = 1 + [2\theta_1^2 - \beta^2]\mu^2 + [-4\theta_1^4 - \tfrac{11}{3}\theta_1^2\beta^2]\mu^4 + \cdots. \end{aligned} \tag{44}$$

This solution contains four arbitrary constants, α, β, v_0 and τ_0. As is shown by equation (44c) the nodes regress, the rate of regression being

$$[\theta_1^2\mu^2 + (\tfrac{3}{2}\theta_1^4 + \tfrac{1}{6}\theta_1^2\beta^2)\mu^4 + \cdots].$$

The generating orbit of these solutions is a circle in the equatorial plane. A circle having any assigned inclination might have been used, e. g.,

$$(45)\quad r = \sqrt{1 - s^2\sin^2\tau}, \qquad q = s\sin\tau, \qquad v = \tan^{-1}[\sqrt{1 - s^2}\tan\tau],$$

where s is the sine of the inclination. The solution thus obtained would have been identical with (44). If we expand (45) as power series in s and then put $s = \beta\mu$, it will be found that the terms thus obtained are identical with the terms independent of θ_1^2 in the solution which has been worked out. It might therefore be of assistance in the physical interpretation of the constants to put $\beta\mu = s$ in the series (44).

§ 10. *Construction of periodic solutions in a meridian plane.*

When the constant c is zero the motion is in a meridian plane. The equations of motion are (21)

$$(46)\quad \begin{aligned} p'' - p\phi'^2 + \frac{1}{p^2} &= -\frac{-\frac{3}{4} + \frac{3}{2}\cos 2\phi + \frac{1}{4}\cos 4\phi}{p^4}\theta_1^2\mu^2 + \cdots, \\ p\phi'' + 2p'\phi' &= -\frac{\frac{1}{2}\sin 2\phi - \frac{1}{4}\sin 4\phi}{p^4}\theta_1^2\mu^2 + \cdots. \end{aligned}$$

We have already shown the existence of periodic solutions of these equations as power series in μ^2, which for $\mu = 0$ reduce to the circle $p = 1$, $\phi = \tau$. Let us put then

$$p = 1 + p_2\mu^2 + p_4\mu^4 + \cdots, \qquad \phi = \tau + \phi_2\mu^2 + \phi_4\mu^4 + \cdots.$$

Substituting these expressions in (46), expanding and collecting the coefficients of the various powers of μ, we find:

$$(47)\quad \begin{aligned} 0 = {} & [p_2'' - 3p_2 - 2\phi_2' - \tfrac{3}{4}\theta_1^2 + \tfrac{3}{2}\theta_1^2\cos 2\tau + \tfrac{1}{4}\theta_1^2\cos 4\tau]\mu^2 + [p_4'' - 3p_4 - 2\phi_4' - 2p_2\phi_2' \\ & - \phi_2'^2 + 3p_2^2 + (3 - 6\cos 2\tau - \cos 4\tau)\theta_1^2 p_2 + (-3\sin 2\tau - \sin 4\tau)\theta_1^2\phi_2]\mu^4 + \cdots, \\ 0 = {} & [\phi_2'' + 2p_2' + \tfrac{1}{2}\theta_1^2\sin 2\tau - \tfrac{1}{4}\theta_1^2\sin 4\tau]\mu^2 + [\phi_4'' + 2p_4' + \phi_2'' p_2 + 2p_2'\phi_2' \\ & + (-2\sin 2\tau + \sin 4\tau)\theta_1^2 p_2 + (\cos 2\tau - \cos 4\tau)\theta_1^2\phi_2]\mu^4 + \cdots. \end{aligned}$$

The initial conditions are

$$p'(0) = \phi(0) = 0.$$

Proceeding to the integration we have:

Coefficient of μ^2.

$$(48)\quad \begin{aligned} &(a) \quad p_2'' - 3p_2 - 2\phi_2' = \tfrac{3}{4}\theta_1^2 - \tfrac{3}{2}\theta_1^2 \cos 2\tau - \tfrac{1}{4}\theta_1^2 \cos 4\tau, \\ &(b) \quad \phi_2'' + 2p_2' = -\tfrac{1}{2}\theta_1^2 \sin 2\tau + \tfrac{1}{4}\theta_1^2 \sin 4\tau. \end{aligned}$$

By integrating (b) once we have

$$(c) \qquad \phi_2' = -2p_2 + \tfrac{1}{4}\theta_1^2 \cos 2\tau - \tfrac{1}{16}\theta_1^2 \cos 4\tau + c_1.$$

Substituting this value of ϕ_2' in (a), we find

$$(d) \qquad p_2'' + p_2 = (2c_1 + \tfrac{3}{4}\theta_1^2) - \theta_1^2 \cos 2\tau - \tfrac{3}{8}\theta_1^2 \cos 4\tau.$$

The integration of this equation gives

$$(e) \qquad p_2 = (2c_1 + \tfrac{3}{4}\theta_1^2) + c_2 \sin \tau + c_3 \cos \tau + \tfrac{1}{3}\theta_1^2 \cos 2\tau + \tfrac{1}{40}\theta_1^2 \cos 4\tau.$$

Since $p_2' = 0$ at $\tau = 0$ we must take $c_2 = 0$. Then substituting this value of p_2 in (c) and integrating, we get

$$(f) \quad \phi_2 = (-3c_1 - \tfrac{3}{2}\theta_1^2)\tau - 2c_3 \sin \tau - \tfrac{5}{24}\theta_1^2 \sin 2\tau - \tfrac{9}{320}\theta_1^2 \sin 4\tau + c_4.$$

From the initial conditions ϕ_2 must be zero when $\tau = 0$. Therefore $c_4 = 0$. Since it must also be periodic, $c_1 = -\tfrac{1}{2}\theta_1^2$. All of the constants of integration are now determined except c_3 which will be determined by the periodicity condition on p_4.

The differential equations for p_4 and ϕ_4 are the same as for p_2 and ϕ_2 except in the right members. The process of integration is therefore the same. In the right members only even multiples of τ occur except terms carrying the undetermined c_3 as a factor. In the equation corresponding to $(48d)$ there will be a term in $\cos \tau$ carrying c_3 as a factor. But the integration of this term will be non-periodic unless $c_3 = 0$. Put then $c_3 = 0$; the integration proceeds just as before and the constants are determined in the same manner. This argument will be repeated in the coefficients of μ^6 and so on for all higher powers:* Therefore no odd multiples of τ can occur in the solution. We have, therefore,

$$(49)\quad \begin{aligned} p &= 1 + [-\tfrac{1}{4} + \tfrac{1}{3} \cos 2\tau + \tfrac{1}{40} \cos 4\tau]\theta_1^2 \mu^2 + \cdots, \\ q &= \tau + [-\tfrac{5}{24} \sin 2\tau - \tfrac{9}{320} \sin 4\tau]\theta_1^2 \mu^2 + \cdots. \end{aligned}$$

* The differential equations arising at the successive steps are of the same type as the first two equations of (32), p. 555, in Professor Moulton's paper on periodic solutions of the problem of three bodies, these Transactions, vol. 7 (1906). Consequently his general formulas for the coefficients of the solutions, equations (42), loc. cit., apply here.

Since the series involve only even multiples of τ the orbits are symmetrical with respect to both the r-axis and the q-axis.

This completes the formal construction of the solutions of which the existence was proved in §§ 5, 6 and 7.

Part II.

ORBITS REËNTRANT ONLY AFTER MANY REVOLUTIONS.

§ 11. *The differential equations.*

The orbits which we have previously considered have had the common property of involving only the period 2π. Since this period is independent of the oblateness of the spheroid the derivation of these orbits has been relatively simple. We shall proceed now to investigate a class of orbits which involves beside the period 2π another period $2\pi/\lambda$, where λ is a function of the oblateness of the spheroid, the inclination of the orbit and the mean distance of the particle. We shall start from the solution which involves an arbitrary inclination. Into this solution four arbitrary constants were introduced, viz., inclination, mean distance, longitude of node and the epoch. Two more arbitraries are necessary for a complete solution, viz., constants corresponding to the eccentricity and to the longitude of perihelion from the node. In what follows we shall introduce the constant corresponding to the eccentricity, but we shall still project the particle from an apse at the node.

We have found for the differential equations a certain solution (44) which we may write

$$r = \phi(\beta, \mu; \tau), \qquad q = \psi(\beta, \mu; \tau), \qquad c^2 = c_0^2,$$

which is symmetric with respect to the equatorial plane. That is to say, at $\tau = 0$ the particle is in the equatorial plane and its motion is perpendicular to the radius vector. Its initial distance is $\phi(0)$. Suppose now we change the initial distance slightly and also the initial velocity so that at $\tau = 0$

$$r(0) = \phi(0) + \alpha, \qquad q(0) = 0,$$

$$r'(0) = 0, \qquad q'(0) = \psi'(0) + \gamma,$$

and give an increment to the constant of areas so that $c^2 = c_0^2 + \epsilon$. Can we determine these three constants, α, γ and ϵ, in such a manner that the series for r and q shall be periodic? These series can be expressed by

$$r = \phi(\beta, \mu; \tau) + \rho, \qquad q = \psi(\beta, \mu; \tau) + \sigma, \qquad c^2 = c_0^2 + \epsilon.$$

If now we substitute these expressions in the differential equations (6) all the terms independent of ρ, σ and ϵ will drop out, and there will remain the fol-

lowing differential equations for ρ and σ:

$$
\begin{aligned}
(a)\quad & \rho'' + \{1 + [(-4\theta_1^2 + \tfrac{3}{2}\beta^2) - \tfrac{9}{2}\beta^2 \cos 2\tau]\,\mu^2 + [(20\theta_1^4 + 6\theta_1^2\beta^2 + \tfrac{3}{8}\beta^4) \\
& \quad + (\theta_1^2\beta^2 - \tfrac{3}{2}\beta^4)\cos 2\tau + \tfrac{9}{8}\beta^4 \cos 4\tau]\,\mu^4 + \cdots\}\,\rho + \{-3\beta\mu \sin\tau \\
& \quad + [(-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3 \sin 3\tau]\,\mu^3 + \cdots\}\,\sigma \\
& \quad + \epsilon\Big\{\{1 + [(-3\theta_1^2 + \tfrac{3}{4}\beta^2) - \tfrac{3}{4}\beta^2 \cos 2\tau]\,\mu^2 + [(15\theta_1^4 - \tfrac{1}{4}\theta_1^2\beta^2 + \tfrac{45}{64}\beta^4) \\
& \quad + (\tfrac{13}{4}\theta_1^2\beta^2 - \tfrac{15}{16}\beta^4)\cos 2\tau + \tfrac{15}{4}\beta^4 \cos 4\tau]\,\mu^4 + \cdots\} \\
& \quad + \{-3 + [(12\theta_1^2 - 3\beta^2) + 3\beta^2 \cos 2\tau]\,\mu^2 + \cdots\}\,\rho + \cdots\Big\} \\
& = \{3 + [(-16\theta_1^2 + 6\beta^2) - 12\beta^2 \cos 2\tau]\mu^2 + [(90\theta_1^4 + \tfrac{59}{4}\theta_1^2\beta^2 + \tfrac{69}{64}\beta^4) \\
(50)\quad & \quad + (\tfrac{1}{4}\theta_1^2\beta^2 - \tfrac{63}{16}\beta^4)\cos 2\tau + \tfrac{183}{64}\beta^4 \cos 4\tau]\,\mu^4 + \cdots\}\,\rho^2 + \{-12\beta\mu \sin\tau \\
& \quad + [(-90\theta_1^2\beta + \tfrac{45}{4}\beta^3)\sin\tau - \tfrac{15}{4}\beta^3 \sin 3\tau]\,\mu^3 + \cdots\}\,\rho\sigma \\
& \quad + \{\tfrac{3}{2} + [(\tfrac{3}{2}\theta_1^2 - \tfrac{33}{8}\beta^2) + \tfrac{33}{8}\beta^2 \cos 2\tau]\,\mu^2 + \cdots\}\,\sigma^2 + \\
& \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(b)\quad & \sigma'' + \{1 + [-\tfrac{3}{2}\beta^2 + \tfrac{3}{2}\beta^2 \cos 2\tau]\,\mu^2 + [-4\theta_1^2\beta^2 + 7\theta_1^2\beta^2 \cos 2\tau]\,\mu^4 + \cdots\}\,\sigma \\
& \quad + \{-3\beta\mu \sin\tau + [(-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3 \sin 3\tau]\,\mu^3 + \cdots\}\,\rho \\
& = \{-6\beta\mu \sin\tau + [(-21\theta_1^2\beta + \tfrac{63}{8}\beta^3)\sin\tau - \tfrac{21}{8}\beta^3 \sin 3\tau]\,\mu^3 + \cdots\}\,\rho^2 \\
& \quad + \{3 + [(3\theta_1^2 - \tfrac{33}{4}\beta^2) + \tfrac{33}{4}\beta^2 \cos 2\tau]\,\mu^2 + \cdots\}\,\rho\sigma + \{\tfrac{9}{2}\beta\mu \sin\tau + \cdots\}\,\sigma^2 + \\
& \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

In the first of these equations the coefficients of all terms containing odd powers of σ involve only odd powers of μ and sines of odd multiples of τ. All other coefficients involve only even powers of μ and cosines of even multiples of τ. In the second equation the coefficient of every odd power of σ involves only even powers of μ and cosines of even multiples of τ. All other coefficients involve only odd powers of μ and sines of odd multiples of τ. These properties play an important rôle throughout the entire discussion.

§ 12. *The equations of variation.*

Considering merely the linear terms of the differential equations (50) for ρ and σ we have the equations of variation:

$$
\begin{aligned}
(a)\quad & \rho'' + \{1 + [(-4\theta_1^2 + \tfrac{3}{2}\beta^2) - \tfrac{9}{2}\beta^2 \cos 2\tau]\,\mu^2 \\
& \quad + [(20\theta_1^4 + 6\theta_1^2\beta^2 + \tfrac{3}{8}\beta^4) + (\theta_1^2\beta^2 - \tfrac{3}{2}\beta^4)\cos 2\tau + \tfrac{9}{8}\beta^4 \cos 4\tau]\,\mu^4 + \cdots\}\,\rho \\
(51)\quad & \quad + \{-3\beta\mu \sin\tau + [(-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3 \sin 3\tau]\,\mu^3 + \cdots\}\,\sigma = 0. \\
(b)\quad & \sigma'' + \{1 + [-\tfrac{3}{2}\beta^2 + \tfrac{3}{2}\beta^2 \cos 2\tau]\,\mu^2 + \cdots\}\,\sigma \\
& \quad + \{-3\beta\mu \sin\tau + [(-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3 \sin 3\tau]\,\mu^3 + \cdots\}\,\rho = 0.
\end{aligned}
$$

These equations admit a solution of the form *

$$\rho_j = A_j e^{\lambda_j \tau} \phi_j(\tau),$$
$$\sigma_j = A_j e^{\lambda_j \tau} \psi_j(\tau) \qquad (j = 1, \cdots, 4)$$

where λ_j is a root of the fundamental equation and where $\phi_j(\tau)$ and $\psi_j(\tau)$ are periodic functions of τ with the period 2π. The four values of the λ_j (real or imaginary) are associated in pairs, equal in value but of opposite sign.† From the fact that τ does not occur explicitly in the original equations (1) it is known *à priori* that one pair of the λ_j has the value 0,‡ and consequently, if we choose the notation so that $\lambda_3 = \lambda_4 = 0$, the two corresponding solutions will have the forms

$$\rho_3 = A_3 \phi_3(\tau), \qquad \rho_4 = A_4[\phi_4(\tau) + \tau\phi_3(\tau)],$$
$$\sigma_3 = A_3 \psi_3(\tau), \qquad \sigma_4 = A_4[\psi_4(\tau) + \tau\psi_3(\tau)].$$

We shall consider first the two solutions in which the λ_j are not zero. Let us first substitute in (51) the forms

$$\rho = e^{i\lambda\tau}\phi(\tau) \qquad (i = \sqrt{-1}),$$
$$\sigma = e^{i\lambda\tau}\psi(\tau).$$

After dividing out the exponential there remains

$$(52) \quad \begin{aligned} &\phi'' + 2i\lambda\phi' + [1 - \lambda^2 + a_2\mu^2 + a_4\mu^4 + \cdots]\phi + [a_1\mu + a_3\mu^3 + \cdots]\psi = 0, \\ &\psi'' + 2i\lambda\psi' + [1 - \lambda^2 + b_2\mu^2 + b_4\mu^4 + \cdots]\psi + [a_1\mu + a_3\mu^3 + \cdots]\phi = 0, \end{aligned}$$

where

$$a_1 = -3\beta \sin\tau,$$
$$a_3 = (-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3 \sin 3\tau,$$
$$a_2 = (-4\theta_1^2 + \tfrac{3}{2}\beta^2) - \tfrac{9}{2}\beta^2 \cos 2\tau,$$
$$a_4 = (20\theta_1^4 + 6\theta_1^2\beta^2 + \tfrac{3}{8}\beta^4) + (\theta_1^2\beta^2 - \tfrac{3}{2}\beta^4)\cos 2\tau + \tfrac{9}{8}\beta^4 \cos 4\tau,$$
$$b_2 = -\tfrac{3}{2}\beta^2 + \tfrac{3}{2}\beta^2 \cos 2\tau,$$
$$b_4 = \text{a sum of cosines of even multiples of } \tau.$$

With respect to equations (52) it is known that ϕ and ψ are periodic with the

* Floquet, Annales Scientifiques de l'École Normale Supérieure, 2d series, vol. 12 (1883), p. 47.

† See *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. 1, p. 193.

‡ Ibid., p. 187.

period 2π, and that λ vanishes with μ since the problem then reduces to the two-body problem in which the characteristic exponents are all zero. It can be shown that ϕ, ψ and λ are expansible as power series in μ of the form

$$\phi = \sum_{j=0}^{\infty} \phi_j \mu^j, \qquad \psi = \sum_{j=0}^{\infty} \psi_j \mu^j, \qquad \lambda = \sum_{j=1}^{\infty} \lambda_j \mu^j.$$

Substituting these expressions in (52) we find:

Coefficient of μ^0.

$$\phi_0'' + \phi_0 = 0,$$

$$\psi_0'' + \psi_0 = 0.$$

Therefore

$$(53) \qquad \begin{aligned} \phi_0 &= \alpha_1^{(0)} \cos\tau + \alpha_2^{(0)} \sin\tau, \\ \psi_0 &= \gamma_1^{(0)} \cos\tau + \gamma_2^{(0)} \sin\tau. \end{aligned}$$

Coefficient of μ.

$$(54) \qquad \begin{aligned} \phi_1'' + \phi_1 &= -2i\lambda_1 \phi_0' - a_1 \psi_0, \\ \psi_1'' + \psi_1 &= -2i\lambda_1 \psi_0' - a_1 \phi_0. \end{aligned}$$

Since the periodicity conditions demand that the coefficients of $\cos\tau$ and $\sin\tau$ in the right members be zero, we must take $\lambda_1 = 0$. We get then

$$(55) \qquad \begin{aligned} \phi_1'' + \phi_1 &= \tfrac{3}{2}\beta\gamma_2^{(0)} - \tfrac{3}{2}\beta\gamma_2^{(0)} \cos 2\tau + \tfrac{3}{2}\beta\gamma_1^{(0)} \sin 2\tau, \\ \psi_1'' + \psi_1 &= \tfrac{3}{2}\beta\alpha_2^{(0)} - \tfrac{3}{2}\beta\alpha_2^{(0)} \cos 2\tau + \tfrac{3}{2}\beta\alpha_1^{(0)} \sin 2\tau. \end{aligned}$$

Integrating we have

$$(56) \qquad \begin{aligned} \phi_1 &= \alpha_1^{(1)} \cos\tau + \alpha_2^{(1)} \sin\tau + \tfrac{3}{2}\beta\gamma_2^{(0)} + \tfrac{1}{2}\beta\gamma_2^{(0)} \cos 2\tau - \tfrac{1}{2}\beta\gamma_1^{(0)} \sin 2\tau, \\ \psi_1 &= \gamma_1^{(1)} \cos\tau + \gamma_2^{(1)} \sin\tau + \tfrac{3}{2}\beta\alpha_2^{(0)} + \tfrac{1}{2}\beta\alpha_2^{(0)} \cos 2\tau - \tfrac{1}{2}\beta\alpha_1^{(0)} \sin 2\tau. \end{aligned}$$

Coefficient of μ^2.

$$(57) \qquad \begin{aligned} \phi_2'' + \phi_2 &= -2i\lambda_2 \phi_0' - a_2 \phi_0 - a_1 \psi_1, \\ \psi_2'' + \psi_2 &= -2i\lambda_2 \psi_0' - b_2 \psi_0 - a_1 \phi_1; \end{aligned}$$

or, expanded,

$$(58) \qquad \begin{aligned} \phi_2'' + \phi_2 &= \tfrac{3}{2}\beta\gamma_2^{(1)} - \tfrac{3}{2}\beta\gamma_2^{(1)} \cos 2\tau + \tfrac{3}{2}\gamma_1^{(1)} \sin 2\tau + (4\theta_1^2 \alpha_2^{(0)} + 2i\lambda_2 \alpha_1^{(0)}) \sin\tau \\ &\quad + (4\theta_1^2 \alpha_1^{(0)} - 2i\lambda_2 \alpha_2^{(0)}) \cos\tau + 3\beta^2 \alpha_2^{(0)} \sin 3\tau + 3\beta^2 \alpha_1^{(0)} \cos 3\tau, \\ \psi_2 + \psi_2 &= \tfrac{3}{2}\beta\alpha_2^{(1)} + (\tfrac{15}{2}\beta^2 \gamma_2^{(0)} + 2i\lambda_2 \gamma_1^{(0)}) \sin\tau + 2i\lambda_2 \gamma_2^{(0)} \cos\tau \\ &\quad + \tfrac{3}{2}\beta\alpha_1^{(1)} \sin 2\tau - \tfrac{3}{2}\beta\alpha_2^{(1)} \cos 2\tau. \end{aligned}$$

In order to satisfy the periodicity condition we must have

$$(59)\qquad \begin{aligned} &2i\lambda_2\alpha_1^{(0)} + 4\theta_1^2\alpha_2^{(0)} = 0, \qquad && 2i\lambda_2\gamma_1^{(0)} + \tfrac{15}{2}\beta^2\gamma_2^{(0)} = 0, \\ &4\theta_1^2\alpha_1^{(0)} - 2i\lambda_2\alpha_2^{(0)} = 0, && + 2i\lambda_2\gamma_2^{(0)} = 0. \end{aligned}$$

The equations of the second column are satisfied by taking

$$\gamma_1^{(0)} = \gamma_2^{(0)} = 0.$$

Solving the other two we find

$$\lambda_2 = \pm 2\theta_1^2, \qquad \alpha_1^{(0)} - ia_2^{(0)} = 0.$$

Equations (59) can also be satisfied by taking

$$\lambda_2 = \alpha_1^{(0)} = \alpha_2^{(0)} = \gamma_2^{(0)} = 0, \qquad \gamma_1^{(0)} = \text{arbitrary},$$

but this would lead to the development of the solutions in which the characteristic exponent is zero, and these solutions will be discussed later.

It was known at the outset that there were two values of λ equal numerically but of opposite sign. We will choose the one with the positive sign. The solution for the negative λ can be derived from the solution for the positive λ. The condition $\alpha_1^{(0)} - i\alpha_2^{(0)} = 0$ still leaves us with an arbitrary constant. Since the equations are linear this constant will enter the solution linearly and may therefore be taken equal to unity. The arbitrary constant is restored in (79) after the solutions are completely developed. We will take then $\alpha_1^{(0)} = 1$ which makes $\alpha_2^{(0)} = -i$. Consequently

$$(60)\qquad \phi_0 = \cos\tau - i\sin\tau, \qquad \psi_0 = 0.$$

Integrating (58) with these values, we get

$$(61)\qquad \begin{aligned} \phi_2 = {} & \tfrac{3}{2}\beta\gamma_2^{(1)} + \alpha_1^{(2)}\cos\tau + \alpha_2^{(2)}\sin\tau + \tfrac{1}{2}\beta\gamma_2^{(1)}\cos 2\tau - \tfrac{1}{2}\beta\gamma_1^{(1)}\sin 2\tau \\ & + 3\beta^2\cos 3\tau - 3\beta^2 i\sin 3\tau, \\ \psi_2 = {} & \tfrac{3}{2}\beta a_2^{(1)} + \gamma_1^{(2)}\cos\tau + \gamma_2^{(2)}\sin\tau + \tfrac{1}{2}\beta\alpha_2^{(1)}\cos 2\tau - \tfrac{1}{2}\beta\alpha_1^{(1)}\sin 2\tau. \end{aligned}$$

Coefficient of μ^3.

$$(62)\qquad \begin{aligned} \phi_3'' + \phi_3 = {} & \tfrac{3}{2}\beta\gamma_2^{(2)} + [-2\lambda_3 + 4\theta_1^2(\alpha_1^{(1)} - i\alpha_2^{(1)})]\cos\tau \\ & + [2i\lambda_3 + 4\theta_1^2(\alpha_2^{(1)} + i\alpha_1^{(1)})]\sin\tau + \tfrac{3}{2}\beta\gamma_1^{(2)}\sin 2\tau - \tfrac{3}{2}\beta\gamma_2^{(2)}\cos 2\tau, \\ \psi_3'' + \psi_3 = {} & [\tfrac{3}{2}\alpha_2^{(2)}\beta - \tfrac{3}{2}i\theta_1^2\beta - \tfrac{21}{16}i\beta^3] + [-2i\lambda_2\gamma_2^{(1)} + \tfrac{3}{4}\beta^2\gamma_1^{(1)}]\cos\tau \\ & + [2i\lambda_2\gamma_1^{(1)} + \tfrac{9}{4}\beta^2\gamma_2^{(1)}]\sin\tau + [-\tfrac{3}{2}\alpha_2^{(2)}\beta + \tfrac{11}{2}i\theta_1^2\beta + \tfrac{21}{16}i\beta^3]\cos 2\tau \\ & + [\tfrac{3}{2}\alpha_1^{(2)}\beta + \tfrac{11}{2}\theta_1^2\beta - \tfrac{9}{16}\beta^3]\sin 2\tau - \tfrac{3}{4}\beta^2\gamma_1^{(1)}\cos 3\tau - \tfrac{3}{4}\beta^2\gamma_2^{(1)}\sin 3\tau. \end{aligned}$$

From the periodicity conditions we must have

$$-2\lambda_3 + 4\theta_1^2(\alpha_1^{(1)} - i\alpha_2^{(1)}) = 0, \qquad \tfrac{3}{4}\beta^2\gamma_1^{(1)} - 2i\lambda_2\gamma_2^{(1)} = 0,$$
$$2i\lambda_3 + 4\theta_1^2(i\alpha_1^{(1)} + \alpha_2^{(1)}) = 0, \qquad 2i\lambda_2\gamma_1^{(1)} + \tfrac{9}{4}\beta^2\gamma_2^{(1)} = 0.$$

The last two equations can be satisfied only if $\gamma_1^{(1)} = \gamma_2^{(1)} = 0$. The first two can be satisfied only if $\lambda_3 = (\alpha_1^{(1)} - i\alpha_2^{(1)}) = 0$. The condition $(\alpha_1^{(1)} - i\alpha_2^{(1)}) = 0$ again gives us an arbitrary constant. Then by (56) $\phi_1 = c(\cos\tau - i\sin\tau)$, but this is the same as ϕ_0 multiplied by $c\mu$. That is, the solution is repeating itself one degree higher in μ and this, of course, should be expected since the equations are linear, so that any solution multiplied by any power of μ must satisfy them. We are at liberty then to choose the arbitrary $c = 0$, which is the same as choosing $\alpha_1^{(1)} = \alpha_2^{(1)} = 0$. Integrating (62) with these values, we find

$$\phi_3 = \tfrac{3}{2}\beta\gamma_2^{(2)} + \alpha_1^{(3)}\cos\tau + \alpha_2^{(3)}\sin\tau + \tfrac{1}{2}\beta\gamma_2^{(2)}\cos 2\tau - \tfrac{1}{2}\beta\gamma_1^{(2)}\sin 2\tau,$$

$$(63)\quad \psi_3 = [\tfrac{3}{2}\alpha_2^{(2)}\beta - \tfrac{3}{2}i\theta_1^2\beta - \tfrac{21}{16}i\beta^3] + \gamma_1^{(3)}\cos\tau + \gamma_2^{(3)}\sin\tau$$
$$+ [\tfrac{1}{2}\alpha_2^{(2)}\beta - \tfrac{11}{6}i\theta_1^2\beta - \tfrac{7}{16}i\beta^3]\cos 2\tau + [-\tfrac{1}{2}\alpha_1^{(2)}\beta - \tfrac{11}{6}\theta_1^2\beta + \tfrac{3}{16}\beta^3]\sin 2\tau.$$

It can be shown by induction at this point that ϕ and λ involve only even powers of μ, and that ψ is an odd series in μ. Furthermore ϕ contains only odd multiples of τ, and ψ only even multiples. Consequently

$$\gamma_1^{(2)} = \gamma_2^{(2)} = \alpha_1^{(3)} = \alpha_2^{(3)} = \gamma_1^{(3)} = \gamma_2^{(3)} = 0,$$

and all

$$\phi_{2j+1} = \psi_{2j} = 0.$$

Coefficient of μ^4.

$$\phi_4'' + \phi_4 = [-2\lambda_4 + 4\theta_1^2(\alpha_1^{(2)} - i\alpha_2^{(2)}) - 16\theta_1^4 - 10\theta_1^2\beta^2]\cos\tau + [2i\lambda_4$$
$$(64)\qquad + 4\theta_1^2(i\alpha_1^{(2)} + \alpha_2^{(2)}) + 16i\theta_1^4]\sin\tau + [6\theta_1^2\beta^2 + 3\beta^2\alpha_1^{(2)} + \tfrac{3}{16}\beta^4]\cos 3\tau$$
$$+ [-6i\theta_1^2\beta^2 + 3\beta^2\alpha_2^{(2)} - \tfrac{45}{16}i\beta^4]\sin 3\tau - \tfrac{21}{16}\beta^4\cos 5\tau + \tfrac{21}{16}i\beta^4\sin 5\tau.$$

Therefore, from the periodicity condition, we must have

$$-2\lambda_4 + 4\theta_1^2(\alpha_1^{(2)} - i\alpha_2^{(2)}) - 16\theta_1^4 - 10\theta_1^2\beta^2 = 0,$$
$$2i\lambda_4 + 4\theta_1^2(i\alpha_1^{(2)} + \alpha_2^{(2)}) + 16i\theta_1^4 = 0.$$

Solving these equations, we find

$$\lambda_4 = -8\theta_1^4 - \tfrac{5}{2}\theta_1^2\beta^2, \qquad \alpha_1^{(2)} - i\alpha_2^{(2)} = \tfrac{5}{4}\beta^2.$$

In this last equation we can choose $\alpha_1^{(2)} = \frac{5}{4}\beta^2$ and $\alpha_2^{(2)} = 0$. This choice of $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ will make the coefficient of $\sin\tau$ in ϕ_2 equal to zero, and since the same thing occurs for each ϕ_j it is evident that this method of choosing will simplify

the solution by making the coefficient of $\sin\tau$ equal to zero for all powers of μ. We have then on integrating

$$
(65)\quad
\begin{aligned}
\phi_4 &= \alpha_4 \cos\tau + \left[-\tfrac{3}{4}\theta_1^2\beta^2 - \tfrac{63}{128}\beta^4\right]\cos 3\tau + i\left[\tfrac{3}{4}\theta_1^2\beta^2 + \tfrac{45}{128}\beta^4\right]\sin 3\tau \\
&\qquad + \tfrac{7}{128}\beta^4\cos 5\tau - \tfrac{7}{128}i\beta^4\sin 5\tau, \\
\psi_3 &= i\left[-\tfrac{3}{2}\theta_1^2\beta - \tfrac{21}{16}\beta^3\right] + i\left[-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3\right]\cos 2\tau + \left[-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3\right]\sin 2\tau, \\
\phi_2 &= \tfrac{5}{4}\beta^2\cos\tau - \tfrac{3}{8}\beta^2\cos 3\tau + \tfrac{3}{8}i\beta^2\sin 3\tau, \\
\psi_1 &= -\tfrac{3}{2}i\beta - \tfrac{1}{2}i\beta\cos 2\tau - \tfrac{1}{2}\beta\sin 2\tau, \\
\phi_0 &= \cos\tau - i\sin\tau, \\
\lambda &= 2\theta_1^2\mu^2 + (-8\theta_1^4 - \tfrac{5}{2}\theta_1^2\beta^2)\mu^4 + \cdots.
\end{aligned}
$$

The coefficient α_4 of $\cos\tau$ in ϕ_4 is determined by the periodicity condition on ϕ_6. That the process of determining the values of the λ_j and the constants of integration arising at each step is general may be shown as follows. Let us suppose that we have computed everything up to and including ϕ_j with the exception of the constants of integration in ϕ_j. We have then

$$\phi_j = \alpha_1^{(j)}\cos\tau + \alpha_2^{(j)}\sin\tau + \text{known terms}.$$

The $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$ enter ψ_{j+1} as follows:

$$
\begin{aligned}
\psi_{j+1}'' + \psi_{j+1} &= 3\beta\sin\tau\cdot\phi_j + \text{known terms}, \\
&= \tfrac{3}{2}\beta\alpha_2^{(j)} - \tfrac{3}{2}\beta\alpha_2^{(j)}\cos 2\tau + \tfrac{3}{2}\beta\alpha_1^{(j)}\sin 2\tau + \text{known terms}.
\end{aligned}
$$

Consequently in so far as it involves $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$

$$\psi_{j+1} = \tfrac{3}{2}\beta\alpha_2^{(j)} + \tfrac{1}{2}\beta\alpha_2^{(j)}\cos 2\tau - \tfrac{1}{2}\beta\alpha_1^{(j)}\sin 2\tau.$$

Similarly, in so far as ϕ_{j+2} depends upon constants as yet undetermined

$$
(66)\quad
\begin{aligned}
\phi_{j+2}'' + \phi_{j+2} &= -2i\lambda_2\phi_j' - 2i\lambda_{j+2}\phi_0' + \left[(4\theta_1^2 - \tfrac{3}{2}\beta^2) + \tfrac{9}{2}\beta^2\cos 2\tau\right]\phi_j + 3\beta\sin\tau\psi_{j+1} \\
&= \left[-2\lambda_{j+2} + 4\theta_1^2(\alpha_1^{(j)} - i\alpha_2^{(j)}) + A_{j+2}\right]\cos\tau \\
&\quad + \left[2i\lambda_{j+2} + 4\theta_1^2(i\alpha_1^{(j)} + \alpha_2^{(j)}) + B_{j+2}\right]\sin\tau \\
&\quad + 3\beta^2\alpha_1^{(j)}\cos 3\tau + 3\beta^2\alpha_2^{(j)}\sin 3\tau,
\end{aligned}
$$

where A_{j+2} and B_{j+2} are the known terms in the coefficients of $\cos\tau$ and $\sin\tau$ respectively. From the periodicity condition we must have

$$
(67)\quad
\begin{aligned}
-2\lambda_{j+2} + 4\theta_1^2(\alpha_1^{(j)} - i\alpha_2^{(j)}) + A_{j+2} &= 0, \\
+2i\lambda_{j+2} + 4\theta_1^2(i\alpha_1^{(j)} + \alpha_2^{(j)}) + B_{j+2} &= 0.
\end{aligned}
$$

The solution of these equations is

$$(68)\qquad \begin{aligned} \lambda_{j+2} &= \tfrac{1}{4}(A_{j+2} + iB_{j+2}), \\ \alpha_1^{(1)} - i\alpha_2^{(j)} &= -\frac{A_{j+2} - iB_{j+2}}{8\theta_1^2}. \end{aligned}$$

As has already been pointed out we can choose $\alpha_1^{(j)} = 0$ and we have then

$$(69)\qquad \alpha_2^{(j)} = -\frac{A_{j+2} - iB_{j+2}}{8\theta_1^2}.$$

In order to show that λ_{j+2} and $\alpha_1^{(j)}$ are real it will be sufficient to show that A_{j+2} is real and B_{j+2} is a pure imaginary. This is readily shown by induction, for up to $j = 4$ inclusive we have

$$\phi_j = \sum_\kappa m_\kappa \cos \kappa\tau + i \sum_\kappa n_\kappa \sin \kappa\tau,$$

$$\psi_j = i \sum_\kappa f_\kappa \cos \kappa\tau + \sum_\kappa g_\kappa \sin \kappa\tau,$$

where m_κ, n_κ, f_κ and g_κ are all real. From the form of the differential equations it follows at once that the same forms hold for $j = 5$, then $j = 6$, and so on. That is A_{j+2} is real while B_{j+2} is a pure imaginary.

It is further to be noticed that A_{j+2} and B_{j+2} do not contain any terms in β independent of θ_1^2, and consequently the θ_1^2 which appears in the denominator of $\alpha_1^{(j)}$ will divide out. This is proved as follows. If θ_1^2 be put equal to zero in the differential equations then equations (52) become the equations of variation of a circular orbit in the ordinary two-body problem, the plane of the circle being inclined to the plane of reference by an angle whose sine is $\beta\mu = s$. The original differential equations (6) can then be written

$$(70)\qquad r'' = \frac{(1-s^2)(1-e^2)}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} = R, \qquad q'' = -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} = Q,$$

where the constant c^2 is given the form $(1-s^2)(1-e^2)$. For these equations we have the solution

$$(71)\qquad \begin{aligned} r &= a\frac{(1-e^2)\sqrt{1-s^2\sin^2(\theta-\Omega)}}{1+e\cos(\theta-\theta_0)}, \\ q &= a\frac{(1-e^2)\,s\sin(\theta-\Omega)}{1+e\cos(\theta-\theta_0)}, \end{aligned}$$

where

$$(\theta-\theta_0) = (\tau-\tau_0) + 2e\sin(\tau-\tau_0) + \cdots.$$

Let us now form the equations of variation by putting

$$r = r_0 + \rho, \qquad q = q_0 + \sigma, \qquad e = e_0 + \epsilon,$$

where r_0, q_0 and e_0 are the values in (71). We find

$$(72) \qquad \begin{aligned} \rho'' &= \frac{\partial R}{\partial r}\rho + \frac{\partial R}{\partial q}\sigma + \frac{\partial R}{\partial e}\epsilon, \\ \sigma'' &= \frac{\partial Q}{\partial r}\rho + \frac{\partial R}{\partial q}\sigma + \frac{\partial Q}{\partial e}\epsilon. \end{aligned}$$

Three solutions of these equations are given by*

$$(73) \qquad \begin{aligned} \rho &= c_1 \frac{\partial r_0}{\partial \Omega}, & \rho &= c_2 \frac{\partial r_0}{\partial \tau_0}, & \rho &= c_3 \frac{\partial r_0}{\partial e_0}, \\ \sigma &= c_1 \frac{\partial q_0}{\partial \Omega}, & \sigma &= c_2 \frac{\partial q_0}{\partial \tau_0}, & \sigma &= c_3 \frac{\partial q_0}{\partial e_0}, \\ \epsilon &= c_1 \frac{\partial e_0}{\partial \Omega}, & \epsilon &= c_2 \frac{\partial e_0}{\partial \tau_0}, & \epsilon &= c_3 \frac{\partial e_0}{\partial e_0}. \end{aligned}$$

If $e_0 \neq 0$ these three solutions are distinct, but the case in which we are interested is when $e_0 = 0$. In this case it is not difficult to see that the first two solutions coincide. Since the equations are linear, the system

$$(74) \qquad \begin{aligned} \rho &= c_4 \left[\frac{\partial r_0}{\partial \Omega} - \frac{\partial r_0}{\partial \tau_0} \right], \\ \sigma &= c_4 \left[\frac{\partial q_0}{\partial \Omega} - \frac{\partial q_0}{\partial \tau_0} \right], \\ \epsilon &= c_4 \left[\frac{\partial e_0}{\partial \Omega} - \frac{\partial e_0}{\partial \tau_0} \right], \end{aligned}$$

is also a solution, but as it vanishes for $e_0 = 0$ it carries e_0 as a factor. We can divide out this factor and absorb it into the arbitrary c_4. For $e_0 = 0$ this solution does not now vanish, and it is moreover distinct from the first solution. Thus we have three distinct solutions even when $e_0 = 0$, but since $\partial Q / \partial e_0 \equiv 0$, and $\partial R / \partial e_0$ carries e_0 as a factor, equations (72) pass over to the equations of variation of a circle when $e_0 = 0$. For these equations we have three solutions which are periodic with the period 2π. The fourth solution is not periodic but involves a term of the form τ times a periodic function.

Let us return now to the solution which we have developed, (65), and consider only the terms which belong to the two-body problem, viz., the terms which are

* See *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 163.

independent of θ_1^2. This solution may be separated into two solutions, one of which is real, the other a pure imaginary. The real solution is the third solution of (73), and the purely imaginary is the second solution. Since both of these solutions are certainly periodic with the period 2π, it follows that no term in β alone can occur in the A_{j+2} and B_{j+2} of (66) for the presence of such terms would give rise to non-periodic terms in the two-body problem. Hence A_{j+2} and B_{j+2} carry θ_1^2 as a factor which may be divided out in equation (69). Further λ_{j+2}, (68), carries θ_1^2 as a factor and therefore λ vanishes with the oblateness of the spheroid.

The solution (65) may now be written

$$\begin{aligned} \rho^{(1)} &= e^{i\lambda\tau}[\phi^{(1)} + i\phi^{(2)}], \\ \sigma^{(1)} &= e^{i\lambda\tau}[\psi^{(1)} + i\psi^{(2)}], \end{aligned} \tag{75}$$

where

$$\begin{aligned} \phi^{(1)} &= \cos\tau + [\tfrac{5}{4}\beta^2\cos\tau - \tfrac{3}{8}\beta^2\cos 3\tau]\,\mu^2 \\ &\quad + [\alpha_4\cos\tau + (-\tfrac{3}{4}\theta_1^2\beta^2 - \tfrac{63}{128}\beta^4)\cos 3\tau + \tfrac{7}{128}\beta^4\cos 5\tau]\mu^4 + \cdots, \\ \phi^{(2)} &= -\sin\tau + [\tfrac{3}{8}\beta^2\sin 3\tau]\,\mu^2 \\ &\quad + [(\tfrac{3}{4}\theta_1^2\beta^2 + \tfrac{45}{128}\beta^4)\sin 3\tau - \tfrac{7}{128}\beta^4\sin 5\tau]\,\mu^4 + \cdots, \\ \psi^{(1)} &= [-\tfrac{1}{2}\beta\sin 2\tau]\,\mu + [(-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3)\sin 2\tau]\,\mu^3 + \cdots, \\ \psi^{(2)} &= [-\tfrac{3}{2}\beta - \tfrac{1}{2}\beta\cos 2\tau]\,\mu \\ &\quad + [(-\tfrac{3}{2}\theta_1^2\beta - \tfrac{21}{16}\beta^3) + (-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3)\cos 2\tau]\,\mu^3 + \cdots. \end{aligned} \tag{76}$$

By putting

$$e^{i\lambda\tau} = \cos\lambda\tau + i\sin\lambda\tau$$

we can write

$$\begin{aligned} \rho^{(1)} &= [\phi^{(1)}\cos\lambda\tau - \phi^{(2)}\sin\lambda\tau] + i[\phi^{(2)}\cos\lambda\tau + \phi^{(1)}\sin\lambda\tau], \\ \sigma^{(1)} &= [\psi^{(1)}\cos\lambda\tau - \psi^{(2)}\sin\lambda\tau] + i[\psi^{(2)}\cos\lambda\tau + \psi^{(1)}\sin\lambda\tau]. \end{aligned} \tag{77}$$

We have thus one solution of the differential equations. A second solution can be derived from it by merely changing the sign of i. Thus

$$\begin{aligned} \rho^{(2)} &= [\phi^{(1)}\cos\lambda\tau - \phi^{(2)}\sin\lambda\tau] - i[\phi^{(2)}\cos\lambda\tau + \phi^{(1)}\sin\lambda\tau], \\ \sigma^{(2)} &= [\psi^{(1)}\cos\lambda\tau - \psi^{(2)}\sin\lambda\tau] - i[\psi^{(2)}\cos\lambda\tau + \psi^{(1)}\sin\lambda\tau]. \end{aligned} \tag{78}$$

By adding and subtracting these two solutions we have finally

$$\begin{aligned} \rho &= A[\rho^{(1)} + \rho^{(2)}] + B[\rho^{(1)} - \rho^{(2)}], \\ \sigma &= A[\sigma^{(1)} + \sigma^{(2)}] + B[\sigma^{(1)} - \sigma^{(2)}], \end{aligned} \tag{79}$$

A and B being two arbitrary constants. As above developed there is a certain arbitrariness in these solutions owing to the manner in which the constants of integration were determined. They may be reduced to a normal form by multiplying each solution by the proper power series in μ^2 with constant coefficients. By this process we can make for $\tau = 0$,

$$\rho^{(1)}(0) + \rho^{(2)}(0) = 1, \qquad \sigma^{(1)}(0) - \sigma^{(2)}(0) = \beta\mu.$$

Since $[\rho^{(1)} - \rho^{(2)}]$ and $[\sigma^{(1)} + \sigma^{(2)}]$ are sine series they vanish for $\tau = 0$.

The third and fourth solutions of the equations of variation, (51), are given by*

$$(80) \qquad \begin{aligned} \rho_3 &= C\frac{\partial r_0}{\partial \tau_0}, & \rho_4 &= D'\frac{\partial(ar_0)}{\partial a}, \\ \sigma_3 &= C\frac{\partial q_0}{\partial \tau_0}, & \sigma_4 &= D'\frac{\partial(aq_0)}{\partial a}. \end{aligned}$$

In forming the partial derivatives with respect to a in the fourth solution it should be remembered that θ_1^2 is an explicit function of a, by equations (4), and that τ is also a function of a implicitly through n. The third solution also can be normalized by giving the arbitrary constant such a form that at $\tau = 0$

$$\rho_3 = 0, \qquad \sigma_3 = C\beta\mu.$$

The fourth solution is non-periodic and has the form

$$(81) \qquad \rho = D\left[\tau\frac{\partial r_0}{\partial \tau_0} + \phi_4\right], \qquad \sigma = D\left[\tau\frac{\partial q_0}{\partial \tau_0} + \psi_4\right],$$

where ϕ_4 and ψ_4 are periodic functions of τ with the period 2π. As in the previous solutions this may be normalized so that at $\tau = 0$

$$\rho_4 = D\beta^2\mu^2, \qquad \sigma_4 = 0.$$

It is also easy to find ϕ_4 and ψ_4 by substituting (81) in the equations of variation and solving for these variables, which must be periodic.

Carrying out the above operations we find the following general solutions

$$\begin{aligned} \rho = A\{&\cos(1-\lambda)\tau + [-\tfrac{1}{4}\beta^2\cos(1-\lambda)\tau + \tfrac{5}{8}\beta^2\cos(1+\lambda)\tau - \tfrac{3}{8}\beta^2\cos(3-\lambda)\tau]\mu^2 \\ &+ [(-\tfrac{1}{2}\alpha_4 + \tfrac{3}{4}\theta_1^2\beta^2 + \tfrac{21}{32}\beta^4)\cos(1-\lambda)\tau + (\tfrac{1}{2}\alpha_4 - \tfrac{35}{64}\beta^4)\cos(1+\lambda)\tau \\ &+ (-\tfrac{3}{4}\theta_1^2\beta^2 - \tfrac{3}{32}\beta^4)\cos(3-\lambda)\tau - \tfrac{9}{128}\beta^4\cos(3+\lambda)\tau \\ &+ \tfrac{7}{128}\beta^4\cos(5-\lambda)\tau]\mu^4 + \cdots\} \end{aligned}$$

* See *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, p. 163.

$$
\begin{aligned}
(82)\qquad & + B\{\tfrac{1}{2}\sin(1-\lambda)\tau + [(-\tfrac{1}{8}\beta^2 - \tfrac{5}{6}\theta_1^2)\sin(1-\lambda)\tau - \tfrac{5}{16}\beta^2\sin(1+\lambda)\tau \\
& \qquad - \tfrac{3}{16}\beta^2\sin(3-\lambda)\tau]\mu^2\cdots\} \\
& + C\{[-\tfrac{1}{2}\beta^2\sin 2\tau]\mu^2 + [(\tfrac{1}{16}\theta_1^2\beta^2 - \tfrac{1}{8}\beta^4)\sin 2\tau + \tfrac{1}{16}\beta^4\sin 4\tau]\mu^4 + \cdots\} \\
& + D\Big\{\{[\tfrac{3}{4}\beta^2 + \tfrac{1}{4}\beta^2\cos 2\tau]\mu^2 + [(\tfrac{9}{8}\theta_1^2\beta^2 - \tfrac{3}{32}\beta^4) \\
& \qquad + (-\tfrac{9}{8}\theta_1^2\beta^2 + \tfrac{1}{8}\beta^4)\cos 2\tau - \tfrac{1}{32}\beta^4\cos 4\tau]\mu^4\cdots\} + \tau\{[\tfrac{3}{4}\beta^4\sin 2\tau]\mu^4 \\
& \qquad + [(\tfrac{91}{16}\theta_1^2\beta^4 + \tfrac{15}{64}\beta^6)\sin 2\tau - \tfrac{3}{32}\beta^6\sin 4\tau]\mu^6 + \cdots\}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\sigma = A\{ & [\tfrac{3}{2}\beta\sin\lambda\tau - \tfrac{1}{2}\beta\sin(2-\lambda)\tau]\mu + [\tfrac{3}{2}\theta_1^2\beta\sin\lambda\tau \\
& \qquad - \tfrac{11}{6}\theta_1^2\beta\sin(2-\lambda)\tau]\mu^3 + \cdots\} \\
& + B\{[\tfrac{3}{4}\beta\cos\lambda\tau + \tfrac{1}{4}\beta\cos(2-\lambda)\tau]\mu + [-\tfrac{1}{2}\theta_1^2\beta\cos\lambda\tau \\
(83)\qquad & \qquad + \tfrac{1}{2}\theta_1^2\beta\cos(2-\lambda)\tau]\mu^3 + \cdots\} \\
& + C\{[\beta\cos\tau]\mu + [0]\mu^3 + \cdots\} \\
& + D\Big\{\{[\tfrac{1}{2}\beta\sin\tau]\mu + [(-\tfrac{7}{4}\theta_1^2\beta + \tfrac{1}{2}\beta^3)\sin\tau]\mu^3 + \cdots\} \\
& \qquad + \tau\{[-\tfrac{3}{2}\beta^3\cos\tau]\mu^3 + [(-\tfrac{53}{16}\theta_1^2\beta^3 + \tfrac{21}{16}\beta^5)\cos\tau]\mu^5 + \cdots\}\Big\}.
\end{aligned}
$$

§ 13. *Non-homogeneous linear differential equations with periodic coefficients.*

If we have a set of homogeneous linear differential equations

$$
(84)\qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij}(t)x_j, \qquad (i=1,\cdots,n),
$$

where the θ_{ij} are periodic functions of t with the period 2π, we know from the writings of Floquet * that, if the roots of the fundamental equation are all distinct, the solution has the form

$$
(85)\qquad x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t}\phi_{ij}(t),
$$

where the A_j are arbitrary constants, the α_j are constants known as the characteristic exponents, and the ϕ_{ij} are periodic functions of t with the period 2π.

Such equations arise in dynamics whenever we study small variations from a known periodic solution. If we confine our attention entirely to the first powers of the variations the equations are linear and homogeneous, and are known as the "equations of variation." They have been studied extensively by Poincaré in *Les Méthodes Nouvelles de la Mécanique Céleste.* If the second and higher

* Annales Scientifiques de l'Ecole Normale Supérieure, 2d series, vol. 12 (1883), p. 47.

powers of the variation are considered equations of the same type arise but they are no longer homogeneous. It is necessary for our purpose to derive the character of the solutions of such equations when the non-homogeneous terms are periodic, even though the period be different from 2π. We will suppose that the equations are the same as (84) with the addition of periodic terms.

Case I.

We will assume first that the period of the non-homogenous terms is 2π. The equations are then

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j + g_i(t) \qquad (i=1, \cdots, n), \tag{86}$$

where $\theta_{ij}(t)$ and $g_i(t)$ are periodic with the period 2π. The solutions of (86) may be written

$$x_i = \sum_{j=1}^{n} A_j e^{a_j t} \phi_{ij}(t) + \psi_i(t) \qquad (i=1, \cdots, n). \tag{87}$$

If in the differential equations we change t into $t + 2\pi$ the equations remain unchanged, but the solutions become

$$x_i = \sum_{j=1}^{n} A_j e^{a_j(t+2\pi)} \phi_{ij}(t + 2\pi) + \psi_i(t + 2\pi). \tag{88}$$

Therefore equations (88) are also solutions of (86), and consequently

$$\frac{d}{dt}\psi_i(t + 2\pi) = \sum_{j=1}^{n} \theta_{ij} \psi_j(t + 2\pi) + g_i(t);$$

also

$$\frac{d}{dt}\psi_i(t) = \sum_{j=1}^{n} \theta_{ij} \psi_j(t) + g_i(t).$$

Forming the difference of these equations we find

$$\frac{d}{dt}[\psi_i(t + 2\pi) - \psi_i(t)] = \sum_{j=1}^{n} \theta_{ij}[\psi_j(t + 2\pi) - \psi_j(t)] \quad (i=1, \cdots, n). \tag{89}$$

These equations are the same as (84) and their solutions have the same form as (85), that is

$$\psi_i(t + 2\pi) - \psi_i(t) = \sum_{j=1}^{n} B_j e^{a_j t} \phi_{ij}(t). \tag{90}$$

The constants B_j in this case are not arbitrary but depend upon the differential equations. They may or may not be zero. Equations (90) may be interpreted as showing that $\psi_i(t)$ is composed of two parts, a periodic part and a non-

periodic part. We may express it therefore in the form

$$\psi_i(t) = \omega_i(t) + \sum_{j=1}^{n} B_j e^{a_j t} \phi_{ij}(t) f_{ij}(t), \tag{91}$$

where $\omega_i(t)$ are periodic with the period 2π, and the $f_{ij}(t)$ are functions which must be determined. Changing t into $t + 2\pi$ in equations (91) and forming the difference $\psi_i(t + 2\pi) - \psi_i(t)$, we find

$$\begin{aligned} \psi_i(t + 2\pi) - \psi_i(t) &= \sum_{j=1}^{n} B_j e^{a_j t} \phi_{ij}(t) [e^{2a_j\pi} f_{ij}(t + 2\pi) - f_{ij}(t)], \\ &= \sum_{j=1}^{n} B_j e^{a_j t} \phi_{ij}(t) \quad \text{from equations (90).} \end{aligned} \tag{92}$$

Comparing the coefficients in these equalities we see that

$$e^{2a_j\pi} f_{ij}(t + 2\pi) - f_{ij}(t) = 1, \tag{93}$$

from which we can determine the character of the functions $f_{ij}(t)$.

For this purpose let us define a new function $\lambda_{ij}(t)$ such that

$$\lambda_{ij}(t) = e^{a_j t} f_{ij}(t) + \frac{e^{a_j t}}{1 - e^{2a_j\pi}}. \tag{94}$$

Then by virtue of the relations (93) the $\lambda_{ij}(t)$ are periodic with the period 2π, for we have

$$\begin{aligned} \lambda_{ij}(t + 2\pi) &= e^{a_j t} \cdot e^{2a_j\pi} f_{ij}(t + 2\pi) + \frac{e^{a_j t} \cdot e^{2a_j\pi}}{1 - e^{2a_j\pi}}, \\ &= e^{a_j t}[f_{ij}(t) + 1] + \frac{e^{a_j t} \cdot e^{2a_j\pi}}{1 - e^{2a_j\pi}}, \\ &= e^{a_j t} f_{ij}(t) + \frac{e^{a_j t}}{1 - e^{2a_j\pi}}, \\ &= \lambda_{ij}(t). \end{aligned}$$

On solving (94) we find

$$f_{ij}(t) = e^{-a_j t} \lambda_{ij}(t) + \frac{1}{e^{2a_j\pi} - 1}, \tag{95}$$

where the λ_{ij} are periodic with the period 2π. This expression for the $f_{ij}(t)$ substituted in (91) gives

$$\psi_i(t) = \omega_i(t) + \sum_{j=1}^{n} B_j \phi_{ij} \lambda_{ij} + \sum_{j=1}^{n} \frac{B_j}{e^{2a_j\pi} - 1} e^{a_j t} \phi_{ij}(t). \tag{96}$$

The terms included under the last summation sign are merely terms of the complementary function. All the other terms are periodic with the period 2π.

This form for the $\psi_i(t)$ fails however if for any j

$$e^{2\alpha_j\pi} = 1,$$

that is, if

$$\alpha_j \equiv 0 \mod \sqrt{-1}.$$

In this event equations (93) for such values of j become

$$f_{ij}(t + 2\pi) - f_{ij}(t) = 1. \tag{97}$$

We will define the corresponding value of λ_{ij} by the relation

$$\lambda_{ij}(t) = f_{ij}(t) - \frac{t}{2\pi}. \tag{98}$$

By virtue of (97) λ_{ij} is periodic with the period 2π, for

$$\begin{aligned}\lambda_{ij}(t + 2\pi) &= f_{ij}(t + 2\pi) - \frac{t + 2\pi}{2\pi},\\ &= f_{ij}(t) - \frac{t}{2\pi},\\ &= \lambda_{ij}(t).\end{aligned}$$

Solving (98) for $f_{ij}(t)$, we obtain

$$f_{ij}(t) = \lambda_{ij}(t) + \frac{t}{2\pi}. \tag{99}$$

Substituting this expression in (91), we find

$$\psi_i(t) = \text{periodic terms} + \frac{t}{2\pi}\sum_j B_j e^{\alpha_j t}\phi_{ij}(t), \tag{100}$$

the summation in the last term to be extended over all j such that $\alpha_j \equiv 0 \mod \sqrt{-1}$.

We have then the general expression for the solution

$$x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t}\phi_{ij}(t) + \text{periodic terms} + \frac{t}{2\pi}\sum_j B_j e^{\alpha_j t}\phi_{ij}(t), \tag{101}$$

where the summation in the last term is to be extended over all j such that $\alpha_j \equiv 0 \mod \sqrt{-1}$.

This result may be stated thus:

THEOREM I. *If the $\theta_{ij}(t)$ and $g_i(t)$ are periodic with the period 2π and if the characteristic exponents α_j are distinct and none of them congruent to zero* mod $\sqrt{-1}$, *then the particular solution is periodic with the period 2π. If the α_j are distinct but some of them congruent to zero* mod $\sqrt{-1}$, *then the particular solution may involve terms of the form t times the corresponding complementary function.*

Case II.

We will suppose that the $g_i(t)$ are periodic and that the period is different from 2π. We will suppose further that $g_i(t)$ are expressible in the form

$$(102)\quad g_i(t)=\sum_j [a_{ij}\cos(j+\beta)t+b_{ij}\sin(j+\beta)t]=e^{\sqrt{-1}\beta t}f_i^{(1)}(t)+e^{-\sqrt{-1}\beta t}f_i^{(2)}(t),$$

where $f_i^{(1)}(t)$ and $f_i^{(2)}(t)$ are periodic with the period 2π and β is any real number not an integer. The differential equations may then be written

$$(103)\qquad \frac{dx_i}{dt}=\sum_{j=1}^{n}\theta_{ij}x_j+e^{\sqrt{-1}\beta t}f_i^{(1)}(t)+e^{-\sqrt{-1}\beta t}f_i^{(2)}(t).$$

Since the equations are linear we can consider the particular solutions depending upon $e^{\sqrt{-1}\beta t}$ and $e^{-\sqrt{-1}\beta t}$ separately. The complete solution will be the sum of the two. Let us consider first

$$(104)\qquad \frac{dx_i}{dt}=\sum_{j=1}^{n}\theta_{ij}x_j+e^{\sqrt{-1}\beta t}f_i^{(1)}(t).$$

If we make the substitution

$$x_i=e^{\sqrt{-1}\beta t}y_i$$

equations (104) become after dividing out the exponential

$$(105)\qquad \frac{dy_i}{dt}+\sqrt{-1}\,y_i=\sum_{j=1}^{n}\theta_{ij}y_j+f_i^{(1)}(t).$$

Since θ_{ij} and $f_i^{(1)}$ are periodic with the period 2π the discussion of the character of the y_i reduces to Case I. The characteristic exponents of equations (105) are $(\alpha_j-\sqrt{-1}\,\beta)$, where the α_j are the characteristic exponents for the homogeneous equations (84). Applying Theorem I we conclude that the y_i are periodic with the period 2π provided

$$\alpha_j\not\equiv\sqrt{-1}\beta \mod \sqrt{-1}\qquad (j=1,\cdots,n).$$

If, however, for any value of j

$$\alpha_j\equiv\sqrt{-1}\beta \mod \sqrt{-1}$$

then, in general, non-periodic terms will arise just as in Case I.

The discussion for the second part of the differential equations

$$(106)\qquad \frac{dx_i}{dt}=\sum_{j=1}^{n}\theta_{ij}x_j+e^{-\sqrt{-1}\beta t}f_i^{(2)}(t)$$

shows similarly that the particular integral is periodic if

$$\alpha_j\not\equiv-\sqrt{-1}\beta\qquad (j=1,\cdots,n).$$

In equations arising in dynamics the characteristic exponents α_j enter in pairs, equal but of opposite sign. Consequently for such equations this last condition

$$\alpha_j \not\equiv -\sqrt{-1}\beta \mod \sqrt{-1} \qquad (j=1,\cdots,n),$$

does not differ from

$$\alpha_j \not\equiv +\sqrt{-1}\beta \mod \sqrt{-1} \qquad (j=1,\cdots,n).$$

We have then the following

THEOREM II. *If the θ_{ij} are periodic with the period 2π, and if the $g_i(t)$ have the form*

$$g_i(t) = \sum_k [a_{ik}\cos(k+\beta)t + b_{ik}\sin(k+\beta)t],$$

and if the characteristic exponents α_j are distinct and none of them congruent to $\pm\sqrt{-1}\beta \mod \sqrt{-1}$, then the particular solution has the form

$$x_i = \sum_m [c_{im}\cos(m+\beta)t + d_{im}\sin(m+\beta)t].$$

If any of the α_j are congruent to $\pm\sqrt{-1}\beta \mod \sqrt{-1}$, then the particular solution will, in general, contain, in addition to periodic terms, terms of the form t times the corresponding complementary function.

Let us suppose that one of the α_j is congruent to $\sqrt{-1}\beta \mod \sqrt{-1}$. Then, in general, the solution will contain non-periodic terms. If now the $g_i(t)$ contain also terms of the form $\frac{\sin}{\cos}(p+q\beta)t$, where p and q are integers, these terms will not, in general, give rise to non-periodic terms in the solution, for the α_j will not, in general, be congruent to $\sqrt{-1}\,q\beta$. Indeed, if β is not rational, for no integral value of q except unity will $\sqrt{-1}\,q\beta \equiv \alpha_j$. But if β (and therefore α_j also) is rational, then certain values of q do exist for which $\alpha_j \equiv \sqrt{-1}\,q\beta$. Let us set

$$\alpha = \frac{i}{j}\sqrt{-1}, \qquad \beta = \frac{I}{J},$$

where i and j are integers relatively prime, and similarly for I and J. By hypothesis

$$\frac{i}{j} - \frac{I}{J} \equiv 0 \mod 1;$$

then

$$\frac{i}{j} - q\frac{I}{J} \not\equiv 0 \mod 1$$

unless $q = 1 + lJ$ (l an integer). In this event a term of the form

$\substack{\cos\\ \sin}(p+q\beta)t$ can also be written $\substack{\cos\\ \sin}(r+\beta)t$ (r an integer) which involves only the first multiple of β, and the solution in general involves non-periodic terms.

Case III.

When two of the characteristic exponents of the homogeneous set are equal the preceding arguments are not applicable. Let us suppose that the homogeneous equations are

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j \qquad (i=1, \cdots, n), \tag{107}$$

but that $\alpha_{n-1} = \alpha_n$. The solution then has in general the form

$$x_i = \sum_{j=1}^{n-2} C_j e^{\alpha_j t} \phi_{ij}(t) + [A + Bt] e^{\alpha_n t} \phi_{in}(t) + B e^{\alpha_n t} \phi_{i(n-1)}(t).$$

Suppose now the given equations are

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j + g_i(t),$$

where the θ_{ij} are the same as in (107) and the g_i are periodic functions of t with the period 2π. Let the particular solutions which depend on the $g_i(t)$ be

$$x_i = \psi_i(t).$$

Just as in Case I we find

$$\frac{d}{dt}[\psi_i(t+2\pi) - \psi_i(t)] = \sum_{j=1}^{n} \theta_{ij} [\psi_j(t+2\pi) - \psi_j(t)], \tag{108}$$

and consequently

$$\psi_i(t+2\pi) - \psi_i(t) = \sum_{j=1}^{n-2} C'_j e^{\alpha_j t} \phi_{ij}(t) + [A' + B't] e^{\alpha_n t} \phi_{in}(t) + B' e^{\alpha_n t} \phi_{i(n-1)}(t). \tag{109}$$

The terms included under the summation sign are obviously the same as those treated in Case I. Neglecting these we may write

$$\psi_i(t) = e^{\alpha_n t} \phi_{in}(t) \cdot f_{in}(t) + e^{\alpha_n t} \cdot \phi_{i(n-1)}(t) f_{i(n-1)}(t) + \text{periodic terms}, \tag{110}$$

where $f_{in}(t)$ and $f_{i(n-1)}(t)$ are functions whose forms are to be determined. Let us form the difference $\psi_i(t+2\pi) - \psi_i(t)$ by means of (110) and compare the result with (109). From (110) we get

$$\begin{aligned} \psi_i(t+2\pi) - \psi_i(t) = {} & [e^{2\alpha_n \pi} f_{in}(t+2\pi) - f_{in}(t)] e^{\alpha_n t} \phi_{in}(t) \\ & + [e^{2\alpha_n \pi} f_{i(n-1)}(t) - f_{i(n-1)}(t)] e^{\alpha_n t} \phi_{i(n-1)}(t). \end{aligned} \tag{111}$$

Comparing this with (109) we see that we must have

$$(112)\qquad \begin{aligned} e^{2\alpha_n\pi} f_{in}(t+2\pi) - f_{in}(t) &= A' + B't, \\ e^{2\alpha_n\pi} f_{i(n-1)}(t+2\pi) - f_{i(n-1)}(t) &= B'. \end{aligned}$$

Let us now define two new functions

$$(113)\qquad \begin{aligned} \lambda_{in}(t) &= e^{\alpha_n t} f_{in}(t) + \left[\frac{A'}{1-e^{2\alpha_n\pi}} + \frac{2\pi B' e^{2\alpha_n\pi}}{(1-e^{2\alpha_n\pi})^2}\right] e^{\alpha_n t} + \frac{B' t e^{\alpha_n t}}{1-e^{2\alpha_n\pi}}, \\ \lambda_{i(n-1)}(t) &= e^{\alpha_n t} f_{i(n-1)}(t) + \frac{B' e^{\alpha_n t}}{1-e^{2\alpha_n\pi}}. \end{aligned}$$

By virtue of the relations (112), $\lambda_{in}(t)$ and $\lambda_{i(n-1)}(t)$ are periodic with the period 2π, and consequently

$$(114)\qquad \begin{aligned} f_{in}(t) &= e^{-\alpha_n t}\lambda_{in}(t) + \left[\frac{A'}{e^{2\alpha_n\pi}-1} - \frac{2\pi B' e^{2\alpha_n\pi}}{(e^{2\alpha_n\pi}-1)^2}\right] + \frac{B' t}{e^{2\alpha_n\pi}-1}, \\ f_{i(n-1)}(t) &= e^{-\alpha_n t}\lambda_{i(n-1)} + \frac{B'}{e^{2\alpha_n\pi}-1}, \end{aligned}$$

where $\lambda_{in}(t)$ and $\lambda_{i(n-1)}(t)$ are periodic with the period 2π. Substituting these expressions in (110), we find

$$(115)\qquad \begin{aligned} \psi_i(t) = &\left[\frac{A'}{(e^{2\alpha_n\pi}-1)} - \frac{2\pi B' e^{2\alpha_n\pi}}{(e^{2\alpha_n\pi}-1)^2} + \frac{B' t}{(e^{2\alpha_n\pi}-1)}\right] e^{\alpha_n t}\phi_{in}(t) \\ &+ \frac{B'}{(e^{2\alpha_n\pi}-1)} e^{\alpha_n t}\phi_{i(n-1)}(t) + \text{periodic terms.} \end{aligned}$$

Comparing these terms with (109) we see that they are merely terms of the complementary function and that therefore

$$x_i = \text{complementary function} + \text{periodic terms.}$$

These results hold provided $\alpha_n \not\equiv 0 \bmod \sqrt{-1}$. In the event $\alpha_n \equiv 0 \bmod \sqrt{-1}$ equations (112) become

$$(116)\qquad f_{in}(t+2\pi) - f_{in}(t) = A' + B't, \qquad f_{i(n-1)}(t+2\pi) - f_{i(n-1)}(t) = B'.$$

It is necessary to give new definitions to the λ functions. We will let

$$(117)\qquad \lambda_{in}(t) = f_{in}(t) - \frac{B'}{4\pi}t^2 - \frac{A' - B'\pi}{2\pi}t, \qquad \lambda_{i(n-1)}(t) = f_{i(n-1)}(t) - \frac{B'}{2\pi}t,$$

which give

$$(118)\qquad f_{in}(t) = \lambda_{in}(t) + \frac{B'}{4\pi}t^2 + \frac{A' - B'\pi}{2\pi}t, \qquad f_{i(n-1)}(t) = \lambda_{i(n-1)}(t) + \frac{B'}{2\pi}t.$$

It is readily verified that $\lambda_{in}(t)$ and $\lambda_{i(n-1)}(t)$ are periodic by virtue of (116). Substituting these expressions in (110), we find

$$(119)\quad \psi_i(t)=e^{\alpha_n t}\left\{\left[\frac{B'}{4\pi}t^2+\frac{A'-B'\pi}{2\pi}t\right]\phi_{in}(t)+\frac{B'}{2\pi}t\phi_{i(n-1)}(t)\right\}+\text{periodic terms.}$$

If we let

$$\frac{A'-B'\pi}{2\pi}=C,\qquad \frac{B'}{2\pi}=D,$$

we get

$$\psi_i(t)=e^{\alpha_n t}\{[Ct+\tfrac{1}{2}Dt^2]\phi_{in}(t)+Dt\phi_{i(n-1)}(t)\}+\text{periodic terms.}$$

This expression is not quite the same as t times the corresponding complementary function for in one term we have the coefficient $\frac{1}{2}D$ instead of D. The theorem for this case then is

THEOREM III. *If θ_{ij} and $g_i(t)$ are periodic with the period 2π, and if two of the α_j are equal but not congruent to zero* mod $\sqrt{-1}$, *then the particular solution consists of terms periodic with the period 2π plus a constant times the corresponding complementary function. But if two of the α_j are equal and congruent to zero so that the corresponding part of the complementary function has the form*

$$e^{\alpha_n t}\{(A+Bt)\phi_{in}(t)+B\phi_{i(n-1)}(t)\},$$

then the particular solution consists of a periodic function plus a term of the form

$$e^{\alpha_n t}\{(Ct+\tfrac{1}{2}Dt^2)\phi_{in}(t)+Dt\phi_{i(n-1)}(t)\}.$$

§ 14. *Special theorems for the equations of variation.*

The foregoing theorems presuppose merely the conditions that the coefficients are periodic with the period 2π. Further facts with regard to the solutions may be established when further facts are specified with regard to the coefficients of the differential equations. Our equations of variation (51) may be written

$$(120)\quad \frac{d\rho_1}{d\tau}=\rho_2,\qquad \frac{d\rho_2}{d\tau}\bar{\theta}_2\rho_1+\theta_3\sigma_1,\qquad \frac{d\sigma_1}{d\tau}=\sigma_2,\qquad \frac{d\sigma_2}{d\tau}=\theta_3\rho_1+\bar{\theta}_4\sigma_1,$$

where the notation *with respect to the* θ's has the following significance: even subscripts denote functions even in τ, and odd subscripts denote functions odd in τ; one dash indicates that only odd multiples of τ are involved, and two dashes indicate that only even multiples of τ are involved. The solutions, equations (82) and (83), may be characterized in the same manner, and are

then

$$
(121)\quad
\begin{aligned}
\rho_1 &= A\alpha_2(\tau) + B\alpha_1(\tau) + C\bar{\alpha}_3(\tau) + D[\bar{\alpha}_4(\tau) + \tau\bar{\alpha}_3(\tau)],\\
\rho_2 &= A\beta_1(\tau) + B\beta_2(\tau) + C\bar{\beta}_4(\tau) + D[\bar{\beta}_3(\tau) + \tau\bar{\beta}_4(\tau)],\\
\sigma_1 &= A\bar{\gamma}_1(\tau) + B\bar{\gamma}_2(\tau) + C\gamma_4(\tau) + D[\gamma_3(\tau) + \tau\bar{\gamma}_4(\tau)],\\
\sigma_2 &= A\bar{\delta}_2(\tau) + B\bar{\delta}_1(\tau) + C\delta_3(\tau) + D[\delta_4(\tau) + \tau\delta_3(\tau)],
\end{aligned}
$$

where the notation is the same as for the θ's with the exception that in the first two solutions every integral multiple of τ is increased by $\pm\lambda\tau$, e. g., $\cos(3+\lambda)\tau$. On these terms the dashes refer only to the integral part of the coefficients of τ.

Suppose now we have the following non-homogeneous equations:

$$
(122)\quad
\begin{aligned}
\frac{d\rho_1}{d\tau} &= \rho_2, & \frac{d\rho_2}{d\tau} &= \bar{\theta}_2\rho_1 + \bar{\theta}_3\sigma_1 + g(\tau),\\
\frac{d\sigma_1}{d\tau} &= \sigma_2, & \frac{d\sigma_2}{d\tau} &= \theta_3\rho_1 + \bar{\theta}_4\sigma_1 + f(\tau),
\end{aligned}
$$

where $g(\tau)$ and $f(\tau)$ are periodic with the period 2π. Since the characteristic exponents are $\sqrt{-1}\lambda$, $-\sqrt{-1}\lambda$, 0, 0, by Theorem III the solution has the form

$$
(123)\quad
\begin{aligned}
\rho_1 &= (\rho_1) + \xi_1 = (\rho_1) + \omega_1(\tau) + a\tau\bar{\alpha}_3 + b[\tfrac{1}{2}\tau^2\bar{\alpha}_3 + \tau\bar{\alpha}_4],\\
\rho_2 &= (\rho_2) + \xi_2 = (\rho_2) + \omega_2(\tau) + a\tau\bar{\beta}_4 + b[\tfrac{1}{2}\tau^2\bar{\beta}_4 + \tau\bar{\beta}_3],\\
\sigma_1 &= (\sigma_1) + \eta_1 = (\sigma_1) + \omega_3(\tau) + a\tau\bar{\gamma}_4 + b[\tfrac{1}{2}\tau^2\bar{\gamma}_4 + \tau\bar{\gamma}_3],\\
\sigma_2 &= (\sigma_1) + \eta_2 = (\sigma_2) + \omega_4(\tau) + a\tau\delta_3 + b[\tfrac{1}{2}\tau^2\delta_3 + \tau\delta_4],
\end{aligned}
$$

where (ρ_i), (σ_i) indicate the complementary function and ξ_i and η_i are the particular integrals, of which the ω_i are the periodic parts. The a and b are constants which depend upon the differential equations. Since the ξ_i and η_i satisfy the differential equations we have

$$
(124)\quad
\begin{aligned}
\frac{d}{d\tau}\xi_1(\tau) &= \xi_2(\tau), & \frac{d}{d\tau}\xi_2(\tau) &= \bar{\theta}_2\xi_1(\tau) + \theta_3\eta_1(\tau) + g(\tau),\\
\frac{d}{d\tau}\eta_1(\tau) &= \eta_2(\tau), & \frac{d}{d\tau}\eta_2(\tau) &= \theta_3\xi_1(\tau) + \bar{\theta}_4\eta_1(\tau) + f(\tau).
\end{aligned}
$$

Changing τ into $-\tau$ in these equations, we get

$$
(125)\quad
\begin{aligned}
\frac{d}{d\tau}\xi_1(-\tau) &= -\xi_2(-\tau), & \frac{d}{d\tau}\xi_2(-\tau) &= -\bar{\theta}_2\xi_1(-\tau) + \theta_3\eta_1(-\tau) - g(-\tau),\\
\frac{d}{d\tau}\eta_1(-\tau) &= -\eta_2(-\tau), & \frac{d}{d\tau}\eta_2(-\tau) &= +\theta_3\xi_1(-\tau) - \bar{\theta}_4\eta_1(-\tau) - f(-\tau).
\end{aligned}
$$

If now we make the additional hypothesis that $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function, equations (124) and (125) may be combined into the following set

$$
(126)\quad
\begin{aligned}
&\frac{d}{d\tau}[\xi_1(\tau)-\xi_1(-\tau)]=[\xi_2(\tau)+\xi_2(-\tau)],\\
&\frac{d}{d\tau}[\xi_2(\tau)+\xi_2(-\tau)]=\bar{\theta}_2[\xi_1(\tau)-\xi_1(-\tau)]+\theta_3[\eta_1(\tau)+\eta_1(-\tau)],\\
&\frac{d}{d\tau}[\eta_1(\tau)+\eta_1(-\tau)]=[\eta_2(\tau)-\eta_2(-\tau)],\\
&\frac{d}{d\tau}[\eta_2(\tau)-\eta_2(-\tau)]=\theta_3[\xi_1(\tau)-\xi_1(-\tau)]+\bar{\theta}_4[\eta_1(\tau)+\eta_1(-\tau)].
\end{aligned}
$$

These equations are the same as (120). Therefore

$$
(127)\quad
\begin{aligned}
\xi_1(\tau)-\xi_1(-\tau)&=A\bar{\alpha}_2(\tau)+B\alpha_1(\tau)+C\bar{\alpha}_3(\tau)+D[\bar{\bar{\alpha}}_4(\tau)+\tau\bar{\bar{\alpha}}_3(\tau)],\\
\xi_2(\tau)+\xi_2(-\tau)&=A\beta_1(\tau)+B\bar{\beta}_2(\tau)+C\bar{\bar{\beta}}_4(\tau)+D[\bar{\bar{\beta}}_3(\tau)+\tau\bar{\bar{\beta}}_4(\tau)],\\
\eta_1(\tau)+\eta_1(-\tau)&=A\bar{\gamma}_1(\tau)+B\bar{\gamma}_2(\tau)+C\gamma_4(\tau)+D[\bar{\gamma}_3(\tau)+\tau\bar{\gamma}_4(\tau)],\\
\eta_2(\tau)-\eta_2(-\tau)&=A\bar{\bar{\delta}}_2(\tau)+B\bar{\bar{\delta}}_1(\tau)+C\delta_3(\tau)+D[\delta_4(\tau)+\tau\delta_3(\tau)].
\end{aligned}
$$

Putting $\tau=0$ in these equations we get from the first and the fourth

$$
(128)\qquad 0=A\alpha_2(0)+D\alpha_4(0),\qquad 0=A\delta_2(0)+D\delta_4(0).
$$

Either $A=D=0$, or the determinant $\alpha_2(0)\delta_4(0)-\delta_2(0)\alpha_4(0)=0$. But it is readily verified that the determinant is not zero. Therefore $A=D=0$. If we suppose that $\xi_2(0)=\eta_1(0)=0$ (we shall be interested only in such cases), it follows from the second and third equations of (127) that

$$
0=B\bar{\beta}_2(0)+C\bar{\bar{\beta}}_4(0),\qquad 0=B\bar{\gamma}_2(0)+C\bar{\gamma}_4(0),
$$

and hence $B=C=0$. Consequently

$$
(129)\quad
\begin{aligned}
\xi_1(\tau)-\xi_1(-\tau)&=0,\qquad \xi_2(\tau)+\xi_2(-\tau)=0,\\
\eta_1(\tau)+\eta_1(-\tau)&=0,\qquad \eta_2(\tau)-\eta_2(-\tau)=0.
\end{aligned}
$$

We have then

THEOREM IV. *If $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function of τ, and if $\xi_2(0)=\eta_1(0)=0$, then $\xi_1(\tau)$ and $\eta_2(\tau)$ are even functions of τ, and $\xi_2(\tau)$ and $\eta_1(\tau)$ are odd functions of τ.*

In the same way it can be shown that if $g(\tau)$ is odd and $f(\tau)$ is even, and if $\xi_1(0) = \eta_2(0) = 0$, then ξ_1 and η_2 are odd and ξ_2 and η_1 are even.

Let us suppose now that $g(\tau)$ is periodic and contains only even multiples of τ, and that $f(\tau)$ is periodic and contains only odd multiples of τ. The general form of the solution will be the same as (123). ξ_1, ξ_2, η_1, and η_2 satisfy the differential equations

$$
(130)\quad
\begin{aligned}
&\frac{d}{d\tau}\xi_1(\tau) = \xi_2(\tau), && \frac{d}{d\tau}\xi_2(\tau) = \bar{\theta}_2\xi_1(\tau) + \bar{\theta}_3\eta_1(\tau) + \bar{g}(\tau),\\
&\frac{d}{d\tau}\eta_1(\tau) = \eta_2(\tau), && \frac{d}{d\tau}\eta_2(\tau) = \theta_3\xi_1(\tau) + \bar{\theta}_4\eta_1(\tau) + \bar{f}(\tau).
\end{aligned}
$$

Let us denote $\xi_i(\tau + \pi)$ by $\xi_i'(\tau)$ and $\eta_i(\tau + \pi)$ by $\eta_i'(\tau)$. Then by changing τ into $\tau + \pi$ in (130) we have

$$
(131)\quad
\begin{aligned}
&\frac{d}{d\tau}\xi_1'(\tau) = \xi_2'(\tau), && \frac{d}{d\tau}\xi_2'(\tau) = \bar{\theta}_2\xi_1'(\tau) - \theta_3\eta_1'(\tau) + \bar{g}(\tau),\\
&\frac{d}{d\tau}\eta_1'(\tau) = \eta_2'(\tau), && \frac{d}{d\tau}\eta_2'(\tau) = -\theta_3\xi_1'(\tau) + \bar{\theta}_4\eta_1'(\tau) - \bar{f}(\tau).
\end{aligned}
$$

From (130) and (131) it follows that

$$
(132)\quad
\begin{aligned}
&\frac{d}{d\tau}[\xi_1 - \xi_1'] = [\xi_2 - \xi_2'], && \frac{d}{d\tau}[\xi_2 - \xi_2'] = \bar{\theta}_2[\xi_1 - \xi_1'] + \theta_3[\eta_1 + \eta_1'],\\
&\frac{d}{d\tau}[\eta_1 + \eta_1'] = [\eta_2 + \eta_2'], && \frac{d}{d\tau}[\eta_2 + \eta_2'] = \theta_3[\xi_1 - \xi_1'] + \bar{\theta}_4[\eta_1 + \eta_1'].
\end{aligned}
$$

The solutions of these equations, which have the form of (120), are

$$
(133)\quad
\begin{aligned}
\xi_1 - \xi_1' &= A\bar{\alpha}_2(\tau) + B\alpha_1(\tau) + C\bar{\alpha}_3(\tau) + D[\bar{\alpha}_4(\tau) + \tau\bar{\alpha}_3(\tau)],\\
\xi_2 - \xi_2' &= A\beta_1(\tau) + B\bar{\beta}_2(\tau) + C\bar{\beta}_4(\tau) + D[\bar{\beta}_3(\tau) + \tau\bar{\beta}_4(\tau)],\\
\eta_1 + \eta_1' &= A\bar{\gamma}_1(\tau) + B\bar{\gamma}_2(\tau) + C\bar{\gamma}_4(\tau) + D[\gamma_3(\tau) + \tau\gamma_4(\tau)],\\
\eta_2 + \eta_2' &= A\bar{\delta}_2(\tau) + B\bar{\delta}_1(\tau) + C\delta_3(\tau) + D[\delta_4(\tau) + \tau\delta_3(\tau)].
\end{aligned}
$$

Forming these expressions directly from (123), we get

$$
(134)\quad
\begin{aligned}
\xi_1 - \xi_1' &= \omega_1(\tau) - \omega_1(\tau + \pi) - [a\pi + \tfrac{1}{2}b\pi^2]\bar{\alpha}_3(\tau) - b\pi[\tau\bar{\alpha}_3(\tau) + \bar{\alpha}_4(\tau)],\\
\xi_2 - \xi_2' &= \omega_2(\tau) - \omega_2(\tau + \pi) - [a\pi + \tfrac{1}{2}b\pi^2]\bar{\beta}_4(\tau) - b\pi[\tau\bar{\beta}_4(\tau) + \bar{\beta}_3(\tau)],\\
\eta_1 + \eta_1' &= \omega_3(\tau) + \omega_3(\tau + \pi) - [a\pi + \tfrac{1}{2}b\pi^2]\bar{\gamma}_4(\tau) - b\pi[\tau\bar{\gamma}_4(\tau) + \bar{\gamma}_3(\tau)],\\
\eta_2 + \eta_2' &= \omega_4(\tau) + \omega_4(\tau + \pi) - [a\pi + \tfrac{1}{2}b\pi^2]\delta_3(\tau) - b\pi[\tau\delta_3(\tau) + \bar{\delta}_4(\tau)].
\end{aligned}
$$

Comparing (133) and (134) we see that

$$A = B = 0, \qquad C = -\left[a\pi + \tfrac{1}{2}b\pi^2\right], \qquad D = -b\pi,$$
$$\omega_1(\tau) - \omega_1(\tau + \pi) = 0, \qquad \omega_3(\tau) + \omega_3(\tau + \pi) = 0,$$
$$\omega_2(\tau) - \omega_2(\tau + \pi) = 0, \qquad \omega_4(\tau) + \omega_4(\tau + \pi) = 0.$$

Therefore $\omega_1(\tau)$ and $\omega_2(\tau)$ contain only even multiples of τ while $\omega_3(\tau)$ and $\omega_4(\tau)$ contain only odd multiples, and by carrying this result into (123) we find

$$(135)\qquad \begin{aligned}
\xi_1 &= \overline{\overline{\omega}}_1(\tau) + a\tau\overline{\alpha}_3(\tau) + b\left[\tfrac{1}{2}\tau^2\overline{\alpha}_3(\tau) + \tau\overline{\alpha}_4(\tau)\right],\\
\xi_2 &= \overline{\overline{\omega}}_2(\tau) + a\tau\overline{\beta}_4(\tau) + b\left[\tfrac{1}{2}\tau^2\overline{\beta}_4(\tau) + \tau\overline{\beta}_3(\tau)\right],\\
\eta_1 &= \overline{\omega}_3(\tau) + a\tau\gamma_4(\tau) + b\left[\tfrac{1}{2}\tau^2\gamma_4(\tau) + \tau\gamma_3(\tau)\right],\\
\eta_2 &= \overline{\omega}_4(\tau) + a\tau\delta_3(\tau) + b\left[\tfrac{1}{2}\tau^2\delta_3(\tau) + \tau\delta_4(\tau)\right],
\end{aligned}$$

and hence we have

THEOREM V. *If $g(\tau)$ contains only even multiples of τ and $f(\tau)$ contains only odd multiples, then ξ_1 and ξ_2 contain only even multiples of τ and η_1 and η_2 contain only odd multiples.*

If in addition to the above hypotheses we suppose that $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function, then ξ_1 and η_2 are even functions of τ and ξ_2 and η_1 are odd functions. Therefore $b = 0$. But if $g(\tau)$ is an odd function and $f(\tau)$ is an even function, then ξ_1 and η_2 are odd functions and ξ_2 and η_1 are even functions, so that in this event $a = 0$.

In the same manner as above we prove

THEOREM VI. *If $g(\tau)$ contains only odd multiples of τ and $f(\tau)$ contains only even multiples, then ξ_1 and ξ_2 contain only odd multiples of τ and η_1 and η_2 contain only even multiples. Furthermore ξ_1, ξ_2, η_1 and η_2 are periodic with the period 2π.*

If $g(\tau)$ is of the form $\sum_j m_j \cos(j \pm \lambda)\tau$ and $f(\tau)$ has the form $\sum_j n_j \sin(j \pm \lambda)\tau$ then, since $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equations, the form of the solution is, by Theorem II,

$$(136)\qquad \begin{aligned}
\xi_1 &= \sum_k p_k^{(1)} \cos(k \pm \lambda)\tau + \sum_k p_k^{(2)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\alpha_1(\tau) + a^{(2)}\tau\overline{\alpha}_2(\tau),\\
\xi_2 &= \sum_k q_k^{(2)} \cos(k \pm \lambda)\tau + \sum_k q_k^{(1)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\overline{\beta}_2(\tau) + a^{(2)}\tau\beta_1(\tau),\\
\eta_1 &= \sum_k r_k^{(2)} \cos(k \pm \lambda)\tau + \sum_k r_k^{(1)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\overline{\gamma}_2(\tau) + a^{(2)}\tau\overline{\gamma}_1(\tau),\\
\eta_2 &= \sum_k s_k^{(1)} \cos(k \pm \lambda)\tau + \sum_k s_k^{(2)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\overline{\delta}_1(\tau) + a^{(2)}\tau\overline{\delta}_2(\tau);
\end{aligned}$$

but since $g(\tau)$ is an even function and $f(\tau)$ is an odd function, ξ_1 and η_2 are

even functions and ξ_2 and η_1 are odd functions. Therefore all the coefficients in (136) which have the upper index (2) are zero. But if $g(\tau)$ were an odd function of τ and $f(\tau)$ an even function then all the coefficients in (136) which have the upper index (1) would be zero. Therefore

THEOREM VII. *If $g(\tau)$ is of the form $\sum_j m_j \cos(j \pm \lambda)\tau$ and $f(\tau)$ has the form $\sum_j n_j \sin(j \pm \lambda)\tau$, where $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equations, then the particular solution has the form*

$$\xi_1 = \sum_a p_a \cos(a \pm \lambda)\tau + A\tau\bar{\alpha}_1(\tau),$$

$$\xi_2 = \sum_b p_b \sin(b \pm \lambda)\tau + A\tau\bar{\beta}_2(\tau),$$

$$\eta_1 = \sum_c p_c \sin(c \pm \lambda)\tau + A\tau\bar{\gamma}_2(\tau),$$

$$\eta_2 = \sum_d p_d \cos(d \pm \lambda)\tau + A\tau\bar{\delta}_1(\tau).$$

Also,

THEOREM VIII. *If $g(\tau)$ has the form $\sum_j m_j \sin(j \pm \lambda)\tau$ and $f(\tau)$ has the form $\sum_j n_j \cos(j \pm \lambda)\tau$, where $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equations, then the particular solution has the form*

$$\xi_1 = \sum_a r_a \sin(a \pm \lambda)\tau + B\tau\bar{\alpha}_2(\tau), \qquad \xi_2 = \sum_b r_b \cos(b \pm \lambda)\tau + B\tau\bar{\beta}_1(\tau),$$

$$\eta_1 = \sum_c r_c \cos(c \pm \lambda)\tau + B\tau\bar{\gamma}_1(\tau), \qquad \eta_2 = \sum_d r_d \sin(d \pm \lambda)\tau + B\tau\bar{\delta}_2(\tau).$$

It is understood that in the above two theorems a, b, c, d, and j are integers.

§ 15. *Integration of the complete differential equations* (50).

It will be convenient hereafter to use the following notation for the solutions of the equations of variation:

$$(137)\qquad \begin{aligned} \rho &= A\alpha_2(\tau) + B\alpha_1(\tau) + C\alpha_3(\tau) + D[\alpha_4(\tau) + \tau\alpha_3(\tau)], \\ \sigma &= A\gamma_1(\tau) + B\gamma_2(\tau) + C\gamma_4(\tau) + D[\gamma_3(\tau) + \tau\gamma_4(\tau)]. \end{aligned}$$

The $\alpha_j(\tau)$ and $\gamma_j(\tau)$ are characterized thus:

$\alpha_2(\tau)$	involves	only	terms	of the	form		$\cos[(2n+1) \pm \lambda]\tau$,
$\gamma_1(\tau)$	"	"	"	"	"	"	$\sin[2n \pm \lambda]\tau$,
$\alpha_1(\tau)$	"	"	"	"	"	"	$\sin[(2n+1) \pm \lambda]\tau$,
$\gamma_2(\tau)$	"	"	"	"	"	"	$\cos[2n \pm \lambda]\tau$,
$\alpha_3(\tau)$	"	"	"	"	"	"	$\sin[2n]\tau$,
$\gamma_4(\tau)$	"	"	"	"	"	"	$\cos[2n+1]\tau$,
$\alpha_4(\tau)$	"	"	"	"	"	"	$\cos[2n]\tau$,
$\gamma_3(\tau)$	"	"	"	"	"	"	$\sin[2n+1]\tau$.

It will be convenient also to write the differential equations for ρ and σ as

$$(138)\quad \begin{aligned} \rho'' + \theta_2\rho + \theta_3\sigma &= \theta_{001}\epsilon + \theta_{101}\epsilon\rho + \theta_{200}\rho^2 + \theta_{110}\rho\sigma + \theta_{020}\sigma^2 + \cdots, \\ \sigma'' + \theta_4\sigma + \theta_3\rho &= \qquad\qquad\qquad\quad \theta_{200}\rho^2 + \theta_{110}\rho\sigma + \theta_{020}\sigma^2 + \cdots, \end{aligned}$$

where all the θ's are periodic with the period 2π, and θ_2 and θ_4 contain only cosines of even multiples of τ, and θ_3 contains only sines of odd multiples of τ. On the right side of the first equation the coefficients of terms carrying odd powers of σ contain only sines of odd multiples of τ. All the other coefficients contain only cosines of even multiples. In the second equation odd powers of σ have coefficients involving only cosines of even multiples. All other coefficients contain only sines of odd multiples.

The initial conditions are

$$\rho = \alpha, \qquad \sigma = 0, \qquad \rho' = 0, \qquad \sigma' = \delta.$$

We will integrate equations (138) as power series in α, δ and ϵ, τ entering into the coefficients. From Poincaré's extension of Cauchy's theorem we know that these series are convergent for any arbitrarily chosen interval for τ, $0 \leqq \tau \leqq T$, provided $|\alpha|$, $|\delta|$ and $|\epsilon|$ are sufficiently small. The equations of variation involve the period $2\pi/\lambda$. The solutions are not periodic unless λ is rational. Hence θ_1^2 must be chosen in advance so that λ is rational. We will suppose then that $\lambda = J/K$, where J and K are integers, relatively prime. Then three solutions of the equations of variation are periodic with the period $2K\pi$.

Since ρ and σ are expansible in powers of α, δ and ϵ, we may write

$$\rho = \rho_{100}\alpha + \rho_{010}\delta + \rho_{001}\epsilon + \rho_{200}\alpha^2 + \rho_{110}\alpha\delta + \rho_{020}\delta^2 + \rho_{101}\alpha\epsilon + \rho_{011}\delta\epsilon + \rho_{002}\epsilon^2 + \cdots,$$

$$\sigma = \sigma_{100}\alpha + \sigma_{010}\delta + \sigma_{001}\epsilon + \sigma_{200}\alpha^2 + \sigma_{110}\alpha\delta + \sigma_{020}\delta^2 + \sigma_{101}\alpha\epsilon + \sigma_{011}\rho\epsilon + \sigma_{002}\epsilon^2 + \cdots.$$

Substituting these expressions in the differential equations (138) and equating the coefficients of similar powers, we have:

Coefficient of α.

$$\rho'_{100} + \theta_2\rho_{100} + \theta_3\sigma_{100} = 0,$$

$$\sigma''_{100} + \theta_4\sigma_{100} + \theta_3\rho_{100} = 0.$$

The solution of these equations is

$$\rho_{100} = A\alpha_2(\tau) + B\alpha_1(\tau) + C\alpha_3(\tau) + D[\alpha_4(\tau) + \tau\alpha_3(\tau)],$$

$$\sigma_{100} = A\gamma_1(\tau) + B\gamma_2(\tau) + C\gamma_4(\tau) + D[\gamma_3(\tau) + \tau\gamma_4(\tau)].$$

In order to satisfy the initial conditions we must have, at $\tau = 0$,

$$\rho_{100} = 1, \qquad \sigma_{100} = 0, \qquad \rho'_{100} = 0, \qquad \sigma'_{100} = 0.$$

From these conditions we find

$$
(139)\qquad
\begin{aligned}
&A\alpha_2(0) + D\alpha_4(0) = 1,\\
&B\alpha_1'(0) + C\alpha_3'(0) = 0,\\
&B\gamma_2(0) + C\gamma_4(0) = 0,\\
&A\gamma_1'(0) + D[\gamma_3'(0) + \gamma_4(0)] = 0.
\end{aligned}
$$

The solution of these conditional equations is

$$
(140)\qquad
\begin{aligned}
&B = 0, \qquad A = \frac{\gamma_3'(0) + \gamma_4(0)}{\Delta} = A_{100}^{(1)},\\
&C = 0, \qquad D = -\frac{\gamma_1'(0)}{\Delta} = A_{100}^{(2)},\\
&\Delta = \alpha_2(0)[\gamma_3'(0) + \gamma_4(0)] - \alpha_4(0)\gamma_1'(0).
\end{aligned}
$$

Hence the solution is

$$
(141)\qquad
\begin{aligned}
\rho_{100} &= A_{100}^{(1)}\alpha_2(\tau) + A_{100}^{(2)}[\alpha_4(\tau) + \tau\alpha_3(\tau)],\\
\sigma_{100} &= A_{100}^{(1)}\gamma_1(\tau) + A_{100}^{(2)}[\gamma_3(\tau) + \tau\alpha_4(\tau)].
\end{aligned}
$$

Coefficient of δ.

$$
(142)\qquad
\begin{aligned}
&\rho_{010}'' + \theta_2\rho_{010} + \theta_3\sigma_{010} = 0,\\
&\sigma_{010}'' + \theta_4\sigma_{010} + \theta_3\rho_{010} = 0.
\end{aligned}
$$

These equations are the same as for the coefficient of α. From the initial conditions we must have at $\tau = 0$

$$
\rho_{010} = 0, \qquad \sigma_{010} = 0, \qquad \rho_{010}' = 0, \qquad \sigma_{010}' = 1.
$$

The solutions are

$$
(143)\qquad
\begin{aligned}
\rho_{010} &= A_{010}^{(1)}\alpha_2(\tau) + A_{010}^{(2)}[\alpha_4(\tau) + \tau\alpha_3(\tau)],\\
\sigma_{010} &= A_{010}^{(1)}\gamma_1(\tau) + A_{010}^{(2)}[\gamma_3(\tau) + \tau\gamma_4(\tau)],
\end{aligned}
$$

where

$$
A_{010}^{(1)} = -\frac{\alpha_4(0)}{\Delta}, \qquad A_{010}^{(2)} = \frac{\alpha_2(0)}{\Delta}.
$$

Coefficient of ϵ.

$$
(144)\qquad
\begin{aligned}
&\rho_{001}'' + \theta_2\rho_{001} + \theta_3\sigma_{001} = \theta_{001},\\
&\sigma_{001}'' + \theta_4\sigma_{001} + \theta_3\rho_{001} = 0.
\end{aligned}
$$

The right member, θ_{001}, is a periodic function of τ with the period 2π. Furthermore it involves only cosines of even multiples of τ. Hence by Theorem V the

solution has the form

$$(145)\quad \begin{aligned} \rho_{001} &= A\alpha_2(\tau) + B\alpha_1(\tau) + C\alpha_3(\tau) + D[\alpha_4(\tau) + \tau\alpha_3(\tau)] + \alpha_5(\tau) + a\tau\alpha_3(\tau), \\ \sigma_{001} &= A\gamma_1(\tau) + B\gamma_2(\tau) + C\gamma_4(\tau) + D[\gamma_3(\tau) + \tau\gamma_4(\tau)] + \gamma_5(\tau) + a\tau\gamma_4(\tau), \end{aligned}$$

where a is a constant depending on θ_{001}, $\alpha_5(\tau)$ is a cosine series involving only even multiples of τ, and $\gamma_5(\tau)$ involves only sines of odd multiples of τ.

From the initial conditions we must have at $\tau = 0$

$$\rho_{001} = 0, \qquad \sigma_{001} = 0, \qquad \rho'_{001} = 0, \qquad \sigma'_{001} = 0.$$

Determining the constants of integration so as to satisfy these conditions we have the solution

$$(146)\quad \begin{aligned} \rho_{001} &= A^{(1)}_{001}\alpha_2(\tau) + A^{(2)}_{001}[\alpha_4(\tau) + \tau\alpha_3(\tau)] + \alpha_6(\tau), \\ \sigma_{001} &= A^{(1)}_{001}\gamma_1(\tau) + A^{(2)}_{001}[\gamma_3(\tau) + \tau\gamma_4(\tau)] + \gamma_6(\tau), \end{aligned}$$

where

$$A^{(1)}_{001} = \frac{1}{\Delta}[\alpha_4\beta'_6 - \alpha_6(\beta_3 + \beta'_4)],$$

$$A^{(2)}_{001} = \frac{1}{\Delta}[\alpha_6\beta'_1 - \alpha_1\beta'_6],$$

$$\alpha_6(\tau) = \alpha_5(\tau) - a\alpha_4(\tau), \qquad \gamma_6(\tau) = \gamma_5(\tau) - a\gamma_3(\tau).$$

It will be seen later that the value of $\alpha_6(\tau)$ for $\tau = 0$ plays an important rôle, so that it is necessary for us to verify that it does not vanish. By hypothesis $\alpha_5(\tau)$ is the periodic part of the particular solution for ρ in the differential equations (144). Let us put in these equations

$$\rho_{001} = \phi + a\tau\alpha_3(\tau), \qquad \sigma_{001} = \psi + a\tau\gamma_4(\tau),$$

where ϕ and ψ are the periodic parts of the particular solution. We find

$$\phi'' + \theta_2\phi + \theta_3\psi = -2a\alpha'_3(\tau) + 1 + [(-3\theta_1^2 + \tfrac{3}{4}\beta^2) - \tfrac{3}{4}\beta^2\cos 2\tau]\mu^2 + \cdots,$$

$$\psi'' + \theta_4\psi + \theta_3\phi = -2a\beta'_3(\tau);$$

or,

$$\phi'' + \theta_2\phi + \theta_3\psi = 1 + [(-3\theta_1^2 + \tfrac{3}{4}\beta^2) - 2(a_0 + \tfrac{3}{4})\beta^2\cos 2\tau]\mu^2,$$

$$\psi'' + \theta_4\psi + \theta_3\psi = -2a_0\beta\sin\tau\mu + \cdots,$$

where we have substituted

$$a = a_0 + a_2\mu^2 + \cdots.$$

The functions ϕ and ψ can be expanded as series of the form

$$\phi = \phi_0 + \phi_2\mu^2 + \cdots, \qquad \psi = \psi_1\mu + \psi_3\mu^3 + \cdots.$$

Then we get

$$\phi_0'' + \phi_0 = 1, \qquad \text{so that} \qquad \phi_0 = 1 + c_0 \cos\tau,$$

$$\psi_1'' + \psi_1 = (3 - 2a_0)\beta \sin\tau + \tfrac{3}{2}\beta c_0 \sin 2\tau.$$

Since ψ_1 is periodic we must have $a_0 = \frac{3}{2}$, and then

$$\psi_1 = c_1 \sin\tau - \tfrac{1}{2}\beta c_0 \sin 2\tau,$$

$$\phi_2'' + \phi_2 = -\tfrac{1}{4}c_0\beta^2 \cos\tau + \text{other terms}.$$

Since ϕ_2 is periodic we must have $c_0 = 0$. Therefore

$$\phi = \alpha_5(\tau) = 1 + \mu^2 P_1(\mu^2), \qquad a = \tfrac{3}{2} + \mu^2 P_2(\mu^2),$$

$$\alpha_4(\tau) = 0 + [\tfrac{3}{2}\beta^2 + \tfrac{1}{2}\beta^2 \cos 2\tau]\mu^2 + \cdots.$$

Consequently

$$\alpha_6(0) = \alpha_5(0) - a\alpha_4(0) = 1 + \mu^2 P_3(\mu^2), \tag{149}$$

an expression which does not vanish identically.

Coefficient of α^2.

$$\begin{aligned} \rho_{200}'' + \theta_2\rho_{200} + \theta_3\sigma_{200} &= R_{200}, \\ \sigma_{200}'' + \theta_4\sigma_{200} + \theta_3\rho_{200} &= S_{200}. \end{aligned} \tag{150}$$

The right members of these equations have the following expressions:

$$\begin{aligned} R_{200} = {} & A_{100}^{(1)2}[\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] \\ & + A_{100}^{(1)}A_{100}^{(2)}[2\theta_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \theta_{110}\{\alpha_2(\tau\gamma_4 + \gamma_3) + \gamma_1(\tau\alpha_3 + \alpha_4)\} \\ & \qquad + 2\theta_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + A_{100}^{(2)2}[\theta_{200}(\tau\alpha_3 + \alpha_4)^2 + \theta_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \theta_{020}(\tau\gamma_4 + \gamma_3)^2] \\ S_{200} = {} & A_{100}^{(1)2}[\bar\theta_{200}\alpha_2^2 + \bar\theta_{110}\alpha_2\gamma_1 + \bar\theta_{020}\gamma_1^2] \\ & + A_{100}^{(1)}A_{100}^{(2)}[2\bar\theta_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \bar\theta_{110}\{\alpha_2(\tau\gamma_4 + \gamma_3) + \gamma_1(\tau\alpha_3 + \alpha_4)\} \\ & \qquad + 2\bar\theta_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + A_{100}^{(2)2}[\bar\theta_{200}(\tau\alpha_3 + \alpha_4)^2 + \bar\theta_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \bar\theta_{020}(\tau\gamma_4 + \gamma_3)^2]. \end{aligned}$$

The initial conditions are

$$\rho_{200} = 0, \qquad \sigma_{200} = 0, \qquad \rho_{200}' = 0, \qquad \sigma_{200}' = 0.$$

Since equations (150) are linear, and their left members have the same form as (51), the solutions will have the form

$$
(151)\quad
\begin{aligned}
\rho_{200} &= A\alpha_2 + B\alpha_1 + C\alpha_3 + D[\tau\alpha_3 + \alpha_4] \\
&\qquad + A_{100}^{(1)^2}\phi_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}\phi_2(\tau) + A_{100}^{(2)^2}\phi_3(\tau), \\
\sigma_{200} &= A\gamma_1 + B\gamma_2 + C\gamma_4 + D[\tau\gamma_4 + \gamma_3] \\
&\qquad + A_{100}^{(1)^2}\psi_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}\psi_2(\tau) + A_{100}^{(2)^2}\psi_3(\tau).
\end{aligned}
$$

Imposing the initial conditions we find

$$B = C = 0,$$

$$
\begin{aligned}
A = {} & \frac{\psi_1'(0)\alpha_4(0) - \phi_1(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta} A_{100}^{(1)^2} \\
& + \frac{\psi_2'(0)\alpha_4(0) - \phi_2(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta} A_{100}^{(1)}A_{100}^{(2)} \\
& + \frac{\psi_3'(0)\alpha_4(0) - \phi_3(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta} A_{100}^{(2)^2},
\end{aligned}
$$

$$
\begin{aligned}
D = {} & \frac{\phi_1(0)\gamma_1'(0) - \psi_1'(0)\alpha_2(0)}{\Delta} A_{100}^{(1)^2} + \frac{\phi_2(0)\gamma_1'(0) - \psi_2'(0)\alpha_2(0)}{\Delta} A_{100}^{(1)}A_{100}^{(2)} \\
& + \frac{\phi_3(0)\gamma_1'(0) - \psi_3'(0)\alpha_2(0)}{\Delta} A_{100}^{(2)^2}.
\end{aligned}
$$

Substituting these values in (151) we have

$$
(152)\quad
\begin{aligned}
\rho_{200} &= A_{100}^{(1)^2}x_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}x_2(\tau) + A_{100}^{(2)^2}x_3(\tau), \\
\sigma_{200} &= A_{100}^{(1)^2}y_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}y_2(\tau) + A_{100}^{(2)^2}y_3(\tau),
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(\tau) = \phi_1(\tau) + {} & \frac{\psi_1'(0)\alpha_4(0) - \phi_1(0)[\gamma_4'(0) + \gamma_3'(0)]}{\Delta}\alpha_2(\tau) \\
& + \frac{\phi_1(0)\gamma_1'(0) - \psi_1'(0)\alpha_2(0)}{\Delta}[\alpha_4(\tau) + \tau\alpha_3(\tau)],
\end{aligned}
$$

$$
\begin{aligned}
y_1(\tau) = \psi_1(\tau) + {} & \frac{\psi_1'(0)\alpha_4(0) - \phi_1(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta}\gamma_1(\tau) \\
& + \frac{\phi_1(0)\gamma_1'(0) - \psi_1'(0)\alpha_2(0)}{\Delta}[\gamma_3(\tau) + \tau\gamma_4(\tau)],
\end{aligned}
$$

and similar expressions for x_2, y_2 and x_3, y_3, which we shall find later [equation (172)] do not interest us. The characters of x_1 and y_1 are known with the exception of ϕ_1 and ψ_1 which we will now investigate. The functions ϕ_1 and ψ_1

are those portions of the solution of the differential equations (150) which depend upon the coefficients of $A_{100}^{(1)2}$. These coefficients are homogeneous of the second degree in $\alpha_2(\tau)$ and $\gamma_1(\tau)$. In R_{200} and S_{200} the expressions

θ_{200}, $\bar{\theta}_{110}$ and θ_{020} contain only cosines of even multiples of τ;

$\bar{\theta}_{200}$, θ_{110} and $\bar{\theta}_{020}$ contain only sines of odd multiples of τ;

$\alpha_2(\tau)$ has the form $\alpha_2 = \sum_k a_k \cos[(2k+1) \pm \lambda]\tau$;

$\gamma_1(\tau)$ has the form $\gamma_1 = \sum_k b_k \sin[2k \pm \lambda]\tau$.

Consequently, so far as the coefficients of $A_{100}^{(1)2}$ are concerned, R_{200} and S_{200} have the form

$$R_{200} = \{\sum_k a_k^{(1)} \cos 2kt + \sum_k a_k^{(2)} \cos[2k \pm 2\lambda]\tau\} A_{100}^{(1)2} + \cdots,$$

$$S_{200} = \{\sum_k b_k^{(1)} \sin(2k+1)\tau + \sum_k b_k^{(2)} \sin[(2k+1) \pm 2\lambda]\tau\} A_{100}^{(1)2} + \cdots.$$

By Theorem II terms involving 2λ give rise only to periodic terms in the solution whose period is $2K\pi$. By Theorem V those parts of ρ_{200} and σ_{200} depending upon the terms in R_{200} and S_{200} which are independent of λ have the form

$$\rho = p_1(\tau) + c\tau\alpha_3(\tau), \qquad \sigma = p_2(\tau) + c\tau\gamma_4(\tau)$$

respectively, where p_1 and p_2 are periodic with the period 2π. Consequently the characters of $x_1(\tau)$ and $y_1(\tau)$ are

$$(153) \qquad x_1(\tau) = \wp_1(\tau) + c_1\tau\alpha_3(\tau), \qquad y_1(\tau) = \wp_2(\tau) + c_1\tau\gamma_4(\tau),$$

where $\wp_1(\tau)$ and $\wp_2(\tau)$ are periodic with the period $2K\pi$.

Coefficient of $\alpha\delta$.

$$(154) \quad \rho''_{110} + \theta_2\rho_{110} + \theta_3\sigma_{110} = R_{110}, \qquad \sigma''_{110} + \theta_4\sigma_{110} + \theta_3\rho_{110} = S_{110},$$

where

$$\begin{aligned} R_{110} = {} & 2A_{100}^{(1)}A_{010}^{(1)}[\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] \\ & + [A_{100}^{(1)}A_{010}^{(2)} + A_{100}^{(2)}A_{010}^{(1)}][2\theta_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \theta_{110}\{\gamma_1(\tau\alpha_3 + \alpha_4) + \alpha_2(\tau\gamma_4 + \gamma_3)\} \\ & \qquad + 2\theta_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + 2A_{100}^{(2)}A_{010}^{(2)}[\theta_{200}(\tau\alpha_3 + \alpha_4)^2 + \theta_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \theta_{020}(\tau\gamma_4 + \gamma_3)^2], \end{aligned}$$

$$\begin{aligned} S_{110} = {} & 2A_{100}^{(1)}A_{010}^{(1)}[\bar{\theta}_{200}\alpha_2^2 + \bar{\theta}_{110}\alpha_2\gamma_1 + \bar{\theta}_{020}\gamma_1^2] \\ & + [A_{100}^{(1)}A_{010}^{(2)} + A_{100}^{(2)}A_{010}^{(1)}][2\bar{\theta}_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \bar{\theta}_{110}\{\gamma_1(\tau\alpha_3 + \alpha_4) + \alpha_2(\tau\gamma_4 + \gamma_3)\} \\ & \qquad + 2\bar{\theta}_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + 2A_{100}^{(2)}A_{010}^{(2)}[\bar{\theta}_{200}(\tau\alpha_3 + \alpha_4)^2 + \bar{\theta}_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \bar{\theta}_{020}(\tau\gamma_4 + \gamma_3)^2]. \end{aligned}$$

The functions R_{110} and S_{110} differ from R_{200} and S_{200} only in the constants A_{ijk}. The initial conditions impose the same conditional equations. Consequently the solutions differ only in the constants A_{ijk}, so that we can express them at once without computation:

$$
(155)\quad
\begin{aligned}
\rho_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}x_1(\tau) + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]x_2(\tau) + 2A^{(2)}_{100}A^{(2)}_{010}x_3(\tau),\\
\sigma_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}y_1(\tau) + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]y_2(\tau) + 2A^{(2)}_{100}A^{(2)}_{010}y_3(\tau),
\end{aligned}
$$

where the $x_i(\tau)$ and $y_i(\tau)$ are the same functions of τ as before.

Coefficient of δ^2.

By symmetry with the coefficient of α^2 it is seen that

$$
(156)\quad
\begin{aligned}
\rho_{020} &= A^{(1)^2}_{010}x_1(\tau) + A^{(1)}_{010}A^{(2)}_{010}x_2(\tau) + A^{(2)^2}_{010}x_3(\tau),\\
\sigma_{020} &= A^{(1)^2}_{010}y_1(\tau) + A^{(1)}_{010}A^{(2)}_{010}y_2(\tau) + A^{(2)^2}_{010}y_3(\tau).
\end{aligned}
$$

Coefficient of ϵ^2.

Since the coefficients of the first powers of α and δ were homogeneous in the A_{ijk} of the first degree, the coefficients of α^2, $\alpha\delta$ and δ^2 are homogeneous in the A_{ijk} of the second degree. The coefficient of the first power of ϵ is not homogeneous in the A_{ijk}, hence the second power is not homogeneous. But if the functions $\alpha_6(\tau)$ and $\gamma_6(\tau)$ were zero the coefficient of the first power of ϵ would be homogeneous. By symmetry therefore, we can at once write down the terms involving the A_{ijk} to the second degree. To these must be added terms in the first degree and one term independent of the A_{ijk}.

The differential equations are

$$
(157)\quad
\begin{aligned}
&\rho''_{002} + \theta_2\rho_{002} + \theta_3\sigma_{002} = R_{002},\\
&\sigma''_{002} + \theta_4\sigma_{002} + \theta_3\rho_{002} = S_{002};\\
&R_{002} = \theta_{200}\rho^2_{001} + \theta_{110}\rho_{001}\sigma_{001} + \theta_{020}\sigma^2_{001} + \theta_{101}\rho_{001},\\
&S_{002} = \bar\theta_{200}\rho^2_{001} + \bar\theta_{110}\rho_{001}\sigma_{001} + \bar\theta_{020}\sigma^2_{001}.
\end{aligned}
$$

The terms involved in these expressions are shown in the following table:

R_{002}

Term	$A^{(1)^2}_{001}$	$A^{(1)}_{001}A^{(2)}_{001}$	$A^{(2)^2}_{001}$	$A^{(1)}_{001}$	$A^{(2)}_{001}$	1	Multiplied by
ρ^2_{001}	a^2_2	$2\alpha_2(\tau a_3 + a_4)$	$(\tau a_3 + a_4)^2$	$2a_2a_6$	$2a_6(\tau a_3 + a_4)$	a^2_6	θ_{200}
$\rho_{001}\sigma_{001}$	$a_2\gamma_1$	$\gamma_1(\tau a_3 + a_4) + a_2(\tau\gamma_4 + \gamma_3)$	$(\tau a_3 + a_4)(\tau\gamma_4 + \gamma_3)$	$\gamma_1 a_6 + a_2\gamma_6$	$a_6(\tau\gamma_4 + \gamma_3) + \gamma_6(\tau a_3 + a_4)$	$a_6\gamma_6$	θ_{110}
σ^2_{001}	γ^2_1	$2\gamma_1(\tau\gamma_4 + \gamma_3)$	$(\tau\gamma_4 + \gamma_3)^2$	$2\gamma_1\gamma_6$	$2(\tau\gamma_4 + \gamma_3)$	γ^2_6	θ_{020}
ρ_{001}				a_2	$(\tau a_3 + a_4)$	a_6	θ_{101}

In order to obtain the S_{002} it is necessary in the above table only to change the θ_{ijk} in the last column into $\bar{\theta}_{ijk}$.

The solutions of equations (157) may be expressed in the form

$$(158)\quad \begin{aligned} \rho_{002} &= A_{001}^{(1)^2} x_1(\tau) + A_{001}^{(1)} A_{001}^{(2)} x_2(\tau) + A_{001}^{(2)^2} x_3(\tau) + A_{001}^{(1)} x_4(\tau) + A_{001}^{(2)} x_5(\tau) + x_6(\tau), \\ \sigma_{002} &= A_{001}^{(1)^2} y_1(\tau) + A_{001}^{(1)} A_{001}^{(2)} y_2(\tau) + A_{001}^{(2)^2} y_3(\tau) + A_{001}^{(1)} y_4(\tau) + A_{001}^{(2)} y_5(\tau) + y_6(\tau). \end{aligned}$$

The coefficients of $A_{001}^{(1)}$ in the differential equations are homogeneous of the first degree in α_2 and γ_1, every term of which involves the first multiple of $\lambda\tau$. Hence the solutions for these terms by Theorem II involve non-periodic terms, and we can write

$$(159)\quad \begin{aligned} x_4(\tau) &= \wp_3(\tau) + c_2\tau\alpha_1(\tau) + c_3\tau\alpha_3(\tau), \\ y_4(\tau) &= \wp_4(\tau) + c_2\tau\gamma_2(\tau) + c_3\tau\gamma_4(\tau), \end{aligned}$$

where $\wp_3(\tau)$ and $\wp_4(\tau)$ are periodic with the period $2K\pi$. As is seen from the table, $x_6(\tau)$ and $y_6(\tau)$ do not involve the λ. They have therefore the form

$$(160)\quad \begin{aligned} x_6(\tau) &= \wp_5(\tau) + c_4\tau\alpha_3(\tau), \\ y_6(\tau) &= \wp_6(\tau) + c_4\tau\gamma_4(\tau). \end{aligned}$$

It will be verified in (172) that we do not need to know the character of x_5 and y_5.

Coefficient of $\alpha\epsilon$.

The differential equations are

$$(161)\quad \begin{aligned} \rho''_{101} + \theta_2\rho_{101} + \theta_3\sigma_{101} &= R_{101}, \\ \sigma''_{101} + \theta_4\sigma_{101} + \theta_3\rho_{101} &= S_{101}. \end{aligned}$$

$$R_{101} = \theta_{200}[2\rho_{100}\rho_{001}] + \theta_{110}[\rho_{100}\sigma_{001} + \rho_{001}\sigma_{100}] + \theta_{020}[2\sigma_{100}\sigma_{001}] + \theta_{101}\rho_{100},$$

$$S_{101} = \bar{\theta}_{200}[2\rho_{100}\rho_{001}] + \bar{\theta}_{110}[\rho_{100}\sigma_{001} + \rho_{001}\sigma_{100}] + \bar{\theta}_{020}[2\sigma_{100}\sigma_{001}].$$

The following table shows the character of the terms entering into these expressions:

R_{101}

	$2A_{100}^{(1)}A_{001}^{(1)}$	$A_{100}^{(1)}A_{001}^{(2)} + A_{100}^{(2)}A_{001}^{(1)}$	$2A_{100}^{(2)}A_{001}^{(2)}$	$A_{100}^{(1)}$	$A_{100}^{(2)}$	Multiplied by
$\rho_{100}\rho_{001}$	α_2^2	$2\alpha_2(\tau\alpha_3 + \alpha_4)$	$(\tau\alpha_3 + \alpha_4)^2$	$2\alpha_2\alpha_6$	$2\alpha_6(\tau\alpha_3 + \alpha_4)$	θ_{200}
$\rho_{100}\sigma_{001} + \rho_{001}\sigma_{100}$	$\alpha_2\gamma_1$	$\gamma_1(\tau\alpha_3 + \alpha_4) + \alpha_2(\tau\gamma_4 + \gamma_3)$	$(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3)$	$\alpha_6\gamma_1 + \alpha_2\gamma_6$	$\alpha_6(\tau\gamma_4 + \gamma_3) + \gamma_6(\tau\alpha_3 + \alpha_4)$	θ_{110}
$\sigma_{100}\sigma_{001}$	γ_1^2	$2\gamma_1(\tau\gamma_4 + \gamma_3)$	$(\tau\gamma_4 + \gamma_3)^2$	$2\gamma_1\gamma_6$	$2\gamma_6(\tau\gamma_4 + \gamma_3)$	θ_{020}
ρ_{100}				α_2	$(\tau\alpha_3 + \alpha_4)$	θ_{101}

In order to obtain S_{101} it is necessary only to change the θ_{ijk} in the last column into $\bar{\theta}_{ijk}$. This table shows that R_{101} and S_{101} differ from R_{002} and S_{002} only in

the constants A_{ijk}. Since the initial conditions impose the same conditional equations as for the coefficient of ϵ^2 we can at once write down the solution

$$
(162)\quad
\begin{aligned}
\rho_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}x_1(\tau) + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]x_2(\tau) \\
&\qquad + 2A^{(2)}_{100}A^{(2)}_{001}x_3(\tau) + A^{(1)}_{100}x_4(\tau) + A^{(2)}_{100}x_5(\tau), \\
\sigma_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}y_1(\tau) + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]y_2(\tau) \\
&\qquad + 2A^{(2)}_{100}A^{(2)}_{001}y_3(\tau) + A^{(1)}_{100}y_4(\tau) + A^{(2)}_{100}y_5(\tau).
\end{aligned}
$$

Coefficient of $\delta\epsilon$.

This coefficient can be obtained by symmetry from the coefficient of $\alpha\epsilon$ by permutation of the first and second subscripts of the A_{ijk}. Therefore

$$
(163)\quad
\begin{aligned}
\rho_{011} &= 2A^{(1)}_{010}A^{(1)}_{001}x_1(\tau) + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]x_2(\tau) \\
&\qquad + 2A^{(2)}_{010}A^{(2)}_{001}x_3(\tau) + A^{(1)}_{010}x_4(\tau) + A^{(2)}_{010}x_5(\tau), \\
\sigma_{101} &= 2A^{(1)}_{010}A^{(1)}_{001}y_1(\tau) + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]y_2(\tau) \\
&\qquad + 2A^{(2)}_{010}A^{(2)}_{001}y_3(\tau) + A^{(1)}_{010}y_4(\tau) + A^{(2)}_{010}y_5(\tau).
\end{aligned}
$$

This concludes the computation of all terms up to the second order inclusive in α, δ and ϵ. In order to establish the existence of the periodic solutions it is not necessary to carry the computation further.

§ 16. *Existence of periodic orbits reëntrant only after many revolutions.*

We have chosen the initial conditions so that at $\tau = 0$ the particle is crossing the ρ-axis orthogonally. It is obvious geometrically that if at any future time it again crosses the ρ-axis perpendicularly the orbit will be a closed one and the motion in it will be periodic. The conditions that the particle shall cross the ρ-axis perpendicularly at $\tau = T$ is that at this epoch

$$\rho' = \sigma = 0.$$

The equations of variation have the period $2K\pi$. Therefore we shall choose $T = K\pi$. Since ρ is an even series in τ, and σ is an odd series, all the purely periodic terms in ρ' and σ are sines and consequently vanish at $\tau = K\pi$. The terms which do not vanish must carry τ as a factor. The conditions for periodicity give us two equations, namely, at $\tau = K\pi$,

$$
(164)\quad
\begin{aligned}
\rho' = 0 &= a_{100}\alpha + a_{010}\delta + a_{001}\epsilon + a_{200}\alpha^2 + a_{110}\alpha\delta \\
&\qquad + a_{020}\delta^2 + a_{101}\alpha\epsilon + a_{011}\delta\epsilon + a_{002}\epsilon^2 + \cdots, \\
\sigma = 0 &= b_{100}\alpha + b_{010}\delta + b_{001}\epsilon + b_{200}\alpha^2 + b_{110}\alpha\delta \\
&\qquad + b_{020}\delta^2 + b_{101}\alpha\epsilon + b_{011}\delta\epsilon + b_{002}\epsilon^2 + \cdots,
\end{aligned}
$$

where a_{ijk} and b_{ijk} are the values derived from the series just computed. Their values are as follows:

$$
(165)\qquad \begin{aligned}
a_{100} &= A^{(2)}_{100}u, & a_{010} &= A^{(2)}_{010}u, & a_{001} &= A^{(2)}_{001}u,\\
b_{100} &= A^{(2)}_{100}v, & b_{010} &= A^{(2)}_{010}v, & b_{001} &= A^{(2)}_{001}v,
\end{aligned}
$$

$$
\begin{aligned}
a_{200} &= A^{(1)2}_{100}\bar{x}_1 + A^{(1)}_{100}A^{(2)}_{100}\bar{x}_2 + A^{(2)2}_{100}\bar{x}_3,\\
b_{200} &= A^{(1)2}_{100}\bar{y}_1 + A^{(1)}_{100}A^{(2)}_{100}\bar{y}_2 + A^{(2)2}_{100}\bar{y}_3,\\
a_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}\bar{x}_1 + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]\bar{x}_2 + 2A^{(2)}_{100}A^{(2)}_{010}\bar{x}_3,\\
b_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}\bar{y}_1 + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]\bar{y}_2 + 2A^{(2)}_{100}A^{(2)}_{010}\bar{y}_3,\\
a_{020} &= A^{(1)2}_{010}\bar{x}_1 + A^{(1)}_{010}A^{(2)}_{010}\bar{x}_2 + A^{(2)2}_{010}\bar{x}_3,\\
b_{020} &= A^{(1)2}_{010}\bar{y}_1 + A^{(1)}_{010}A^{(2)}_{010}\bar{y}_2 + A^{(\ )2}_{010}\bar{y}_3,\\
a_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}\bar{x}_1 + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]\bar{x}_2 + 2A^{(2)}_{100}A^{(2)}_{001}\bar{x}_3 + A^{(1)}_{100}\bar{x}_4 + A^{(2)}_{100}\bar{x}_5,\\
b_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}\bar{y}_1 + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]\bar{y}_2 + 2A^{(2)}_{100}A^{(2)}_{001}\bar{y}_3 + A^{(1)}_{100}\bar{y}_4 + A^{(2)}_{100}\bar{y}_5,\\
a_{011} &= 2A^{(1)}_{010}A^{(1)}_{001}\bar{x}_1 + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]\bar{x}_2 + 2A^{(2)}_{010}A^{(2)}_{001}\bar{x}_3 + A^{(1)}_{010}\bar{x}_4 + A^{(2)}_{010}\bar{x}_5,\\
b_{011} &= 2A^{(1)}_{010}A^{(1)}_{001}\bar{y}_1 + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]\bar{y}_2 + 2A^{(2)}_{010}A^{(2)}_{001}\bar{y}_3 + A^{(1)}_{010}\bar{y}_4 + A^{(2)}_{010}\bar{y}_5,\\
a_{002} &= A^{(1)2}_{001}\bar{x}_1 + A^{(1)}_{001}A^{(2)}_{001}\bar{x}_2 + A^{(2)2}_{001}\bar{x}_3 + A^{(1)}_{001}\bar{x}_4 + A^{(2)}_{001}\bar{x}_5 + \bar{x}_6,\\
b_{002} &= A^{(1)2}_{001}\bar{y}_1 + A^{(1)}_{001}A^{(2)}_{001}\bar{y}_2 + A^{(2)2}_{001}\bar{y}_3 + A^{(1)}_{001}\bar{y}_4 + A^{(2)}_{001}\bar{y}_5 + \bar{y}_6,
\end{aligned}
$$

where

$$
(166)\qquad \begin{aligned}
u &= K\pi\frac{d\alpha_3}{d\tau}, & v &= k\pi\gamma_4,\\
\bar{x}_i &= \frac{dx_i}{d\tau}, & \bar{y}_i &= y_i, & &\text{at } \tau = K\pi.
\end{aligned}
$$

Let us solve now the equation (164), $\rho' = 0$, for ϵ as a power series in α and δ. We obtain

$$
(167)\qquad \epsilon = \epsilon_{10}\alpha + \epsilon_{01}\delta + \epsilon_{20}\alpha^2 + \epsilon_{11}\alpha\delta + \epsilon_{02}\delta^2 + \cdots,
$$

where the coefficients ϵ_{ij} have the following values:

$$
\epsilon_{10} = -\frac{a_{100}}{a_{001}},
$$

$$
\epsilon_{01} = -\frac{a_{010}}{a_{001}},
$$

$$
\epsilon_{20} = -\frac{a_{200}a^2_{001} - a_{100}a_{101}a_{001} + a_{002}a^2_{100}}{a^3_{001}},
$$

$$\epsilon_{11} = -\frac{a_{110}a_{001}^2 - a_{011}a_{100}a_{001} - a_{101}a_{010}a_{001} + 2a_{002}a_{100}a_{010}}{a_{001}^3},$$

$$\epsilon_{02} = -\frac{a_{020}a_{001}^2 - a_{011}a_{010}a_{001} + a_{002}a_{010}^2}{a_{001}^3}.$$

Solving now equation (164), $\sigma = 0$, for ϵ in terms of α and δ, we obtain

$$\epsilon = \bar{\epsilon}_{10}\alpha + \bar{\epsilon}_{01}\delta + \bar{\epsilon}_{20}\alpha^2 + \bar{\epsilon}_{11}\alpha\delta + \bar{\epsilon}_{02}\delta^2 + \cdots, \tag{168}$$

where the $\bar{\epsilon}_{ij}$ have the same expressions in the b_{klm} as the ϵ_{ij} have in the a_{klm}. Subtracting (167) from (168), we have

$$0 = [\bar{\epsilon}_{10} - \epsilon_{10}]\alpha + [\bar{\epsilon}_{01} - \epsilon_{01}]\delta + [\bar{\epsilon}_{20} - \epsilon_{20}]\alpha^2 + [\bar{\epsilon}_{11} - \epsilon_{11}]\alpha\delta + [\bar{\epsilon}_{02} - \epsilon_{02}]\delta^2 + \cdots. \tag{169}$$

We must now examine the coefficients of this series.

$$\begin{aligned} [\bar{\epsilon}_{10} - \epsilon_{10}] &= \frac{a_{100}}{a_{001}} - \frac{b_{100}}{b_{001}} = \frac{A_{100}^{(2)}u}{A_{001}^{(2)}u} - \frac{A_{100}^{(2)}v}{A_{001}^{(2)}v} = 0, \\ [\bar{\epsilon}_{01} - \epsilon_{10}] &= \frac{a_{010}}{a_{001}} - \frac{b_{010}}{b_{001}} = \frac{A_{010}^{(2)}u}{A_{001}^{(2)}u} - \frac{A_{010}^{(2)}v}{A_{001}^{(2)}v} = 0. \end{aligned} \tag{170}$$

Both of the linear terms therefore vanish. The computation of the second degree terms is somewhat more complicated. It simplifies matters somewhat if we observe that the $\bar{\epsilon}_{ij}$ have the same expressions in v and the $\bar{y}_i$ as the ϵ_{ij} have in u and $\bar{x}_i$. It is sufficient therefore to compute one and derive the other from it. From (165) we have

$$-\epsilon_{20} = \frac{a_{200}}{a_{001}} - \frac{a_{100}a_{101}}{a_{001}^2} + \frac{a_{002}a_{100}^2}{a_{001}^3}.$$

Substituting in these fractions the values of the a_{ijk} from (165) we get

$$\begin{aligned} \frac{a_{200}}{a_{001}} &= \frac{1}{A_{001}^{(2)^3}}\left[A_{100}^{(1)^2}A_{001}^{(2)^2}\frac{\bar{x}_1}{u} + A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(2)^2}\frac{\bar{x}_2}{u} + A_{100}^{(2)^2}A_{001}^{(2)^2}\frac{\bar{x}_3}{u}\right], \\ -\frac{a_{100}a_{101}}{a_{001}^2} &= \frac{1}{A_{001}^{(2)^3}}\left[-2A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(1)}A_{001}^{(2)}\frac{\bar{x}_1}{u} + [-A_{100}^{(2)^2}A_{001}^{(1)}A_{001}^{(2)} - A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(2)^2}]\frac{\bar{x}_2}{u}\right. \\ &\qquad \left. - 2A_{100}^{(2)^2}A_{001}^{(2)^2}\frac{\bar{x}_3}{u} - A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(2)}\frac{\bar{x}_4}{u} - A_{100}^{(2)^2}A_{001}^{(2)}\frac{\bar{x}_5}{u}\right], \\ \frac{a_{002}a_{100}^2}{a_{001}^3} &= \frac{1}{A_{001}^{(2)^3}}\left[A_{100}^{(2)^2}A_{001}^{(1)^2}\frac{\bar{x}_1}{u} + A_{100}^{(2)^2}A_{001}^{(1)}A_{001}^{(2)}\frac{\bar{x}_2}{u} + A_{100}^{(2)^2}A_{001}^{(2)^2}\frac{\bar{x}_3}{u}\right. \\ &\qquad \left. + A_{100}^{(2)^2}A_{001}^{(1)}\frac{\bar{x}_4}{u} + A_{100}^{(2)^2}A_{001}^{(2)}\frac{\bar{x}_5}{u}A_{100}^{(2)^2}\frac{\bar{x}_6}{u}\right]. \end{aligned} \tag{171}$$

For the sum of these three expressions, there results

$$(172)\qquad -\epsilon_{20}=\frac{1}{A_{001}^{(2)3}}\left[\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]^2\frac{\bar{x}_1}{u}\right.$$
$$\left.-\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]A_{100}^{(2)}\frac{\bar{x}_4}{u}+A_{100}^{(2)2}\frac{\bar{x}_6}{u}\right],$$

the coefficients of $\bar{x}_2/u$, $\bar{x}_3/u$ and $\bar{x}_5/u$ vanishing identically. This is the reason why it was not necessary to compute x_2, x_3, and x_5 in (152) and (158).

Changing the $\bar{x}_i$ into $\bar{y}_i$ and u into v gives us $-\bar{\epsilon}_{20}$. Hence

$$(173)\qquad [\bar{\epsilon}_{20}-\epsilon_{20}]=\frac{1}{A_{001}^{(2)3}}\left\{\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]^2\left[\frac{\bar{x}_1}{u}-\frac{\bar{y}_1}{v}\right]\right.$$
$$\left.-A_{100}^{(2)}\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right]+A_{100}^{(2)2}\left[\frac{\bar{x}_6}{u}-\frac{\bar{y}_6}{v}\right]\right\}.$$

But $[\bar{x}_1/u-\bar{y}_1/v]$ and $[\bar{x}_6/u-\bar{y}_6/v]$ both vanish since

$$\bar{x}_1=c_1u,\qquad \bar{y}_1=c_1v,\qquad \bar{x}_6=c_4u,\qquad \bar{y}_6=c_4v,$$

as is readily seen from (153) and (160). It is also seen on referring to (159) that $[\bar{x}_4/u-\bar{y}_4/v]$ does not vanish, but is equal to

$$\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}=c_2\left[\frac{1}{u}\frac{d\alpha_1}{d\tau}-\frac{\gamma_2}{v}\right]_{\tau=K\pi}.$$

Hence

$$(174)\qquad [\bar{\epsilon}_{20}-\epsilon_{20}]=-\frac{A_{100}^{(2)}\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]}{A_{001}^{(2)3}}\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right].$$

Without repeating the details of the computation we find in a similar way

$$(175)\quad [\bar{\epsilon}_{11}-\epsilon_{11}]=-\frac{A_{010}^{(2)}\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]+A_{100}^{(2)}\left[A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}\right]}{A_{001}^{(2)3}}\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right],$$

$$(176)\quad [\bar{\epsilon}_{02}-\epsilon_{02}]=-\frac{A_{010}^{(2)}\left[A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}\right]}{A_{001}^{(2)3}}\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right].$$

Substituting these values in (169), we find that the second degree terms in α and δ are factorable. Thus (169) becomes

$$(177)\qquad 0=\frac{1}{A_{001}^{(2)3}}\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right]\left[A_{100}^{(2)}\alpha+A_{010}^{(2)}\delta+\cdots\right]\left[\left(A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right)\alpha\right.$$
$$\left.+\left(A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}\right)\delta+\cdots\right].$$

There are therefore two real solutions for δ as power series in α. Substituting these values of δ in (168) we find the two corresponding values of ϵ. In this manner we find:

First solution.

$$(178)\qquad \begin{aligned} \delta &= -\frac{A^{(1)}_{100}A^{(2)}_{001}-A^{(2)}_{010}A^{(1)}_{001}}{A^{(1)}_{010}A^{(2)}_{001}-A^{(2)}_{010}A^{(1)}_{001}}\alpha+\cdots=\frac{\gamma'_5-a\gamma'_3}{\alpha_5-a\alpha_4}\alpha+\cdots=\frac{\gamma'_6(0)}{\alpha_6(0)}\alpha+\cdots,\\ \epsilon &= \frac{A^{(1)}_{100}A^{(2)}_{010}-A^{(2)}_{100}A^{(1)}_{010}}{A^{(1)}_{010}A^{(2)}_{001}-A^{(2)}_{010}A^{(1)}_{001}}\alpha+\cdots=\frac{1}{\alpha_5-a\alpha_4}\alpha+\cdots=\frac{1}{\alpha_6(0)}\alpha+\cdots. \end{aligned}$$

Second solution.

$$(179)\qquad \begin{aligned} \delta &= -\frac{A^{(2)}_{100}}{A^{(2)}_{010}}\alpha+\cdots=\frac{\gamma'_1(0)}{\alpha_2(0)}\alpha+\cdots=\gamma'_2(0)\cdot\alpha+\cdots,\\ \epsilon &= -\frac{A^{(2)}_{100}}{A^{(2)}_{001}}\alpha-\frac{A^{(2)}_{010}}{A^{(2)}_{001}}\delta+\cdots=0\cdot\alpha+\cdots, \end{aligned}$$

where α_6 and γ_6 are the quantities defined in (147), and $\gamma'_6(0)$ is the value of $d\gamma_6/d\tau$ for $\tau=0$. Thus one solution for ϵ begins with the first power of α while the other certainly does not begin before the second, but in both solutions δ begins with the first power of α.

§ 17. *Construction of the solutions having the period $2K\pi$.*

We have just proved the existence of series for ρ, σ and ϵ proceeding in powers of the initial value of ρ (let us call this value e).* The series for ρ and σ are periodic in τ with the period $2K\pi$, and since this condition holds for all values of e sufficiently small each coefficient is separately periodic. The series for ρ is even in τ, and the series for σ is odd in τ. These series have the form

$$(180)\qquad \begin{aligned} \rho &= \rho_1 e+\rho_2 e^2+\rho_3 e^3+\cdots,\\ \sigma &= \sigma_1 e+\sigma_2 e^2+\sigma_3 e^3+\cdots,\\ \epsilon &= \epsilon_1 e+\epsilon_2 e^2+\epsilon_3 e^3+\cdots. \end{aligned}$$

We may substitute these series in the differential equations and integrate the coefficients of each power of e step by step, and determine the constants in such a way that ρ and σ shall be periodic, and satisfy the initial conditions

$$(181)\qquad \rho(0)=e,\qquad \sigma(0)=0,\qquad \rho'(0)=0,\qquad \sigma'(0)=\delta,$$

where δ is a constant at present unknown but to be determined in the process.

Substituting the series (180) in the differential equations (138), we find for the coefficient of the first power of e

$$(182)\qquad \rho_1''+\theta_2\rho_1+\theta_3\sigma_1=\theta_{001}\epsilon_1,\qquad \sigma_1''+\theta_4\sigma_1+\theta_3\rho_1=0.$$

* The reason for changing α to e is that this parameter corresponds to the eccentricity in the two-body problem.

Since by the condition of orthogonality ρ must be even in τ and σ odd in τ, the solution complying with the condition is

$$(183)\qquad \begin{aligned} \rho_1 &= A^{(1)}\alpha_2(\tau) + D^{(1)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + \epsilon_1[a\tau\alpha_3(\tau) + \alpha_5(\tau)], \\ \sigma_1 &= A^{(1)}\gamma_1(\tau) + D^{(1)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + \epsilon_1[a\tau\gamma_4(\tau) + \gamma_5(\tau)], \end{aligned}$$

where $\alpha_5(\tau)$ contains only cosines of even multiples of τ and $\gamma_5(\tau)$ contains only sines of odd multiples of τ.

In order that this solution shall be periodic it is necessary and sufficient that

$$D^{(1)} = -a\epsilon_1.$$

Upon imposing this condition the solution (183) becomes [see eq. (146)]

$$(184)\qquad \begin{aligned} \rho_1 &= A^{(1)}\alpha_2(\tau) + \epsilon_1[\alpha_5(\tau) - a\alpha_4(\tau)] = A^{(1)}\alpha_2(\tau) + \epsilon_1\alpha_6(\tau), \\ \sigma_1 &= A^{(1)}\gamma_1(\tau) + \epsilon_1[\gamma_5(\tau) - a\gamma_3(\tau)] = A^{(1)}\gamma_1(\tau) + \epsilon_1\gamma_6(\tau). \end{aligned}$$

It remains to impose the initial condition that $\rho_1 = 1$ at $\tau = 0$. From this condition we get

$$(185)\qquad 1 = A^{(1)}\bar{\alpha}_2 + \epsilon_1\bar{\alpha}_6,$$

where $\bar{\alpha}_2$ and $\bar{\alpha}_6$ denote the values of these functions when $\tau = 0$.

Coefficient of e^2.

$$(186)\qquad \begin{aligned} \rho_2'' + \theta_2\rho_2 + \theta_3\sigma_2 &= \theta_{001}\epsilon_2 + \theta_{101}\epsilon_1\rho_1 + \theta_{200}\rho_1^2 + \theta_{110}\rho_1\sigma_1 + \theta_{020}\sigma_1^2 = R_2, \\ \sigma_2'' + \theta_4\sigma_2 + \theta_3\rho_2 &= \qquad\qquad \bar{\theta}_{200}\rho_1^2 + \bar{\theta}_{110}\rho_1\sigma_1 + \bar{\theta}_{020}\sigma_1^2 = S_2. \end{aligned}$$

Every term of R_2 and S_2 contains either $A^{(1)}$, ϵ_1 or ϵ_2 as a factor. Arranged in this manner we have the expansions

$$\begin{aligned} R_2 = {} & A^{(1)^2}[\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] \\ & + A^{(1)}\epsilon_2[2\theta_{200}\alpha_2\alpha_6 + \theta_{110}(\gamma_1\alpha_6 + \alpha_2\gamma_6) + 2\theta_{020}\gamma_1\gamma_6 + \theta_{101}\alpha_2] \\ & + \epsilon_1^2[\theta_{200}\alpha_6^2 + \theta_{110}\alpha_6\gamma_6 + \theta_{020}\gamma_6^2 + \theta_{101}\alpha_6] + \epsilon_2[\theta_{001}], \\ S_2 = {} & A^{(1)^2}[\bar{\theta}_{200}\alpha_2^2 + \bar{\theta}_{110}\alpha_2\gamma_1 + \bar{\theta}_{020}\gamma_1^2] + A^{(1)}\epsilon_1[2\bar{\theta}_{200}\alpha_2\alpha_6 + \bar{\theta}_{110}(\gamma_1\alpha_6 + \alpha_2\gamma_6) + 2\bar{\theta}_{020}\gamma_1\gamma_6] \\ & + \epsilon_1^2[\bar{\theta}_{200}\alpha_6^2 + \bar{\theta}_{110}\alpha_6\gamma_6 + \bar{\theta}_{020}\gamma_6^2]. \end{aligned}$$

In order to understand the character of the solution of these equations we must examine the character of the various terms. The coefficient of $A^{(1)^2}$ in both R_2 and S_2 is homogeneous of the second degree in α_2 and γ_1. Its expansion therefore involves terms carrying $2\lambda\tau$ and terms independent of λ. By Theorem II the solution for the terms in $2\lambda\tau$ is periodic. The terms independent of λ are cosines of even multiples of τ in R_2 and sines of odd multiples of τ in

S_2. These terms have the same character as those in the coefficients of ϵ_1^2 and ϵ_2 and may be considered under the discussion of those terms.

The coefficients of $A^{(1)}\epsilon_1$ in both R_2 and S_2 are homogeneous of the first degree in α_2 and γ_1, all terms of which carry the first multiple of $\lambda\tau$. By Theorem VII the solution for ρ_2 will carry the term $\tau\alpha_1(\tau)$, and for σ_2 the term $\tau\gamma_2(\tau)$. Non-periodic terms of this character do not arise elsewhere in the solution, hence, in order to avoid them, we must take either $A^{(1)} = 0$ or $\epsilon_1 = 0$. If we choose $A^{(1)} = 0$, then, by (185), ϵ_1 is determined so that we must have $\epsilon_1 = 1/\bar{\alpha}_6$, thus agreeing with the first solution (178) of the existence proof. But if we choose $\epsilon_1 = 0$ so that by (185) $A^{(1)} = 1/\bar{\alpha}_2$ we are in agreement with the second solution (179) of the existence proof. We will commence by developing the first solution. This necessitates the choice $A^{(1)} = 0$, $\epsilon_1 = 1/\bar{\alpha}_6$.

First solution.

Since $A^{(1)} = 0$ all terms in R_2 and S_2 which carry $\lambda\tau$ or any multiple of it vanish. There remains

$$(187)\qquad \begin{aligned} R_2 &= \epsilon_1^2\left[\theta_{200}\alpha_6^2 + \theta_{110}\alpha_6\gamma_6 + \theta_{020}\gamma_6^2 + \theta_{101}\alpha_6\right] + \epsilon_2\theta_{001}, \\ S_2 &= \epsilon_1^2\left[\bar{\theta}_{200}\alpha_6^2 + \bar{\theta}_{110}\alpha_6\gamma_6 + \bar{\theta}_{020}\gamma_6^2\right]. \end{aligned}$$

We have also

$$(188)\qquad \rho_1 = \frac{\alpha_6(\tau)}{\bar{\alpha}_6}, \qquad \sigma_1 = \frac{\gamma_6(\tau)}{\bar{\alpha}_6}, \qquad \epsilon_1 = \frac{1}{\bar{\alpha}_6}.$$

It is easy to characterize R_2 and S_2. R_2 contains only cosines of even multiples of τ, and S_2 contains only sines of odd multiples of τ. Since ρ_2 is even in τ and σ_2 odd, the solution satisfying this condition is

$$\rho_2 = A^{(2)}\alpha_2(\tau) + D^{(2)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + [\eta_2(\tau) + a_2\tau\alpha_3(\tau)] + \epsilon_2[\alpha_5(\tau) + a\tau\alpha_3(\tau)],$$

$$\sigma_2 = A^{(2)}\gamma_1(\tau) + D^{(2)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + [\zeta_2(\tau) + a_2\tau\gamma_4(\tau)] + \epsilon_2[\gamma_5(\tau) + a\tau\gamma_4(\tau)].$$

In this solution the terms are grouped according to their origin. The first two terms are the complementary function. The third arises from the terms carrying ϵ_1^2 as a factor. The fourth arises from the terms carrying ϵ_2 as a factor. a_2 is a constant depending upon the coefficients of ϵ_1^2 in the differential equations. $\alpha_5(\tau)$ and $\gamma_5(\tau)$ are the same functions as in the coefficient of the first power of e. By Theorems IV and V, $\eta_2(\tau)$ and $\zeta_2(\tau)$ are periodic functions of τ with the period 2π and so constituted that $\eta_2(\tau)$ contains only cosines of even multiples of τ, and $\zeta_2(\tau)$ contains only sines of odd multiples.

In order that ρ_2 and σ_2 shall be periodic we must have

$$D^{(2)} = -a_2 - a\epsilon_2,$$

and this makes

$$\rho_2 = A^{(2)}\alpha_2(\tau) + \epsilon_2\alpha_6(\tau) + \eta_2(\tau) - a_2\alpha_4(\tau),$$

$$\sigma_2 = A^{(2)}\gamma_1(\tau) + \epsilon_2\gamma_6(\tau) + \zeta_2(\tau) - a_2\gamma_3(\tau).$$

In order that we may satisfy the initial conditions we must have $\rho_2(0) = 0$, and this determines ϵ_2 so that

$$\epsilon_2 = \frac{a_2\bar{\alpha}_4 - \bar{\eta}_2 - A^{(2)}\bar{\alpha}_2}{\bar{\alpha}_6}.$$

$A^{(2)}$ is not yet determined, but it is obvious that it must be zero in order to satisfy the periodicity condition on the coefficient of e^3 in which the non-periodic parts arising from terms involving the first multiple of $\lambda\tau$ carry $A^{(2)}$ as a factor. Hence the only way to avoid non-periodic terms of this character is to choose $A^{(2)} = 0$. Anticipating this step then we have

$$\begin{aligned} \rho_2 &= \frac{a_2\bar{\alpha}_4 - \bar{\eta}_2}{\bar{\alpha}_6}\alpha_6(\tau) + \eta_2(\tau) - a_2\alpha_4(\tau), \\ \sigma_2 &= \frac{a_2\bar{\alpha}_4 - \bar{\eta}_2}{\bar{\alpha}_6}\gamma_6(\tau) + \zeta_2(\tau) - a_2\gamma_3(\tau). \end{aligned} \tag{191}$$

Hence ρ_2 contains only cosines of even multiples of τ, and σ_2 contains only sines of odd multiples of τ.

It remains only to show that this process of integration can be carried on indefinitely. Assuming that up to and including ρ_{i-1} and σ_{i-1} every ρ_j and σ_j is periodic with the period 2π and that the ρ_j contain only cosines of even multiples of τ and σ_j only sines of odd multiples of τ, except that ρ_{i-1} contains the term $A^{(i-1)}\alpha_2(\tau)$ and σ_{i-1} contains the term $A^{(i-1)}\gamma_1(\tau)$, it will be shown that the same conditions obtain for the next succeeding step. For ρ_i and σ_i we have from the differential equations (138)

$$\begin{aligned} \rho_i'' + \theta_2\rho_i + \theta_3\sigma_i = \theta_{001}\epsilon_i + A^{(i-1)}[\theta_{101}\epsilon_1\alpha_2 + 2\theta_{200}\rho_1\alpha_2 \\ + \theta_{110}(\rho_1\gamma_1 + \sigma_1\alpha_2) + 2\theta_{030}\sigma_1\gamma_1] + \Phi_i, \\ \sigma_i'' + \theta_4\sigma_i + \theta_3\rho_i = A^{(i-1)}[2\bar{\theta}_{200}\rho_1\alpha_2 + \bar{\theta}_{110}(\rho_1\gamma_1 + \sigma_1\alpha_2) + 2\bar{\theta}_{020}\sigma_1\gamma_1] + \Psi_i. \end{aligned} \tag{192}$$

From the properties of the differential equations it is readily seen that Φ_i contains only known terms all of which are cosines of even multiples of τ, and that Ψ_i contains only known terms all of which are sines of odd multiples of τ. The coefficients of $A^{(i-1)}$ are homogeneous of the first degree in α_2 and γ_1, and consequently each term involves a first multiple of $\lambda\tau$. Their solution gives rise to non-periodic terms of the form $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$. They carry $A^{(i-1)}$ as a

factor, and since terms of this type arise nowhere else in the solution we can make them disappear only by putting $A^{(i-1)} = 0$. The solution for (192) then has the form

$$
(193)\quad \begin{aligned}
\rho_i &= A^{(i)}\alpha_2(\tau) + D^{(i)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + [\eta_i(\tau) + a_i\tau\alpha_3(\tau)] + \epsilon_i[\alpha_5(\tau) + a\tau\alpha_3(\tau)],\\
\sigma_i &= A^{(i)}\gamma_1(\tau) + D^{(i)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + [\zeta_i(\tau) + a_i\tau\gamma_4(\tau)] + \epsilon_i[\gamma_5(\tau) + a\tau\gamma_4(\tau)],
\end{aligned}
$$

where $\eta_i(\tau)$ and $\zeta_i(\tau)$ are periodic with the period 2π, and by Theorem V, $\eta_i(\tau)$ contains only cosines of even multiples of τ, and $\zeta_i(\tau)$ contains only sines of odd multiples of τ.

For ρ_i and σ_i to be periodic it is necessary and sufficient that

$$D^{(i)} = -a_i - a\epsilon_i,$$

which makes

$$
(194)\quad \begin{aligned}
\rho_i &= A^{(i)}\alpha_2(\tau) + \eta_i(\tau) - a_i\alpha_4(\tau) + \epsilon_i\alpha_6(\tau),\\
\sigma_i &= A^{(i)}\gamma_1(\tau) + \zeta_i(\tau) - a_i\gamma_3(\tau) + \epsilon_i\gamma_6(\tau).
\end{aligned}
$$

From the initial conditions we must have $\rho_i(0) = 0$. This condition determines ϵ_i to be

$$\epsilon_i = \frac{a_i\bar{\alpha}_4 - \bar{\eta}_i - A^{(i)}\bar{\alpha}_2}{\bar{\alpha}_6}.$$

Thus the constants are uniquely determined. The ρ_i and σ_i have the properties assumed for those having lower subscripts, and the process of integration can be continued indefinitely. Every $A^{(j)}$ is zero. Since no terms involving the $\lambda\tau$ enter, the solution has the period 2π. But the orbits represented belong to the class of generating orbits from which we started. In other words, we set out with a generating orbit for which the initial distance was, let us say, r_0, and we have found another generating orbit for which the initial distance is $r_0 + e$ (e arbitrary). There is nothing surprising in this, for r_0 is a function of an arbitrary constant β. Let us suppose we had started with a definite value of β, let us say β_0. This gives us a definite generating orbit with a definite initial distance r_0. Let us seek now the generating orbit for which the initial distance is $r_0 + e$. If e is sufficiently small we can evidently give an increment ϵ to β which will increase r_0 by the amount e. We have

$$r_0 = f(\beta_0),$$

$$r_0 + e = f(\beta_0 + \epsilon).$$

Expanding the right member of the last equation by Taylor's theorem we have

$$e = \frac{\partial f}{\partial \beta_0}\epsilon + \tfrac{1}{2}\frac{\partial^2 f}{\partial \beta_0^2}\epsilon^2 + \cdots,$$

and by inversion, since

$$\left.\frac{\partial f}{\partial \beta}\right|_{\beta=\beta_0} \neq 0,$$

$$\epsilon = \frac{1}{\dfrac{\partial f}{\partial \beta_0}} e + \cdots,$$

which we might write

$$\epsilon = e_1 e + e_2 e^2 + \cdots.$$

Then by substituting in the generating orbit

$$\beta = \beta_0 + e_1 e + e_2 e^2 + \cdots$$

and arranging as a power series in e, we would obtain the generating orbit which we sought. As these are the same conditions that are imposed when we seek new orbits through the equations of variation it is to be anticipated that one of the class of generating orbits satisfies the conditions.

Second solution.

We return now to equation (186) where the two methods of satisfying the periodicity condition presented themselves, and we shall continue with the second solution. We choose $\epsilon_1 = 0$, which we found determines $A^{(1)}$ to be $A^{(1)} = 1/\bar{\alpha}_2$. From (82) it is seen that $\bar{\alpha}_2 = \alpha_2(0) = 1$, and therefore $A^{(1)} = 1$. Hence in the second solution

$$\rho_1 = \alpha_2(\tau), \qquad \sigma_1 = \gamma_1(\tau). \tag{195}$$

Using these values of $A^{(1)}$ and ϵ_1, R_2 and S_2 of (186) become

$$\begin{aligned} R_2 &= [\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] + \epsilon_2\theta_{001}, \\ S_2 &= [\bar{\theta}_{200}\alpha_2^2 + \bar{\theta}_{101}\alpha_2\gamma_1 + \bar{\theta}_{020}\gamma_1^2]. \end{aligned} \tag{196}$$

All of the terms in these expressions except $\epsilon_2\theta_{001}$ are of the second degree in α_2 and γ_1. Therefore they involve terms carrying $2\lambda\tau$ and terms independent of $\lambda\tau$. θ_{001} also is independent of $\lambda\tau$. In the solution the terms depending upon $2\lambda\tau$ are periodic by Theorem II. As for the terms independent of λ, R_2 contains only cosines of even multiples of τ and S_2 contains only sines of odd multiples of τ. These terms give rise to non-periodic terms in the solution which has the form

$$\begin{aligned} \rho_2 &= A^{(2)}\alpha_2(\tau) + D^{(2)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + \phi_2(\lambda, \tau) \\ &\qquad + [\eta_2(\tau) + a_2\tau\alpha_3(\tau)] + \epsilon_2[\alpha_5(\tau) + a\tau\alpha_3(\tau)]; \\ \sigma_2 &= A^{(2)}\gamma_1(\tau) + D^{(2)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + \psi_2(\lambda, \tau) \\ &\qquad + [\zeta_2(\tau) + a_2\tau\gamma_4(\tau)] + \epsilon_2[\gamma_5(\tau) + a\tau\gamma_4(\tau)]. \end{aligned} \tag{197}$$

In these equations $\phi_2(\lambda, \tau)$ and $\psi_2(\lambda, \tau)$ are the periodic terms involving λ, η_2 and ζ_2 are the periodic terms with the period 2π; a_2 the constant belonging to the non-periodic part, and the coefficients of ϵ_2 are the solutions arising from the coefficient of ϵ_2 in the differential equations. In order that this solution (197) shall be periodic it is necessary that

$$D^{(2)} = -a_2 - a\epsilon_2,$$

and this reduces ρ_2 and σ_2 to

$$(198) \qquad \begin{aligned} \rho_2 &= A^{(2)}\alpha_2(\tau) + \phi_2(\lambda, \tau) + \eta_2(\tau) - a_2\alpha_4(\tau) + \epsilon_2\alpha_6(\tau), \\ \sigma_2 &= A^{(2)}\gamma_1(\tau) + \psi_2(\lambda, \tau) + \zeta_2(\tau) - a_2\gamma_3(\tau) + \epsilon_2\gamma_6(\tau). \end{aligned}$$

In order that we may satisfy the initial conditions we must have $\rho_2(0) = 0$. Hence, since $\alpha_2(0) = 1$,

$$A^{(2)} = -\bar{\phi}_2 - \bar{\eta}_2 + a_2\bar{\alpha}_4 - \epsilon_2\bar{\alpha}_6.$$

The constant ϵ_2 is determined by the periodicity condition for the coefficient of e^3.

Coefficient of e^3.

$$(199) \qquad \rho_3'' + \theta_2\rho_3 + \theta_3\sigma_3 = R_3, \qquad \sigma_3'' + \theta_4\sigma_3 + \theta_3\rho_3 = S_3,$$

where

$$\begin{aligned} R_3 = \theta_{001}\epsilon_3 + \theta_{101}\epsilon_2\rho_1 + 2\theta_{200}\rho_1\rho_2 + \theta_{110}[\sigma_2\rho_1 + \sigma_1\rho_2] + 2\theta_{020}\sigma_1\sigma_2 + \theta_{300}\rho_1^3 \\ + \theta_{210}\rho_1^2\sigma_1 + \theta_{120}\rho_1\sigma_1^2 + \theta_{030}\sigma_1^3, \\ S_3 = \qquad 2\bar{\theta}_{200}\rho_1\rho_2 + \bar{\theta}_{110}[\sigma_2\rho_1 + \sigma_1\rho_2] + 2\bar{\theta}_{020}\sigma_1\sigma_2 + \bar{\theta}_{300}\rho_1^3 \\ + \bar{\theta}_{210}\rho_1^2\sigma_1 + \bar{\theta}_{120}\rho_1\sigma_1^2 + \bar{\theta}_{030}\sigma_1^3. \end{aligned}$$

In classifying the terms which belong to the expansion of R_3 and S_3 we bear in mind:

First. The θ_{ijk} in R_3 involve only cosines of even multiples of τ, except those which are coefficients of odd powers of σ (i. e., j is odd). These involve only sines of odd multiples. The reverse is the case in the $\bar{\theta}_{ijk}$ of S_3. If j is even, $\bar{\theta}_{ijk}$ involves only sines of odd multiples of τ. If j is odd, $\bar{\theta}_{ijk}$ involve only cosines of even multiples of τ.

Second. ρ_1 and ρ_2 are cosine series. The terms independent of λ involve only even multiples of τ.

σ_1 and σ_2 are sine series. The terms independent of λ involve only odd multiples of τ.

It is seen then that among the terms in R_3 which are independent of λ only cosines of even multiples of τ enter; and among the terms in S_3 which are independent of λ only sines of odd multiples enter. In the process of integration

therefore two types of non-periodic terms arise, first, those coming from the terms which involve the first multiple of $\lambda\tau$, and second, those coming from the terms independent of λ. It is important therefore to separate the various terms into three classes:

(a) terms independent of λ,

(b) terms involving first multiple of $\lambda\tau$ only,

(c) terms involving multiples of $\lambda\tau$ higher than the first.

The solution for these last terms is periodic. We rewrite then the differential equations (199) as follows:

$$\rho_3'' + \theta_2\rho_3 + \theta_3\sigma_3 = \epsilon_3\theta_{001} + \epsilon_2 f_1(\lambda, \tau) + f_2(\lambda, \tau) + f_3(\tau) + f_4(\kappa\lambda, \tau),$$

$$\sigma_{33}'' + \theta_4\sigma_3 + \theta_3\rho_3 = \epsilon_2 g_1(\lambda, \tau) + g_2(\lambda, \tau) + g_3(\tau) + g_4(\kappa\lambda, \tau).$$

The coefficients of ϵ_2 are, explicitly,

$$(200)\qquad \begin{aligned} f_1(\lambda, \tau) &= \theta_{101}\alpha_2 + 2\theta_{200}\alpha_2\alpha_6 + \theta_{110}(\alpha_2\gamma_6 + \gamma_1\alpha_6) + 2\theta_{020}\gamma_1\gamma_6, \\ g_1(\lambda, \tau) &= 2\bar{\theta}_{200}\alpha_2\alpha_6 + \bar{\theta}_{110}(\alpha_2\gamma_6 + \gamma_1\alpha_6) + 2\bar{\theta}_{020}\gamma_1\gamma_6. \end{aligned}$$

These terms are homogeneous in the first degree in α_2 and γ_1, and consequently involve only terms which carry the first multiple of $\lambda\tau$. They are of importance since they carry the undetermined constant ϵ_2 as a factor. The solution for these terms has, by Theorem II, the form

$$\rho = F_1(\lambda, \tau) + b_1\tau\alpha_1(\tau), \qquad \sigma = G_1(\lambda, \tau) + b_1\tau\gamma_2(\tau),$$

where $F_1(\lambda, \tau)$ and $G_1(\lambda, \tau)$ are periodic and involve only terms carrying the first multiple of $\lambda\tau$. b_1 is a constant depending upon $f_1(\lambda, \tau)$ and $g_1(\lambda, \tau)$. It is found, by a calculation not difficult, to have the value

$$b_1 = 3 + \mu^2\wp(\mu^2).$$

The functions $f_2(\lambda, \tau)$ and $g_2(\lambda, \tau)$ have the same characters as $f_1(\lambda, \tau)$ and $g_1(\lambda, \tau)$. They are considered separately since they are independent of ϵ_2. Their solutions may be written

$$\rho = F_2(\lambda, \tau) + b_2\tau\alpha_1(\tau), \qquad \sigma = G_2(\lambda, \tau) + b_2\tau\gamma_2(\tau).$$

The terms $f_3(\tau)$ and $g_3(\tau)$ are independent of λ. $f_3(\tau)$ contains only cosines of even multiples of τ, while $g_3(\tau)$ contains only sines of odd multiples of τ. The solution for these terms has the form

$$\rho = F_3(\tau) + b_3\tau\alpha_3(\tau), \qquad \sigma = G_3(\tau) + b_3\tau\gamma_4(\tau).$$

Finally, $f_4(k\lambda, \tau)$ and $g_4(k\lambda, \tau)$ involve only terms which carry multiples of $\lambda\tau$ higher than the first. The solution for these terms is

$$\rho = F_4(k\lambda, \tau), \qquad \sigma = G_4(k\lambda, \tau).$$

The entire solution is then

$$
(201)\quad
\begin{aligned}
\rho_3 = {} & A^{(3)}\alpha_2(\tau) + D^{(3)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + \epsilon_3[\alpha_5(\tau) + a\tau\alpha_3(\tau)] \\
& + \epsilon_2[F_1(\lambda, \tau) + b_1\tau\alpha_1(\tau)] + [F_2(\lambda, \tau) + b_2\tau\alpha_1(\tau)] \\
& + [F_3(\tau) + b_3\tau\alpha_3(\tau)] + F_4(k\lambda, \tau), \\
\sigma_3 = {} & A^{(3)}\gamma_1(\tau) + D^{(3)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + \epsilon_3[\gamma_5(\tau) + a\tau\gamma_4(\tau)] \\
& + \epsilon_2[G_1(\lambda, \tau) + b_1\tau\gamma_2(\tau)] + [G_2(\lambda, \tau) + b_2\tau\gamma_2(\tau)] \\
& + [G_3(\tau) + b_3\tau\gamma_4(\tau)] + G_4(k\lambda, \tau).
\end{aligned}
$$

All of the functions $\alpha_i(\tau)$, $\gamma_i(\tau)$, $F_i(\tau)$, and $G_i(\tau)$ are periodic. In order that ρ_3 and σ_3 shall be periodic it is necessary and sufficient that the coefficient of $\tau\alpha_3(\tau)$ and $\tau\gamma_3(\tau)$, and the coefficient of $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$ be zero. That is,

$$D^{(3)} = -b_3 - a\epsilon_3,$$

and

$$\epsilon_2 = -\frac{b_2}{b_1},$$

by which condition the value of ϵ_2 is determined. In order to satisfy the initial conditions we must have $\rho_3 = 0$ at $\tau = 0$, and this determines $A^{(3)}$,

$$A^{(3)} = b_3\alpha_4(0) - \epsilon_3\alpha_6(0) + \frac{b_2}{b_1}F_1(0) - F_2(0) - F_3(0) - F_4(0).$$

Thus all the constants are determined except ϵ_3, and the solution is

$$
(202)\quad
\begin{aligned}
\rho_3 &= A^{(3)}\alpha_2(\tau) - b_3\alpha_4(\tau) + \epsilon_3\alpha_6(\tau) - \frac{b_2}{b_1}F_1(\lambda, \tau) + F_2(\lambda, \tau) + F_3(\tau) + F_4(k\lambda, \tau), \\
\sigma_3 &= A^{(3)}\gamma_1(\tau) - b_3\gamma_3(\tau) + \epsilon_3\gamma_6(\tau) - \frac{b_2}{b_1}G_1(\lambda, \tau) + G_2(\lambda, \tau) + G_3(\tau) + G_4(k\lambda, \tau).
\end{aligned}
$$

The constant ϵ_3 will be determined in satisfying the periodicity condition for the coefficient of e^4. This process of integration can be continued indefinitely. ρ_3 and σ_3 have the same properties that have been stated for ρ_2 and σ_2. It is evident from the properties of the differential equations that these properties persist for ρ_4 and σ_4, and so on indefinitely. The coefficient for ϵ_{i-1} in so far as it carries the first multiples of $\lambda\tau$ is always the same as for ϵ_2. Therefore the constant ϵ_{i-1} can always be determined so as to avoid non-periodic terms of the type $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$. The constant $D^{(i)}$ of integration can always be determined so as to destroy non-periodic terms of the type $\tau\alpha_3(\tau)$ and $\tau\gamma_4(\tau)$. The constant $A^{(i)}$ can always be determined so as to satisfy the initial conditions. The analysis of the type of terms entering is the same as for the subscript 3.

We have therefore a periodic solution with the period $2K\pi$ which is different from the class of generating orbits from which we set out, for the particle makes many revolutions before its orbit reënters. After integrating the equation $dv/d\tau = c/r^2$ the solution will contain five arbitrary constants (subject to the condition that λ must be rational), corresponding to mean distance, eccentricity, inclination, node, and epoch.

UNIVERSITY OF CHICAGO,
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THE THEOREM OF THOMSON AND TAIT AND NATURAL FAMILIES OF TRAJECTORIES*

BY

EDWARD KASNER

In most dynamical investigations relating to conservative forces, in particular those connected with the principle of least action and the developments of Hamilton and Jacobi, it is essential to group the possible motions according to the value of the constant h representing the total energy. The trajectories corresponding to any given value of h are said to form a *natural* † family. In the case of a particle moving in a three-dimensional conservative field, a natural family consists of ∞^4 curves. Examples are the ∞^4 straight lines of space, corresponding to zero force; and the ∞^4 vertical parabolas whose directrices are located in a fixed horizontal plane, corresponding to gravity assumed constant. In a paper published in the preceding volume of these Transactions,‡ the general geometric character of natural families was expressed in terms of osculating circles; and a certain reciprocity, analogous to that of Scheffers for plane isogonal trajectories, was established.

In the present paper, which may be read independently, we start from the remarkable theorem due to THOMSON and TAIT which states that the ∞^2 curves of a natural family which meet any surface orthogonally are necessarily orthogonal to ∞^1 surfaces, that is, form a normal congruence. Our main object is to show that *this property belongs exclusively to natural families.* The new result may be regarded as a converse of Thomson and Tait's theorem and stated as follows:

If a quadruply-infinite system of curves in space is such that ∞^2 curves of the system meet an arbitrary surface orthogonally and always form a normal congruence (that is, admit ∞^1 orthogonal surfaces), then the system is of the natural type.

Natural families present themselves not only in the study of dynamical tra-

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† The term *natural* was first used in this sense by PAINLEVÉ, Journal de Mathématiques, vol. 10, 1894. Of course no connection is to be inferred with natural or intrinsic coördinates, that is, to invariance under the displacement group.

‡ *Natural families of trajectories: conservative fields of forces,* these Transactions, vol. 10 (1909), pp. 201–219. This will be cited as *earlier paper.*

jectories, but also, for example, in the discussion of brachistochrones, catenaries, and optical path curves.* The many analogies that have been observed between these theories are based on the fact that they all lead to variation problems of the particular form †

$$\int F(x, y, z)\,ds = \text{minimum} \qquad (ds = \sqrt{dx^2 + dy^2 + dz^2}),$$

where F is any point function and ds is the element of length. From the purely mathematical point of view a natural family may be defined as the totality of extremals connected with such a variation problem.

In the kinetic interpretation F represents the speed v, as found from the energy equation

$$v^2 = 2(W + h),$$

where W denotes the work function (negative potential) and the mass is assumed to be unity. The integral above then represents the action, $\int v ds$, and the result is the principle of least action in the form given by Jacobi. The complete theorem of Thomson and Tait ‡ is as follows: "If from all points of an arbitrary surface particles not mutually influencing one another be projected normally with the proper velocities [so as to make the sum of the kinetic and potential energies have a given value]; points which they reach with equal actions lie on a surface cutting the paths at right angles." The ∞^1 orthogonal surfaces thus appear as surfaces of equal action. §

In the optical interpretation, which is even more concrete, the function F represents the index of refraction ν, varying from point to point, and the integral $\int \nu ds$ is proportional to the time. We thus have Fermat's principle of least time. Thus the paths of light in an isotropic (not necessarily homogeneous) medium (for example, the atmosphere) form a natural family. Every natural family may be obtained in this way. The ∞^1 surfaces orthogonal to a congruence of paths present themselves as wave fronts (surfaces of equal time). ‖

* Cf. APPELL, *Mécanique rationnelle*, vol. 2, p. 216. For other interpretations see the introduction to the author's earlier paper.

† This form is characterized by the fact that the transversal relation reduces to orthogonality. The theorem of Thomson and Tait thus presents itself as a special case of Kneser's general theory of transversals. Cf. BOLZA, *Variationsrechnung*, pp. 131, 691.

‡ *Treatise on natural philosophy* (edition of 1879), part 1, section 332. Cf. DARBOUX, *Théorie des surfaces*, vol. 2, p. 468; APPELL, *Mécanique rationnelle*, vol. 2, p. 431.

§ A second converse which the author has obtained is as follows: If particles are projected normally from a base surface into a conservative field, and if the ∞^2 trajectories generated form a normal congruence, then the initial velocities must be such that the total energy is constant. This is true only "in general." In the case of central forces it is in fact possible to obtain normal congruences for which h is not constant.

‖ Usually the sets of wave surfaces so obtained will not form Lamé families, that is, cannot be regarded as members of triply orthogonal systems. The author has shown that the only natural families leading to Lamé families are those composed of circles; these circles must be orthogonal to a fixed sphere (of real, imaginary, or zero radius). The corresponding optical and dynamical statements are quite simple.

In § 1 the general condition that ∞^2 curves shall form a normal congruence is written out for the case of parametric representation. In our applications the curves are orthogonal to some base surface Σ by assumption. It is therefore expedient to expand the general condition in powers of the parameter t which, starting with the value zero for any point of Σ, varies along the orthogonal curve through that point. The result is then of the form

$$\Omega_1 t + \Omega_2 t^2 + \Omega_3 t^3 + \cdots = 0.$$

For a normal congruence *all* the coefficients vanish. If the leading coefficient Ω_1 vanishes the congruence may be described as "approximately of normal character." If the first n coefficients vanish we describe the congruence as "approximately normal to the nth degree of approximation." A congruence orthogonal to a surface Σ is approximately normal when and only when it is possible to construct ∞^2 curves osculating the given curves at the points of Σ and forming a normal congruence (in the exact sense).

In § 2 the converse theorem stated in italics above is proved. The result is then strengthened in two directions. In the first place it is sufficient to require that the ∞^2 curves, belonging to the quadruply-infinite system, which are orthogonal to an arbitrary surface Σ shall form an approximately normal congruence. The discussion then shows that they necessarily form an exact normal congruence. In the second place it is sufficient to make this demand for suitable manifolds (of course infinite) of base surfaces, instead of for all surfaces.

Special cases of the theorem of Thomson and Tait arise where the selected base Σ degenerates into a point, considered as a small sphere, or into a curve, considered as a thin tube. We show in § 4 that the case of base points is not sufficient to characterize the natural type. Systems of ∞^4 curves exist, which, though not natural, have the property that the curves passing through an arbitrary point always form a normal congruence.

If for every curve as base the corresponding congruence is to be normal, the system is necessarily natural. If only the approximately normal character described above is demanded, the more general class of *velocity systems* is obtained (§ 3). This type was defined in our earlier paper in terms of motion (loci of equal speed) in connection with an arbitrary field of force.* Only for conservative fields are the velocity systems of the natural type. The general velocity system is represented by equations of the form

$$y'' = (\psi - y'\phi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\phi)(1 + y'^2 + z'^2),$$

where ϕ, ψ, and χ are arbitrary point functions, and is characterized by the property that the osculating circles constructed at any point p for the ∞^2 curves

* These Transactions, vol. 10 (1909), p. 376.

passing through that point meet at another point P, thus forming a bundle (property A).*

Natural families are distinguished from other velocity systems by an additional peculiarity, property B, or its equivalent D, of our earlier paper.† In § 4 of the present paper it is shown that this is simply expressed by the requirement that the curves through any point form a normal congruence. *Thus the natural type is characterized by these two properties: the curves through any point admit ∞^1 orthogonal surfaces, and the osculating circles constructed at the common point form a bundle.*

In the last section (§ 5) a contribution is made to the general theory of quadruply-infinite systems of curves in space, the only assumption being that ∞^2 curves pass through every point, one in each direction. It is shown that with any system there are associated certain surfaces which when employed as bases give rise to congruences of approximately normal character.‡ These surfaces satisfy a certain partial differential equation of the second order. In the particular case of the velocity type this equation reduces to one of the first order, and in the still more special case of the natural type it vanishes identically.

Although the discussion of the present paper is restricted to ordinary space of three dimensions, the results are, in all probability, valid for higher spaces also. In the case of two dimensions, however, the geometric part of Thomson and Tait's theorem becomes trivial since any system of ∞^1 curves necessarily admits orthogonal curves; the converse proposition is then not true.§

§ 1. Conditions for an exact or approximate normal congruence.

The condition that a congruence of curves, represented in the usual form by

$$\frac{dx}{A(x, y, z)} = \frac{dy}{B(x, y, z)} = \frac{dz}{C(x, y, z)},$$

shall be normal is that the corresponding total equation

$$Adx + Bdy + Cdz = 0$$

shall be exact; this gives the familiar result

$$A(B_z - C_y) + B(C_x - A_z) + C(A_y - B_x) = 0.$$

* The analogous systems in the plane, $y'' = (\psi - y'\phi)(1 + y'^2)$, were first studied by Scheffers, Leipziger Berichte (1898), in connection with isogonal trajectories.

† Property B states the mutual orthogonality of the three hyperosculating circles existing in any bundle. Property D may be expressed in simplified form as follows: The transformation T from the point p to the point P is such that the three lineal elements at p each of which is converted by T into a cocircular element at P are mutually orthogonal.

‡ In general no exactly normal congruences can be obtained.

§ The results of our earlier paper hold for all dimensions. That the present results remain valid in three-dimensional spaces of constant curvature may be shown by conformal representation.

A congruence of curves defines a field of ∞^3 direction elements in space. If these elements are given in terms of three arbitrary parameters u, v, w (instead of x, y, z) the field takes the form

$$x = X(u, v, w), \quad y = Y(u, v, w), \quad z = Z(u, v, w),$$
$$dx : dy : dz = \bar{X}(u, v, w) : \bar{Y}(u, v, w) : \bar{Z}(u, v, w).$$

The condition for a normal congruence then becomes

$$\bar{X}\{(\bar{Y}XY)-(\bar{Z}ZX)\}+\bar{Y}\{(\bar{Z}YZ)-(\bar{X}XY)\}+\bar{Z}\{(\bar{X}ZX)-(\bar{Y}YZ)\}=0,$$

where the parentheses denote jacobians of the enclosed functions with respect to the parameters, so that, for example,

$$(\bar{Y}XY) \equiv \begin{vmatrix} \bar{Y}_u & \bar{Y}_v & \bar{Y}_w \\ X_u & X_v & X_w \\ Y_u & Y_v & Y_w \end{vmatrix}.$$

In the applications below our congruence admits one orthogonal surface Σ by assumption, and we may take our parameters relative to it as follows: Let u, v denote Gaussian parameters on Σ, and let w be the parameter which varies as the point describes one of the orthogonal curves, assuming it to be zero as the point starts out from the surface. (In particular w might represent the distance from Σ along the curve.) The congruence then takes the form

$$\begin{aligned} X &= x + A_1 w + \tfrac{1}{2} A_2 w^2 + \cdots, \\ Y &= y + B_1 w + \tfrac{1}{2} B_2 w^2 + \cdots, \\ Z &= z + C_1 w + \tfrac{1}{2} C_2 w^2 + \cdots, \\ \bar{X} &= A_1 + A_2 w + \tfrac{1}{2} A_3 w^2 + \cdots, \\ \bar{Y} &= B_1 + B_2 w + \tfrac{1}{2} B_3 w^2 + \cdots, \\ \bar{Z} &= C_1 + C_2 w + \tfrac{1}{2} C_3 w^2 + \cdots, \end{aligned}$$

the last three functions being found by differentiating the first three with respect to w. Here the coefficients are all functions of u and v; for x, y, z are to be replaced by the functions defining the surface Σ; A_1, B_1, C_1 are proportional to the direction cosines of the normal to the surface; and the remaining coefficients are arbitrary.

The condition for a normal congruence, expanded in powers of w, now takes the form

$$\Omega_1 w + \Omega_2 w^2 + \Omega_3 w^3 + \cdots = 0,$$

where Ω_i depends for its value upon the quantities $A_1, B_1, C_1, \cdots, A_{i+1}, B_{i+1}, C_{i+1}$, involving only partial derivatives of the first order. The absolute term neces-

sarily vanishes, since the curves are orthogonal to Σ for which $w=0$ by assumption.

In case the congruence is normal *all* the coefficients Ω will vanish. If the first coefficient Ω_1 vanishes then the congruence may be said to be normal to the first degree of approximation. In this case the curves may be regarded as orthogonal to some surface infinitesimally close to Σ. In general the congruence will be said to be normal to the nth degree of approximation (or to terms of the nth order) provided

$$\Omega_1 = \Omega_2 = \cdots = \Omega_n = 0.$$

In the following applications we take up first the general case where Σ is a proper surface, and then the special cases where Σ reduces to a curve and to a point respectively. In the case of surface and curve we shall make use of the explicit value of Ω_1 only; in the case of a point it will appear that Ω_1 vanishes identically and it will be necessary to calculate Ω_2 and Ω_3.

§ 2. General case where the base is a proper surface.

Consider an arbitrary quadruply infinite system of curves in space, assuming that one passes through each point in each direction. Such a system may be defined by a pair of differential equations of the second order

$$y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z'), \tag{1}$$

where F and G are uniform functions which we assume to be analytic in the five arguments. Denoting the initial values of x, y, z, y', z', which may be taken at random, by x, y, z, p, q respectively, and employing X, Y, Z as current coördinates, we may write the solutions of (1) in the form

$$\begin{aligned} Y &= y + p(X-x) + \tfrac{1}{2}F(X-x)^2 + \tfrac{1}{6}M(X-x)^3 + \cdots, \\ Z &= z + q(X-x) + \tfrac{1}{2}G(X-x)^2 + \tfrac{1}{6}N(X-x)^3 + \cdots. \end{aligned} \tag{2}$$

Here F and G are expressed as functions of x, y, z, p, q, and M and N, found by differentiating (1), are given by

$$\begin{aligned} M &\equiv F_x + pF_y + qF_z + FF_p + GF_q, \\ N &\equiv G_x + pG_y + qG_z + FG_p + GG_q. \end{aligned} \tag{3}$$

The terms of higher order will not be needed in our discussion. Equations (2) involve five arbitrary parameters but of course represent only ∞^4 curves.

Consider now an arbitrary surface Σ

$$z = f(x, y). \tag{4}$$

At each point of this surface and normal to it a definite curve of the given family (1) may be constructed. A certain congruence will thus be determined. We wish to express the condition that this shall be of the normal type, that is, that the ∞^2 curves shall admit a family of orthogonal surfaces.

The direction normal to the surface Σ at any point is given by

$$1 : p : q = f_x : f_y : -1,$$

so that

$$p = P(x, y), \qquad q = Q(x, y), \tag{5}$$

where

$$P \equiv \frac{f_y}{f_x}, \qquad Q \equiv -\frac{1}{f_x}. \tag{5'}$$

These functions are connected by the relation

$$PQ_x - QP_x - Q_y \equiv 0. \tag{5''}$$

The equations of the ∞^2 curves corresponding to the given initial conditions may now be written

$$\begin{aligned} Y &= y + Pt + \tfrac{1}{2}\bar{F}t^2 + \tfrac{1}{6}\bar{M}t^3 + \cdots, \\ Z &= f + Qt + \tfrac{1}{2}\bar{G}t^2 + \tfrac{1}{6}\bar{N}t^3 + \cdots, \end{aligned} \tag{6}$$

where t takes the place of $X - x$ in (2), and where the bars indicate that the substitution (4), (5) has been carried out, so that, for example,

$$\bar{F}(x, y) \equiv F(x, y, f, P, Q). \tag{6'}$$

The coefficients of the powers of t in (6) are thus functions of the two parameters x, y.

The congruence (6) defines a field of lineal elements, which, in terms of the three parameters t, x, y, is given by

$$\begin{aligned} X &= x + t, \\ Y &= y + Pt + \tfrac{1}{2}\bar{F}t^2 + \tfrac{1}{6}\bar{M}t^3 + \cdots, \\ Z &= f + Qt + \tfrac{1}{2}\bar{G}t^2 + \tfrac{1}{6}\bar{N}t^3 + \cdots, \\ Y' &= P + \bar{F}t + \tfrac{1}{2}\bar{M}t^2 + \cdots, \\ Z' &= Q + \bar{G}t + \tfrac{1}{2}\bar{N}t^2 + \cdots. \end{aligned} \tag{7}$$

The condition for a normal congruence is *

$$(Y'XY) - (Z'ZX) + Y'(Z'YZ) - Z'(Y'YZ) = 0, \tag{8}$$

*In the present case $\bar{X} : \bar{Y} : \bar{Z} = 1 : Y' : Z'$. We may also use the convenient form due to Beltrami. Cf. BIANCHI-LUKAT, *Differentialgeometrie*, p. 340.

where the parentheses denote jacobians taken with respect to t, x, y. For example:

$$(Y'XY) = \begin{vmatrix} \bar{F} + \bar{M}t + \cdots & P_x + \bar{F}_x t + \cdots & P_y + \bar{F}_y t + \cdots \\ 1 & 1 & 0 \\ P + \bar{F}t + \cdots & P_x t + \cdots & 1 + P_y t + \cdots \end{vmatrix}.$$

Expanding our results in powers of t, and writing down only the terms of first degree, we find

$$\begin{aligned}
&(Y'XY) = \bar{F} - PP_y - P_x + t\{\bar{M} - \bar{F}_x - P\bar{F}_y\} + \cdots, \\
&(Z'ZX) = Q_x f_y - Q_y f_x + QQ_y - Gf_y + t\{-\bar{N}f_y + \bar{G}_x f_y - \bar{G}_y f_x + Q\bar{G}_y\} + \cdots, \\
&(Z'YZ) = QQ_x + P(Q_y f_x - Q_x f_y) - \bar{G}f_x + t\{-\bar{N}f_x + \bar{F}(Q_y f_x - Q_x f_y) \\
&\qquad + \bar{G}(P_x f_y - P_y f_x) + QG_x + P(G_y f_x - G_x f_y) + Q(Q_x P_y - P_x Q_y)\} + \cdots, \\
&(Y'YZ) = QP_x + PP_y f_x - PP_x f_y - \bar{F}f_x \\
&\qquad + t\{-\bar{M}f_x + Q\bar{F}_x - \bar{F}Q_x + \bar{G}P_x + P(\bar{F}_y f_x - \bar{F}_x f_y + Q_x P_y - P_x Q_y)\} + \cdots.
\end{aligned} \tag{8'}$$

Substituting these values in (8), we find that the part free from t is

$$\frac{1}{Q}(1 + P^2 + Q^2)(PQ_x - QP_x - Q_y), \tag{9}$$

which vanishes in consequence of (5″). This is as it should be, since our ∞^2 curves are orthogonal to Σ by construction.

The terms containing the first power of t give

$$\begin{aligned}
&(1 + P^2 + Q^2)(P\bar{G}_x - Q\bar{F}_x - \bar{G}_y) \\
&\qquad + 2\bar{F}\{(P^2 + Q^2)(Q_x - PQ_y) + 2\bar{G}fPP_y - (P^2 + Q^2)P_x\} = 0.
\end{aligned} \tag{10}$$

From (6′) we find

$$\bar{F}_x = F_x + F_z f_x + F_p P_x + F_q Q_x,$$

with corresponding results for $\bar{G}_x$ and $\bar{G}_y$. Substituting these values, and observing from (5) and (5′) that

$$f_x = -\frac{1}{Q}, \qquad f_y = -\frac{P}{Q}, \qquad Q_y = PQ_x - QP_x,$$

we may reduce (10) to

$$\begin{aligned}
&2F\{PQP_x + Q^2 Q_x\} + 2G\{PP_y - (P^2 + Q^2)P_x\} - (1 + P^2 + Q^2) \times \\
&\{QF_x - F_z - PG_x + G_y + (QF_p - QG_q - PG_p)P_x + G_p P_y + QF_q Q_x\} = 0.
\end{aligned} \tag{10'}$$

This is then a necessary condition in order that the ∞^2 *curves belonging to the quadruply infinite system* (1) *and orthogonal to the surface* (4) *shall form a normal congruence.* The result is to hold in virtue of (4) and (5).

It is of course not a sufficient condition. It merely expresses the fact that the curves orthogonal to Σ are also orthogonal to some consecutive surface, that is, that the congruence is approximately normal to the first degree.

Our main problem is to find all systems (1) which have the orthogonality property with respect to *every* base surface Σ. It is then necessary that (10′) should be true for an arbitrary function $f(x, y)$. The function can be so selected that for any chosen values of x and y the quantities, f, P, Q, P_x, P_y, Q_x, shall take on arbitrary numerical values; for the only relation to be fulfilled is (5″) and this merely determines P_y. The condition (10′) must therefore hold identically. Arranging it in the form

$$(10'') \qquad (1 + P^2 + Q^2)C_0 + QC_1Q_x + C_2P_y - C_3P_x = 0,$$

and equating coefficients to zero, we find

$$(11) \qquad \begin{aligned} C_0 &\equiv qF_x - F_z - pG_x + G_y = 0, \\ C_1 &\equiv (1 + p^2 + q^2)F_q - 2qF = 0, \\ C_2 &\equiv (1 + p^2 + q^2)G_p - 2pG = 0, \\ C_3 &\equiv (1 + p^2 + q^2)\{qG_q + pG_p - qF_p\} + 2pqF - 2(p^2 + q^2)G = 0. \end{aligned}$$

Integration of the second and third of these partial differential equations gives

$$F = f_1(p, x, y, z)(1 + p^2 + q^2), \qquad G = g_1(q, x, y, z)(1 + p^2 + q^2),$$

where f_1 and g_1 denote unknown functions of the four arguments indicated. Substituting these values in the fourth equation, we find

$$f_{1p} = g_{1q},$$

and therefore

$$f_1 = \psi - p\phi, \qquad g_1 = \chi - q\phi,$$

where ϕ, ψ, χ are functions of x, y, z only. The general solution of the last three equations of the set (11) is therefore

$$(12) \qquad F = (\psi - p\phi)(1 + p^2 + q^2), \qquad G = (\chi - q\phi)(1 + p^2 + q^2).$$

We have still to satisfy the first equation of (11), which now reduces to

$$(13) \qquad \psi_z - \chi_y + p(\chi_x - \phi_z) + q(\phi_y - \psi_x) = 0.$$

The functions ϕ, ψ, χ must therefore satisfy the equations

$$(13') \qquad \psi_z - \chi_y = 0, \qquad \chi_x - \phi_z = 0, \qquad \phi_y - \psi_x = 0,$$

and hence are expressible as the derivatives of a common function in the form

$$(13'') \qquad \phi = L_x, \qquad \psi = L_y, \qquad \chi = L_z.$$

The solutions of the set (11) are therefore

$$(14) \quad F = (L_y - pL_x)(1 + p^2 + q^2), \qquad G = (L_z - pL_x)(1 + p^2 + q^2),$$

involving an arbitrary function L of x, y, z. The resulting system (1) is thus recognized to be a natural family.* This gives our fundamental

THEOREM I. *If a quadruply infinite system of curves in space, one passing in each direction through each point, has the property that those ∞^2 members of it which are orthogonal to an arbitrarily selected surface always form a normal congruence (that is, admit ∞^1 orthogonal surfaces), then the system must be a natural family.*

That natural families of trajectories actually have this property is known from Thomson and Tait's theorem. In the above discussion use has been made not of the complete condition for a normal congruence, but only of condition (10′) derived from the terms of the first order in t. We may therefore state a stronger converse result as follows:

THEOREM II. *The only systems of ∞^4 curves which have the property that the curves orthogonal to any surface are always orthogonal to some infinitesimally neighboring surface are those of the natural type.*

If a congruence of curves meets two neighboring surfaces orthogonally it need not meet ∞^1 surfaces orthogonally and therefore it approximates but need not coincide with a normal congruence. The above theorem shows however that if the weak requirement of approximate normal character be imposed on *all* the congruences obtained from the given quadruply infinite system, they will *all* be exactly normal.

We may further strengthen our theorem by demanding the orthogonality property for some instead of all surfaces. Our fundamental equations (11) resulted from the fact that x, y, z, f, P, Q, P_x, P_y, Q_x, might receive arbitrary numerical values. It will therefore be sufficient to take a manifold of surfaces sufficiently large to leave these quantities, or the equivalent quantities

$$(15) \qquad x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy},$$

unrestricted. Since these quantities define a differential surface element of the second order, we may state the result as follows:

*See these Transactions, vol. 10 (1909), p. 204.

THEOREM III. *If in theorem II, and, a fortiori, in theorem I, we consider instead of all surfaces, a manifold of surfaces containing all the* ∞^8 *differential elements of second order* (15), *the results are still valid.*

The smallest manifolds of this sort are composed of ∞^6 surfaces. The totality of quadrics would be more than sufficient. The paraboloids

$$z = a + bx + cy + a_1x^2 + b_1xy + c_1y^2$$

would be sufficient. In the discussion of base curves and points that follows, smaller manifolds of bases will present themselves.

2. CASE WHERE THE BASE IS A CURVE.

We now consider those curves of a given system

$$y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z') \tag{1}$$

which meet an arbitrary curve

$$y = f(x), \qquad z = g(x) \tag{16}$$

orthogonally, and impose the condition for a normal congruence (to first approximation).

The ∞^2 possible initial elements (x, y, z, p, q) may be written in terms of two parameters x and p as follows:

$$y = f(x), \quad z = g(x), \quad q = Rp + S,$$

where

$$R \equiv -\frac{f'}{g'}, \qquad S \equiv -\frac{1}{g'}. \tag{17}$$

This results from the orthogonality of the directions $(1 : p : q)$ and $(1 : f' : g')$, expressed by

$$1 + pf' + qg' = 0, \tag{18'}$$

and giving

$$q = S + pR. \tag{18}$$

The congruence orthogonal to the base curve (16) may be written as a direction field in the form

$$\begin{aligned} X &= x + t, \\ Y &= f + pt + \tfrac{1}{2}\bar{F}t^2 + \cdots, \\ Z &= g + qt + \tfrac{1}{2}\bar{G}t^2 + \cdots, \\ Y' &= p + \bar{F}t + \tfrac{1}{2}\bar{M}t^2 + \cdots, \\ Z' &= S + pR + \bar{G}t + \tfrac{1}{2}\bar{N}t^2 + \cdots, \end{aligned} \tag{19}$$

where the parameters are t, x, and p. The bars indicate that the substitution (16), (18) has been effected, so that, for example,

$$\bar{F}(x, p) = F(x, f, g, p, S + pR). \tag{19'}$$

The requisite jacobians taken with respect to t, x, p are

$$\begin{aligned}
(Y'XY) &= f' - p + t(f' - p)\bar{F}_p + \cdots, \\
(Z'ZX) &= R(S + pR - g') + t(S + pR - g')\bar{G}_p + \cdots, \\
(Z'YZ) &= R\{pg' - f'(S + pR)\} + t[(S + pR)(S' + pR') \\
&\qquad + gR\bar{F} - g'\bar{G} + \{pg' - f'(S + pR)\}\bar{G}_p] + \cdots, \\
(Y'YZ) &= pg' - f'(S + pR) + t[p(S' + pR') \\
&\qquad + f'R\bar{F} - f'\bar{G} + \{pg' - f'(S + pR)\}\bar{F}_p] + \cdots.
\end{aligned}$$

Developing the condition for a normal congruence in powers of t, we find that the absolute term vanishes identically, while the coefficient Ω_1 of the first power gives

$$\begin{aligned}
&(R - pS)\bar{F}_p - \{1 + pS(S + pR)\}G_p - 2R^2(S + pR)\bar{F} + pG - 2R\bar{F} \\
&\quad + R(S + pR)\bar{G} - \{p + R(S + pR)\}\{p\bar{G}_p - (S + pR)\bar{F}_p - G\} = 0.
\end{aligned} \tag{20}$$

Eliminating S by means of (18), and substituting the values

$$\bar{F}_p = F_p + RF_q, \qquad \bar{G}_p = G_p + RG_q,$$

derived from (19′), we may reduce (20) to

$$\begin{aligned}
2pG &= (1 + p^2 + q^2)G_p + R[(1 + p^2 + q^2)(F_p - G_q) - 2pF + 2qG] \\
&\qquad + R^2[(1 + p^2 + q^2)F_q - 2qF] = 0.
\end{aligned} \tag{20'}$$

If this is to hold for all curves (16) it must be an identity. For the curve (16) can be so selected that x, y, z, p, q, R receive arbitrarily assigned numerical values. Hence we have the three partial differential equations

$$\begin{aligned}
(1 + p^2 + q^2)F_q - 2qF &= 0, \\
(1 + p^2 + q^2)G_p - 2pG &= 0, \\
(1 + p^2 + q^2)(F_p - G_q) - 2pF + 2qG &= 0.
\end{aligned} \tag{20''}$$

This set is seen to be equivalent to the last three of the equations (11). The general solution is

$$F = (\psi - p\phi)(1 + p^2 + q^2), \qquad G = (\chi - p\phi)(1 + p^2 + q^2),$$

involving three arbitrary functions of x, y, z. The quadruply infinite system (1) thus takes the form

$$(21)\quad y'' = (\psi - y'\phi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\phi)(1 + y'^2 + z'^2).$$

This is recognized to be a *velocity system*, according to the terminology of the author's earlier paper. It is characterized geometrically by the property that the circles of curvature constructed at any point for those curves of the system which pass through that point form a bundle, i. e., have another point in common (property A). We have made use of merely one term in the complete condition for a normal congruence. Hence our result is

THEOREM IV. *Every velocity system has the property that those members of it which meet an arbitrarily selected base curve* orthogonally are also orthogonal to some neighboring surface (necessarily tubular). Conversely, if a quadruply infinite system has this property for all base curves it will be of the velocity type.*

The converse result remains valid if instead of all base curves we take a manifold which allows x, f, g, f', g' to take arbitrary values. The smallest number of base curves that would be sufficient is ∞^4. Two adequate manifolds of curves are made use of in the following

THEOREM V. *The converse result of theorem* IV *remains valid if the only bases employed are the ∞^4 straight lines of space; or, the ∞^4 curves of the system itself.*

§ 4. CASE WHERE THE BASE IS A POINT.

When the base is a point orthogonality means merely incidence. Consider then the ∞^2 curves of the system (1) which pass through a given point (x, y, z). Define the initial direction by $y' = p$, $z' = q$, where p and q are now arbitrary; the resulting congruence takes the form

$$(22)\quad \begin{aligned} X &= x + t, \\ Y &= y + pt + \tfrac{1}{2}Ft^2 + \tfrac{1}{6}Mt^3 + \cdots, \\ Z &= z + qt + \tfrac{1}{2}Gt^2 + \tfrac{1}{6}Nt^3 + \cdots, \\ Y' &= p + Ft + \tfrac{1}{2}Mt^2 + \cdots, \\ Z' &= q + Gt + \tfrac{1}{2}Nt^2 + \cdots. \end{aligned}$$

Our jacobians, calculated to terms of the third order in t, instead of merely

* This curve may or may not belong to the given system.

the first order as hitherto, are found to be

$$(Y'XY) = \tfrac{1}{2}F_q t^2 + \tfrac{1}{3}M_q t^3 + \cdots,$$

$$(Z'ZY) = \tfrac{1}{2}G_p t^2 + \tfrac{1}{3}N_p t^3 + \cdots,$$

$$-(Z'YZ) = qt + \tfrac{1}{2}\{pG_p + qF_p + 2qG_q\}t^2 + \tfrac{1}{2}\{q(F_pG_q - F_qG_p) - G_1 + qN_q + \tfrac{2}{3}pN_p + \tfrac{1}{3}qM_p\}t^3 + \cdots,$$

$$-(Y'YZ) = pt + \tfrac{1}{2}\{pG_q + qF_q + 2pF_p\}t^2 + \tfrac{1}{2}\{p(F_pG_q - F_qG_p) - F_1 + pM_p + \tfrac{2}{3}qM_q + \tfrac{1}{3}pN_q\}t^3 + \cdots,$$

where

$$(22') \qquad F_1 \equiv F_x + pF_y + qF_z, \qquad G_1 \equiv G_x + pG_y + qG_z.$$

Expanding the condition for a normal congruence, we find that there are no terms independent of t, the coefficient Ω_1 of t vanishes identically, the coefficient Ω_2 of t^2 gives

$$(23) \quad (1+q^2)F_q - (1+p^2)G_p + pq(F_p - G_q) + 2(pG - qF) = 0,$$

and the coefficient Ω_3 of t^3 gives

$$(24') \quad \begin{aligned}(1+q^2)M_q - (1+p^2)N_p + pq(M_p - N_q) + 3(pG - qF)(F_p + G_q)\\ + 3p(G_x + pG_y + qG_z) - 3q(F_x + pF_y + qF_z) = 0.\end{aligned}$$

By means of (23) and (22′) the equation (24′) may be written in the more convenient form

$$(24) \quad (1+q^2)F_{1q} - (1+p^2)G_{1p} + pq(F_{1p} - G_{1q}) + 3(pF_1 - qG_1) = 0.$$

In order that a quadruply infinite system

$$(1) \qquad y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z')$$

shall have the property that the ∞^2 *curves through an arbitrary point form a normal congruence, it is necessary that the functions* $F(x, y, z, p, q)$ *and* $G(x, y, z, p, q)$ *fulfill the conditions* (23) *and* (24).

As stated above, the coefficient of the first power of t vanishes identically; in fact, if we think of the point as a small sphere, the curves passing through it, for any system (1), may be considered as meeting the sphere twice orthogonally. Equation (23) is the condition for approximation of the second degree; while (24) and (23) are the conditions for approximation of the third degree.

It is easily verified that all velocity systems

$$y'' = (\psi - y'\phi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\phi)(1 + y'^2 + z'^2),$$

satisfy equation (23). Substituting in (24), however, we find a restriction, which reduces to

$$\psi_z - \chi_y + p(\chi_x - \phi_z) + q(\phi_y - \psi_x) = 0.$$

Since this is to hold identically, the result is a natural family. In this case we know that the congruences are actually of the normal type. Hence

THEOREM VI. *The only velocity systems which have the property that the curves through an arbitrary point form a normal* congruence are the natural families.*

By combining this with the first part of theorem V, we obtain the following

THEOREM VII. *If a quadruply infinite system is such that the congruences constructed by taking as base every straight line and every point of space are normal, then the same will be true when the base is any surface or any curve, and the system must be natural.*†

It is easily seen that conditions (23) and (24), expressing the approximate fulfillment of the condition for a normal congruence in the case of base points, are not of themselves sufficient to characterize natural families. In fact the general solution of these two partial equations for the determination of F and G, one of the first and the other of the second order, would involve arbitrary functions of four arguments, and thus would include the natural type, which involves a function of only three arguments, as a very special case.

This question now arises: Can a system of ∞^4 curves, other than the known natural type, be such that the ∞^2 curves through an arbitrary point form a normal congruence? That the answer is in the affirmative we show by an example. The direct attack on the problem, according to the method hitherto pursued in this paper, would lead to great difficulties; it would require the discussion of the endless set of equations, of successively higher order, obtained by adjoining to (23) and (24) the conditions derived from the powers of t beyond the third.

To obtain an appropriate example, we consider the systems whose equations are of the special form

$$y'' = F(p, q), \qquad z'' = G(p, q), \tag{25}$$

the variables x, y, z being absent. Excluding the case where F and G both vanish, such systems are not natural. They are characterized by the fact that they admit the group of all translations. It will therefore be sufficient to test the character of the curves through a single point, for instance, the origin.

For the functions appearing in (25), we observe, from (22′), that the derived functions F_1 and G_1 vanish and hence condition (24) is satisfied. Condition

* It is sufficient to demand the third degree of approximation.

† It is sufficient to demand the first approximation in the case of the straight lines, and the third in the case of the points.

(23) may be fulfilled by taking either F or G at random and calculating the other function; this can easily be effected by quadratures. Thus all systems allowing the complete group of translations and such that the curves through an arbitrary point fulfill the normal condition up to terms of the third order may be obtained by quadratures.

It will be sufficient to consider the special case where G is assumed to vanish, that is, where the curves are assumed to lie in planes perpendicular to the xz plane. The systems are then found to be of the form

$$y'' = p^2 f\left(\frac{1+q^2}{p^2}\right), \qquad z'' = 0, \tag{26}$$

where f is an arbitrary function of one argument.

Consider now the particular system

$$y'' = p^2, \qquad z'' = 0, \tag{27}$$

included in this form. The integral curves are

$$y = \log \frac{a_1}{bx + a}, \qquad z = bx + b_1,$$

involving four arbitrary constants. The ∞^2 curves passing through the origin are

$$y = \log \frac{a}{x + a}, \qquad z = bx.$$

The differential equations of this congruence are

$$\frac{dx}{x} = \frac{dy}{e^y - 1} = \frac{dz}{z};$$

the corresponding total equation,

$$xdx + (e^y - 1)dy + zdz = 0,$$

is actually integrable, giving as the ∞^1 orthogonal surfaces

$$x^2 + z^2 + 2(e^y - y) = \text{constant}.$$

Our example (27), which evidently is not of the natural type, enables us to state

THEOREM VIII. *A quadruply infinite system may have the property that the curves through an arbitrary point form a normal congruence, without necessarily being a natural system.*

Not all systems of the form (26) have the required property. For consider the particular system

$$y'' = \tfrac{1}{2}(1 + q^2), \qquad z'' = 0, \tag{28}$$

whose integral curves are the parabolas

$$y = (1 + b^2)x^2 + ax + a_1, \qquad z = bx + b_1.$$

The congruence through the origin is

$$y = (1 + b^2)x^2 + ax, \qquad z = bx,$$

with the differential equations

$$\frac{dx}{x} = \frac{dy}{y + x^2 + z^2} = \frac{dz}{z}.$$

Since the corresponding total equation

$$xdx + zdz + (y + x^2 + z^2)dy = 0$$

is not integrable, the congruence is not normal.

This example shows that a system may be such that the curves through an arbitrary point form a congruence which is normal to the third degree of approximation without being exactly normal.

§ 5. The family of surfaces related to an arbitrary system.

Given an arbitrary quadruply infinite system of curves

$$y'' = F(x, y, z, p, q), \qquad z'' = G(x, y, z, p, q), \tag{29}$$

where p and q stand for y' and z' respectively, it is evident that in general there will exist no surface such that the ∞^2 curves of the system which meet the surface orthogonally form a normal congruence. If, however, it is merely required that the ∞^2 curves shall be of the normal character to the first approximation, that is, shall be orthogonal to some consecutive surface, then solutions will always exist.

The requisite condition on the surface

$$z = f(x, y) \tag{30}$$

is furnished by equation (10′). In the present discussion F and G are considered as the given functions and f is to be found. Rearranging the equation in terms of second derivatives of f, we write it

$$B_0 + B_1 f_{xx} + B_2 f_{xy} + B_3 f_{yy} = 0, \tag{31}$$

where the coefficients are functions of x, y, z, p, q, determined by F and G as follows:

$$\begin{aligned}
qB_0 &\equiv (1 + p^2 + q^2)(qF_x - F_z - pG_x + G_y),\\
B_1 &\equiv (1 + p^2 + q^2)(pqF_p + q^2F_q - p^2G_p - pqG_q) + 2(p^2 + q^2)(pG - qF),\\
B_2 &\equiv (1 + p^2 + q^2)(2pG_p + qG_q - qF_p) + 2pqF - 2(q^2 + 2p^2)G,\\
B_3 &\equiv 2pG - (1 + p^2 + q^2)G_p.
\end{aligned} \tag{31′}$$

Here p and q take the place of P and Q respectively in the discussion of § 2, so that we must substitute

$$p = \frac{f_y}{f_x}, \qquad q = -\frac{1}{f_x}. \tag{32}$$

After the substitution (30), (32), the coefficients B_i become known functions of x, y, f, f_x, f_y. Denoting these new functions by $\bar{B}_i$, we have

THEOREM IX. *With any quadruply infinite system of curves* (29), *there is associated a certain family of surfaces, namely, the solutions of the Monge partial differential equation of the second order*

$$\bar{B}_0 + \bar{B}_1 f_{xx} + \bar{B}_2 f_{xy} + \bar{B}_3 f_{yy} = 0, \tag{33}$$

where the coefficients are found from F and G by differentiations and eliminations. The ∞^2 curves of the given system which are orthogonal to any surface of this family are also orthogonal to some consecutive surface, that is, form an approximate normal congruence.

In case the system is of the natural type, equation (33) is of course an identity in f. In fact the set of equations obtained by putting the four coefficients $\bar{B}_i$, or the corresponding coefficients B_i, separately equal to zero is equivalent to the set (11) whose solution yielded the natural systems in the discussion of § 2.

When will (33) reduce to an equation of the first order? The conditions for this are

$$B_1 = B_2 = B_3 = 0,$$

which, in the notation of equations (11), are seen to be equivalent to

$$C_1 = C_2 = C_3 = 0.$$

The system must therefore be of the velocity type

$$y'' = (\psi - p\phi)(1 + p^2 + q^2), \qquad z'' = (\chi - q\phi)(1 + p^2 + q^2).$$

In this case, (33) reduces to

$$(\psi_z - \chi_y) f_x + (\chi_x - \phi_z) f_y - (\phi_y - \psi_x) = 0. \tag{34}$$

The integral surfaces are generated by the curves (characteristics)

$$\frac{dx}{\psi_z - \chi_y} = \frac{dy}{\chi_x - \phi_z} = \frac{dz}{\phi_y - \psi_x}. \tag{35}$$

These are the curl lines connected with the vector field* ϕ, ψ, χ.

THEOREM X. *For a general system of curves the associated equation* (33) *is actually of the second order, the only exceptions arising as follows: for*

* This represents the force which generates the velocity system.

a natural system, the equation holds identically; for any other velocity system the equation is merely of the first order. In the first case the associated family of surfaces is thus larger, and in the second smaller, than usual.

This deficiency refers to proper surfaces. We have already seen that, in the velocity type, an arbitrary *curve* employed as base gives rise to a congruence of approximately normal character. Thus, in addition to the proper surfaces (34), all curves may be regarded as limiting cases belonging to our family (33). In general we may with LIE replace the concept surface by the more general concept of union (*Elementverein*), that is, manifold of ∞^2 surface elements in united position. We might in fact have started out from this point of view and thereby should have avoided the separate discussion of curves and points. However this would have required the introduction of appropriate systems of homogeneous coördinates for differential surface elements.

One of our former results may be stated in the form: If the equation (33) is satisfied by all quadrics, or merely by the paraboloids

$$z = a + bx + cy + a_1 x^2 + b_1 xy + c_1 y^2,$$

it must hold identically, so that the corresponding system must be natural. We now discuss the analogous question for spheres.

The ∞^4 spheres of space may be defined by the set of partial differential equations

$$\frac{f_{xx}}{1+f_x^2} = \frac{f_{xy}}{f_x f_y} = \frac{f_{yy}}{1+f_y^2}. \tag{36}$$

If then the associated family of surfaces is to include the spheres, equation (33) must be satisfied in consequence of (36). Eliminating f_{xx} and f_{yy} by means of (36), and making use of (32), we find that (33) reduces to

$$pB_0 + \{(1+q^2)B_1 + pB_2 + (p^2+q^2)B_3\} f_{xy} = 0.$$

This must be true for all values of f_{xy}; hence

$$B_0 = 0, \qquad (1+q^2)B_1 + pB_2 + (p^2+q^2)B_3 = 0,$$

that is,

$$qF_x - F_z - pG_x + G_y = 0, \tag{37}$$

$$(1+q^2)F_q - (1+p^2)G_p + pq(F_p - G_q) + 2(pG - qF) = 0. \tag{38}$$

The latter is recognized as the condition that all points shall give rise to approximately (of first degree) normal congruences. The other equation, (37), is easily seen to be the condition that (33) shall be satisfied by all planes.

If for each point and for each plane as base, the congruence of orthogonal curves is approximately of normal character, then the same is true for an arbitrary sphere. The systems of this sort are defined by equations (37), (38).*

* The result involves arbitrary functions of four arguments, hence is more general than the natural type.

It is easily seen that the only velocity systems satisfying condition (37) are those of the natural type. Making use of theorem V, we thus have

THEOREM XI. *If a system of curves is such that the ∞^2 curves orthogonal to any straight line or to any plane are orthogonal to some neighboring surface then the system must be natural.*

If for an arbitrary curve as base the congruence of orthogonal curves is (exactly) normal, then the same must be true for any point as base. For the curve may be made to shrink up to the point. Hence, by combining the converse result of theorem IV with theorem VI, we have

THEOREM XII. *Natural systems are characterized by the requirement that the curves of the system which are orthogonal to an arbitrarily selected curve (inside or outside the system) form a normal congruence.*

It is even sufficient to take as base curves merely the ∞^6 circles of space.

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THE INTRODUCTION OF IDEAL ELEMENTS AND A NEW DEFINITION OF PROJECTIVE n-SPACE*

BY

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Introduction.

The ideal elements of projective geometry are usually introduced by means of the parallel and congruence axioms. The idea of defining the ideal elements without the assumption of the parallel axiom is due to KLEIN.† It was developed by PASCH;‡ by SCHUR;§ by BONOLA;‖ and by VEBLEN.¶ In all of these developments a three-dimensional geometry is assumed. The problem of defining the ideal elements in a plane geometry satisfying only order relations is closely connected with the problem of finding the necessary and sufficient condition that a plane may be a part of a three-space in which the axioms of order are satisfied. This condition is stated by HILBERT** to be the validity of the Desargues theorem in the plane. The Desargues theorem may be proved in a three-dimensional space satisfying only order relations, but can not be proved in the corresponding plane geometry, without additional assumptions, e. g., of the parallel and congruence axioms.

In this paper plane axioms of order will be assumed in the form given them by VEBLEN, the undefined elements being taken as the *point*, and a relation among points, called *order*. The first eight axioms are identical, except for notation, with his. Another axiom, one of closure, is then introduced, limiting the set of *points* considered to a plane. Two more axioms are then introduced, forms of the Desargues theorem, and of its converse, in terms of the set of points satisfying only order relations.

* Presented to the Society (Chicago) under a different title, April 22, 1905.

† *Ueber die sogenannte Nicht-Euklidische Geometrie*, Mathematische Annalen, vol. 6 (1872), p. 132.

‡ *Vorlesungen über neuere Geometrie*, pp. 40–72.

§ *Ueber die Einführung der sogenannten idealen Elemente in die projective Geometrie*, Mathematische Annalen, vol. 39 (1891), pp. 113–128.

‖ *Sulla introduzione degli enti improprii in Geometria projettiva*, Giornale di Matematiche, vol. 38 (1900), p. 105.

¶ *A system of axioms for geometry*, these Transactions, vol. 5 (1904), pp. 343–384.

** *Grundlagen der Geometrie, Festschrift* (1899), § 30.

On the basis of these eleven axioms, a new set of elements, called *pencils*, including in particular the pencils of lines centered at points, are defined, and their properties deduced. This set of pencils is shown to satisfy the incidence axioms and the general Desargues theorem of projective geometry, but not necessarily the Pascal theorem, well known * to be independent of the Desargues theorem, and even stronger assumptions. It is also shown that if we leave out of this set of pencils a subset called a *range*, the remaining elements satisfy the original axioms, I to XI.

In Section IV, a new set of elements called three-points, including in particular the projective points (i. e. pencils) of the plane, is defined, and this set is shown to form a projective three-space containing the plane set as a subset.

In Section V, an extension is made to n-space by generalizing the method of Section IV.

The notation used, though sometimes cumbersome, is very convenient in giving a concise expression for the proofs; and by displaying the configurational character of many of the figures used, it has been found of value in devising the proofs of the theorems. The *points* which are characterized by Axioms I–XI will be denoted by the symbol P_k, the P simply indicating that the element belongs to the class of points, and the subscript being a mark, not necessarily a single digit or letter, and the order of the digits or letters in the subscript immaterial, used to distinguish the various points; for instance, conveniently, the notation P_{xy}, $x, y = 1, \cdots, 5$, $x \neq y$, and $P_{xy} = P_{yx}$, will be used for the configuration of ten points called the Desargues configuration; the notation l_k will be used for lines, the k again being a mark of one or more digits or letters, the order of which is immaterial; the notation l_{123} will be used where convenient for a line containing points P_{12}, P_{13}, P_{23}. Such implications of the notation, while useful, are avoided in the statement of the proofs, as far as possible, but the notation is used consistently, even the axioms themselves being stated in it, in order to accustom the reader to it. Other forms of notation will be explained, as they occur.

The independence of the Axioms X and XI from Axioms I–IX may be shown from the Non-Desarguesian geometry of MOULTON † if the "line of break" of this geometry is a line of our points. The relation of Axioms X and XI on the basis of Axioms I–IX has not been determined.

This problem was first suggested to the author by Professor O. VEBLEN, to whom, and to Professor E. H. MOORE, he tenders his grateful acknowledgment for many helpful suggestions and criticisms.

* *Grundlagen der Geometrie, Festschrift*, § 31. The form of Pascal's theorem here referred to is, of course, that for the degenerate conic. (Theorem of Pappus.)

† *A simple Non-Desarguesian geometry*, Transactions of the American Mathematical Society, vol. 3 (1902), pp. 192–195.

The Axioms and Definitions.

AXIOM I. There exist at least two distinct points.

AXIOM II. If points P_1, P_2, P_3 are in the order $P_1P_2P_3$, they are in the order $P_3P_2P_1$.

AXIOM III. If points P_1, P_2, P_3 are in the order $P_1P_2P_3$, they are not in the order $P_2P_3P_1$.

AXIOM IV. If points P_1, P_2, P_3 are in the order $P_1P_2P_3$, then P_1 is distinct from P_3.

AXIOM V. If P_1 and P_2 are any two distinct points, there exists a point P_3, such that P_1, P_2, P_3 are in the order $P_1P_2P_3$.

If P_1 and P_2 stand for the same points, we write $P_1 = P_2$; if for distinct points, $P_1 \neq P_2$.

DEFINITION 1. The line P_1P_2, or conveniently, l_{12}, consists of P_1 and P_2 and all points P_k in any of the orders, $P_1P_kP_2$, $P_1P_2P_k$, $P_kP_1P_2$. The subscripts 1, 2 in the notation l_{12} refer to the points by which the line is determined.

The points P_k in the order $P_1P_kP_2$ constitute the *segment* P_1P_2.

AXIOM VI. If points P_3 and $P_4 (P_3 \neq P_4)$ lie on the line l_{12} (i. e., P_1P_2) then P_1 lies on the line l_{34} (i. e., P_3P_4).

AXIOM VII. If there exist three distinct points, there exist three points P_1, P_2, P_3 not in any one of the orders $P_1P_2P_3$, $P_1P_3P_2$, or $P_3P_1P_2$.

DEFINITION 2. Points of the same line are *collinear*. Three distinct non-collinear points, P_1, P_2, P_3 are the *vertices* of a *triangle* $P_1P_2P_3 = T_{123}$, whose *sides* are the segments P_1P_2, P_1P_3, P_2P_3, and whose *boundary* consists of its vertices and the points on its sides. Two triangles having the same vertices are identical.

AXIOM VIII. (Triangle transversal axiom.) If three distinct points $P_1P_2P_3$ do not lie on the same line, and P_4 and P_5 are two points in the orders $P_2P_3P_4$ and $P_3P_5P_1$, respectively, then a point P_6 exists in the order $P_1P_6P_2$, and such that P_4, P_5, P_6 lie on the same line.

DEFINITION 3. If P_1, P_2, P_3 are three non-collinear points, the *plane* $P_1P_2P_3$ consists of all the points collinear with any two points of the sides of the triangle $P_1P_2P_3$.

AXIOM IX. The set of points P_k here considered constitutes a plane.

DEFINITION 4. A triangle is said *to lie on three lines* if it has a vertex on each of these lines.

DEFINITION 5. Three distinct lines l_{340}, l_{341}, l_{342} are *in pencil* if there exists on these lines a pair of triangles having no side common $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$, P_{30} and P_{40}, $P_{30} \neq P_{40}$, being points of the line l_{340}, etc., such that the lines $P_{30}P_{31}$ and $P_{40}P_{41}$ meet* in a point P_{01}, the lines $P_{31}P_{32}$ and

* Two lines *meet* only if distinct.

$P_{41}P_{42}$ meet in a point P_{12}, and the lines $P_{30}P_{32}$ and $P_{40}P_{42}$ meet in a point P_{02}, and such that the points P_{01}, P_{02}, P_{12} are collinear.

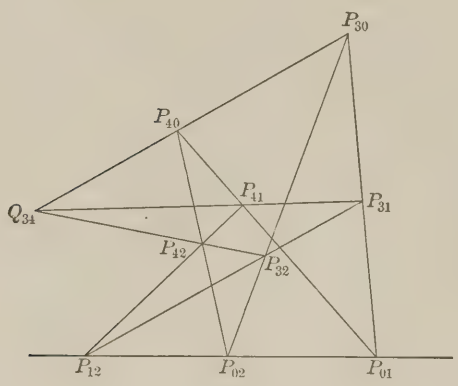

FIG. 1.

The figure of Definition 5 (see Fig. 1) will be called a Desargues configuration, and the notation

$$(1)\qquad \begin{array}{ccc} P_{12} & P_{02} & P_{01} \\ \hline P_{40} & P_{41} & P_{42} \\ P_{30} & P_{31} & P_{32} \\ & Q_{34} & \end{array}$$

will be used when by hypothesis the triads of points, $P_{12}P_{02}P_{01}$, $P_{12}P_{41}P_{42}$, $P_{12}P_{31}P_{32}$, $P_{02}P_{40}P_{42}$, $P_{02}P_{30}P_{32}$, $P_{01}P_{40}P_{41}$, $P_{01}P_{30}P_{31}$ are collinear, and the triads $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$ are vertices of triangles; by the above definition, then, the lines $P_{40}P_{30}$, $P_{41}P_{31}$, and $P_{42}P_{32}$ will be *in pencil*, Q_{34}.

DEFINITION 6. The set of all lines *in pencil* with a given pair, together with this pair, constitute a pencil, determined by the two lines, and by Theorem VI, determinable by *any* two lines in it. The notation Q_k will be used for pencils, the mark k, as before, being any convenient symbol of one, two, or more digits or letters, these usually referring to the lines by which it is determined, and the order of the digits of the subscript being immaterial as far as denoting the pencil is concerned. A line is said to *belong* to a pencil if it is one of the lines of the pencil.

AXIOM X. If l_{340}, l_{341}, l_{342} are three lines in a pencil, and $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$ are any two triangles on these lines, P_{30} and P_{40} lying on l_{340} (but not necessarily distinct), etc., such that the lines

$$\begin{array}{lll} P_{30}P_{31} = l_{301} & \text{and} & P_{40}P_{41} = l_{401} \text{ meet in a point } P_{01}, \\ P_{31}P_{32} = l_{312} & \text{and} & P_{41}P_{42} = l_{412} \text{ meet in a point } P_{12}, \\ P_{30}P_{32} = l_{302} & \text{and} & P_{40}P_{42} = l_{402} \text{ meet in a point } P_{02}, \end{array}$$

and P_{01}, P_{12}, P_{02} are distinct and determinate, then P_{01}, P_{12}, and P_{02} are collinear.

The figure of Axiom X (see Fig. 2) will be represented by the notation

$$(2)\qquad \begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline P_{12} & P_{02} & P_{01} \end{array}$$

which indicates that by hypothesis the lines $P_{30}P_{40}$, $P_{31}P_{41}$, $P_{32}P_{42}$ are in pencil Q_{34}, and the triads of points P_{30} $P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$ are the vertices of triangles on these lines, and the triads of points $P_{30}P_{31}P_{01}$, $P_{40}P_{41}P_{01}$, $P_{30}P_{32}P_{02}$, $P_{40}P_{42}P_{02}$, $P_{31}P_{32}P_{12}$, $P_{41}P_{42}P_{12}$, exist and are respectively collinear. Axiom X states, then, that the points $P_{12}P_{02}P_{01}$ are collinear.

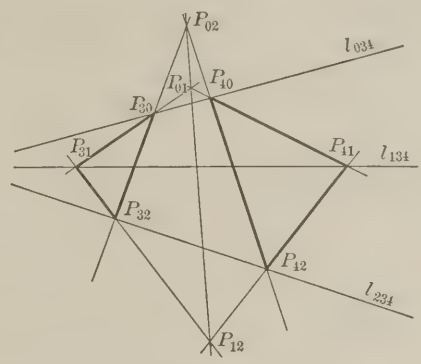

FIG. 2.

DEFINITION 7. Three distinct pencils Q_{01}, Q_{12}, Q_{02} are *in range*, if there exist two triangles, $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$, whose sides are all distinct and such that $P_{30}P_{31}$ and $P_{40}P_{41}$ belong to Q_{01}, $P_{31}P_{32}$ and $P_{41}P_{42}$ belong to Q_{12}, and $P_{30}P_{32}$ and $P_{40}P_{42}$ belong to Q_{02}; and such that the lines $P_{30}P_{40}$, $P_{31}P_{41}$, and $P_{32}P_{42}$ are in a pencil Q_{34}.

The notation used for the figure of Definition 7 will be identical with that of Axiom X, except that for P_{12}, P_{02}, P_{01} corresponding symbols Q_{12}, Q_{02}, Q_{01} will be used, thus:

$$\begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline Q_{12} & Q_{20} & Q_{01} \end{array}$$

It will be observed that whenever the hypothesis of Axiom X is fulfilled, so is the hypothesis in Definition 7, and to the points $P_{12}P_{02}P_{01}$ of Axiom X correspond pencils $Q_{12}Q_{02}Q_{01}$ which by Definition 7 are in range; for whether the pairs of lines $P_{30}P_{32}$ and $P_{40}P_{42}$, $P_{31}P_{32}$ and $P_{41}P_{42}$, and $P_{30}P_{31}$ and $P_{40}P_{41}$, determine points or not, they *do* determine *pencils* (cf. Corollary to Theorem VII).

AXIOM XI. If three pencils Q_{01}, Q_{02}, Q_{12} are in range, *every* pair of triangles $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$, such that

lines $P_{30}P_{31}$ and $P_{40}P_{41}$ belong to Q_{01},
lines $P_{31}P_{32}$ and $P_{41}P_{42}$ belong to Q_{12},
lines $P_{30}P_{32}$ and $P_{40}P_{42}$ belong to Q_{02},

has the property that the lines $P_{30}P_{40}$, $P_{31}P_{41}$ and $P_{32}P_{42}$ are in a pencil Q_{34}.

The figure of Axiom XI will be denoted by the notation:

$$\begin{array}{ccc} Q_{12} & Q_{20} & Q_{01} \\ \hline P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ & Q_{34} & \end{array}$$

The inversion of the notational form is to call attention to the change in the hypothesis. That Definitions 6 and 7 and Axioms X and XI all involve the Desargues configuration explains the similarity of the notation.

DEFINITION 8. The set of all pencils in range with a given pair of ranges, together with this pair, constitutes a *range*. The notation r_k will be used for ranges, the mark k being again any convenient digit or letter, or combination of them in which the order is immaterial, so far as denoting the range is concerned.

From Axioms I–IX, such theorems as the following may be deduced (cf. VEBLEN, loc. cit.):

a) Between any two distinct points of a line there is another point (P_1 is between P_2 and P_3 if they are in the order $P_2 P_1 P_3$).

b) If a line, not a side of a triangle, meets the perimeter of the triangle in one point not a vertex, then it meets the perimeter in another distinct point.

c) To any four points of a line the notation P_1, P_2, P_3, P_4, can be so assigned that they are in the order, $P_1 P_2 P_3 P_4$, i. e. in the orders, $P_1 P_2 P_3$, $P_1 P_3 P_4$, $P_1 P_2 P_4$, and $P_2 P_3 P_4$.

d) Any line divides the plane into two regions which may be denoted by $+$ and $-$, and have the property that any segment joining a point of $+$ to a point $-$ contains a point of the line.

e) Two non-intersecting lines divide a plane into three regions, which can be conveniently denoted by $++$, $+-$, $--$.

f) Two intersecting lines divide a plane into four regions, which may be denoted by $++$, $+-$, $-+$, $--$.

g) Three lines forming an actual triangle divide the plane into seven regions, which may be denoted by $+++$, $++-$, $+-+$, $+--$, $--+$, $-++$, $-+-$.

SECTION I.

Consequences of Axioms I–X.

Our main object in Section I is to show that a line is uniquely determined by a point and the fact that it is in pencil with two other given lines neither of which passes through the point. This is done without any use of Axiom XI.

THEOREM I. If the lines l_{034}, l_{134}, l_{234} are in pencil Q_{34}, and l_{034} and l_{134} meet in a point P_{30}, then l_{234} passes through P_{30}. (Fig. 3.)

Proof. Suppose the theorem untrue. Take P_{32}, any point of l_{234} (P_{32} not a point of l_{034} or l_{134}). Take on l_{134} points P_{71} and P_{61} each distinct from P_{30}, not on l_{234}, in the order $P_{71} P_{30} P_{61}$, and such there is no point of l_{234} on the segment $P_{61} P_{71}$. The line l_{034} cuts the side $P_{71} P_{61}$ of the triangle $P_{71} P_{61} P_{32}$. Hence it cuts either the segment $P_{32} P_{71}$ or the segment $P_{32} P_{61}$ in a point P_t. Call the side cut $P_{32} P_{31}$, so that either P_{71} or P_{61} is P_{31}, and call the other P_{81}. Choose a point P_{02} in the order $P_{32} P_{30} P_{02}$, and a point P_y in the order

$P_{31}P_{32}P_y$. The segment $P_{02}P_y$ contains a point of the segment $P_{32}P_{81}$ or does not. If it does, use P_y as P_{01} below. If not, take a point P_r on the segment $P_{32}P_{81}$, and the line $P_{02}P_r$ will meet the segment $P_{32}P_y$ in a point

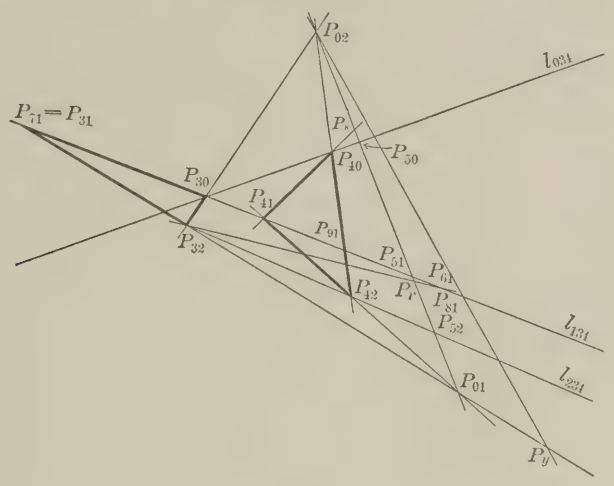

FIG. 3.

P_{01}, which is in the order $P_{02}P_rP_{01}$. Then the segment $P_{02}P_{01}$ will meet l_{034}, l_{134}, and l_{234} in points P_{50}, P_{51}, P_{52}, respectively, in the order $P_{02}P_{50}P_{51}P_rP_{52}P_{01}$. Take P_{41} in the order $P_{30}P_{41}P_{51}$. Then the segment $P_{41}P_{01}$ meets l_{234} in a point P_{42} of the segment $P_{32}P_{52}$. The segment $P_{42}P_{02}$ contains points, P_{40} on the line l_{034}, and P_{91} on the line l_{134}, in the order $P_{02}P_{40}P_{91}P_{42}$, and P_{91} is in the order $P_{30}P_{41}P_{91}P_{51}$. Therefore lines $P_{41}P_{40}$ and $P_{02}P_{01}$ determine a point P_s in the orders $P_{41}P_{40}P_s$ and $P_{02}P_sP_{50}$. We have then the Desargues configuration

$$\begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline P_{01} & P_{02} & P_{41}, \end{array}$$

in which the hypothesis of Axiom X is satisfied, but P_{01}, P_{02}, P_{41} are not collinear. Hence the assumption that l_{234} does not pass through P_{02} is contradictory to Axiom X. Hence the theorem is true.

THEOREM II. Any three lines l_{034}, l_{134}, l_{234} incident at a point P_{34} are in a pencil Q_{34}. (Fig. 4.)

Proof. Choose $P_{30} \neq P_{34}$ on l_{034} and $P_{61} \neq P_{34}$ on l_{134}, and $P_{32} \neq P_{34}$ on l_{234}, in the order $P_{32}P_{61}P_{30}$. Then choose P_{02} in the order $P_{32}P_{61}P_{30}P_{02}$; P_{31} in order $P_{34}P_{61}P_{31}$; and P_{12} in the order $P_{31}P_{32}P_{12}$, and such that segment $P_{32}P_{12}$ contains no point of line $P_{34}P_{02}$. Then, by consideration of

triangle $P_{02}P_{32}P_{12}$, $P_{02}P_{12}$ contains points P_{50}, P_{51}, in order $P_{02}P_{50}P_{51}P_{12}$, and P_{50} is on the segment $P_{34}P_{30}$, and P_{51} is on the segment $P_{34}P_{61}$. $P_{31}P_{30}$ meets $P_{02}P_{12}$ in a point P_{01}, in the order $P_{02}P_{01}P_{50}$. Take P_{41} on the segment $P_{34}P_{51}$. The segment $P_{41}P_{01}$ contains P_{40} on the segment $P_{34}P_{50}$. $P_{02}P_{40}$ meets $P_{41}P_{12}$ in a point P_{42}. Then $P_{32}P_{42}$ is a line in pencil with l_{034} and l_{134}. By Theorem I, the line $P_{32}P_{42}$ must pass through P_{34}, but this line contains the points P_{32} and P_{34}, and hence is the line l_{234}.

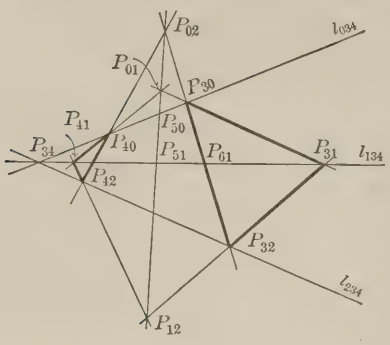

FIG. 4.

COROLLARY. If three lines are in pencil they meet in a point, or else no two of them have a point in common.

DEFINITION 8. A line l_k is *between* the non-intersecting lines l_m and l_n, if every segment P_mP_n (P_m on l_m, P_n on l_n) contains a point P_k of l_k.

DEFINITION 9. If three lines are in pencil and no two of them intersect, we will call the pencil *non-intersecting*.

THEOREM III. If l_{034}, l_{134}, l_{234} are three lines of a *non-intersecting* pencil, Q_{34}, then one of them is always *between* the other two.

LEMMA. If l_k, l_m, l_n are three non-intersecting lines, and if l_m cuts *one* segment joining points of l_k and l_n, then it cuts every such segment, and hence is between l_k and l_n. The proof follows readily from Axiom VIII and is omitted.

Proof of Theorem III. (Fig. 5.) Take P_{30} on l_{034}, P_{31} on l_{134}, and P_{32} on l_{234}. Then not more than one of the lines l_{034}, l_{134}, l_{234} enters the triangle $P_{30}P_{31}P_{32}$, since by hypothesis the pencil is non-intersecting. Then, if *one* does, it cuts the opposite side of the triangle and hence by the lemma it is between the other two lines. We will now prove that the case of all three of the lines l_{034}, l_{134}, l_{234} exterior to the triangle can not occur. Suppose that it did occur. Let the plane be divided by the sides of the triangle into seven regions as in Theorem g, p. 146.

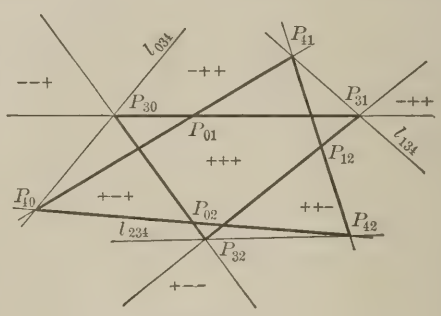

FIG. 5.

Choose P_{40} on l_{034} in $+-+$, and P_{41} on l_{134} in $-++$, and P_{42} on l_{234} in $++-$. Then $P_{40}P_{41}$ meets segment $P_{30}P_{31}$ in a point P_{01}, and $P_{41}P_{42}$ meets segment $P_{31}P_{32}$ in a point P_{12}, and $P_{42}P_{40}$ meets segment $P_{32}P_{30}$ in a point P_{02} and by hypothesis we have (2); but the points P_{12}, P_{02}, P_{01} are not collinear since they are on distinct sides of the triangle. Hence our supposition that no one of the lines enters the triangle is impossible.

DEFINITION 10. If three lines of a pencil are incident at a point, then their common point is the *center* of the pencil.

THEOREM IV. If l_{034}, l_{134}, and l_{234} are three lines in pencil, and such that, if the pencil be non-intersecting,* l_{134} is between l_{034} and l_{234}, and l_{934} ($l_{934} \neq l_{234}$) is a line meeting l_{234} in a point P_{32} (P_{32} not on l_{034} and not on l_{134}), then l_{034}, l_{134}, and l_{934} are not in pencil.

Proof. (*A*) If the pencil has a center, the theorem follows from Theorem I.

(*B*) Suppose the pencil has no center. (Fig. 6.)

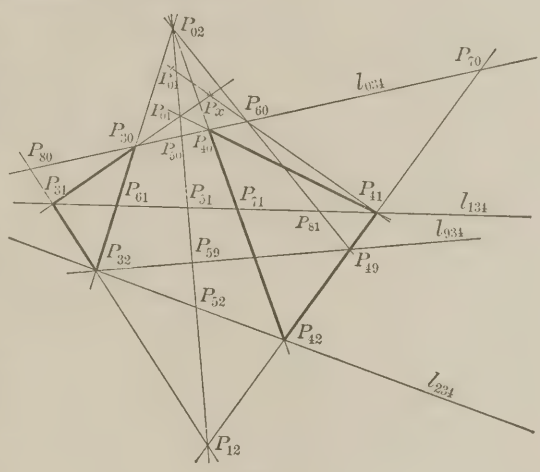

FIG. 6.

Suppose there were a line l_{934} satisfying the conditions of the hypothesis. Then l_{934} can not meet either l_{034} or l_{134}, by Theorem I, and also l_{134} will be between l_{034} and l_{934}. Let P_{80} and P_{70} be any two distinct points on l_{034}. Let P_{12} be any point in the order $P_{80}P_{32}P_{12}$. The segment $P_{80}P_{32}$ meets l_{134} in a point P_{31}. Since neither l_{234} nor l_{934} meets either l_{034} or l_{134}, the line $P_{70}P_{12}$ meets l_{134} in a point P_{41}, l_{234} in a point P_{42}, and l_{934} in a point P_{49}, and these points are in the order $P_{70}P_{41}P_{42}P_{12}$ and $P_{70}P_{41}P_{49}P_{12}$. Take P_{50} on the segment $P_{80}P_{70}$. Then take P_{02} in the order $P_{12}P_{50}P_{02}$ and the line $P_{12}P_{50}$ will meet l_{234} in a point P_{52}, l_{934} in a point P_{59}, and l_{134} in a point P_{51}, and these points are in the orders $P_{12}P_{52}P_{51}P_{50}P_{02}$ and $P_{12}P_{59}P_{51}P_{50}P_{02}$. $P_{02}P_{32}$ contains points P_{30} on l_{034} and P_{61} on l_{134}, in the order $P_{02}P_{30}P_{61}P_{32}$. $P_{02}P_{42}$ meets l_{034} in P_{40} and l_{134} in P_{71}, in the order $P_{02}P_{40}P_{71}P_{42}$. The line $P_{31}P_{30}$ meets $P_{52}P_{50}$ in a point P_{01}, in the orders $P_{31}P_{30}P_{01}$ and $P_{02}P_{01}P_{50}$. $P_{41}P_{40}$ also meets the line $P_{12}P_{02}$ in a point which by Axiom X is the same as P_{01}, from the Desargues configuration (2).

Now the line $P_{02}P_{49}$ meets l_{134} in P_{81}, and l_{034} in P_{60}, and these points are in the order $P_{02}P_{60}P_{81}P_{49}$. The line $P_{41}P_{60}$ meets the segment $P_{02}P_{50}$ in P'_{01}. But P'_{01} can not be P_{01}, since P_{40} and P_{60} are distinct. Also $P_{41}P_{60}$ and $P_{31}P_{30}$, since each enters the triangle $P_{02}P_{30}P_{60}$ at a vertex, meet in a point P_x. But this point is not collinear with P_{12} and P_{02}, since $P_{31}P_{30}$ and $P_{41}P_{40}$ meet $P_{12}P_{02}$ in distinct points, P_{01} and P'_{01}. Then we should have the Desargues configuration

* Note that the proof of the theorem as given would not be valid except for this restriction. The result is proved in general in Theorems VII and VIII.

$$\begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{60} & P_{41} & P_{49} \\ \hline P_{12} & P_{02} & P_x \end{array}$$

In this configuration, if l_{034}, l_{134}, and l_{934} are in pencil Q_{34}, as we have supposed, then by Axiom X the points P_{12}, P_{02}, P_x must be collinear. But we have proved that they are not collinear, and hence the lines l_{034}, l_{134}, and l_{934} are not in pencil.

THEOREM V. If l_{045}, l_{145}, and l_{245} are three lines in a non-intersecting pencil, and l_{045}, l_{145}, and l_{345} ($l_{245} \neq l_{345}$) are three lines in a non-intersecting pencil, then two of these lines are always between the other two.

Proof. Since no two of the lines meet, it is only necessary to find a single segment joining points of two of them, and containing points of the remaining two, as the lemma to Theorem III shows.

Let the regions into which the plane is divided by the lines l_{045} and l_{145} be denoted by $++$; $-+$; $--$ (cf. Theorem g, p. 146), the first sign referring to division by l_{045} and the second to division by l_{145}, and the notation so used that a point in the $++$ region is on the opposite side of l_{045} from l_{145}, and a point in the $--$ region is on the opposite side of l_{145} from l_{045}. The remaining points not on the lines are in $-+$.

Since neither l_{245} nor l_{345} can meet either l_{045} or l_{145} each must lie wholly in some one of the regions $++$, $-+$, or $--$. We have then the following cases, with the conclusion as given:

Case	l_{245} in	l_{345} in	Conclusion
1	$++$	$++$	See below.
2	$++$	$-+$	l_{045} and l_{345} between other two lines.
3	$++$	$--$	l_{045} and l_{145} " " " "
4	$-+$	$++$	l_{045} and l_{245} " " " "
5	$-+$	$-+$	l_{245} and l_{345} " " " "
6	$-+$	$--$	l_{145} and l_{245} " " " "
7	$--$	$++$	l_{145} and l_{045} " " " "
8	$--$	$-+$	l_{145} and l_{345} " " " "
9	$--$	$--$	See below.

In cases 2 to 8 inclusive, the conclusion is obvious and cases 1 and 9 differ only in notation. The proof is made only for case 9. (Fig. 7.)

Take any points P_{62} and P_{73} on l_{245} and l_{345} respectively, and P_{67} any point in $++$. If P_{62}, P_{73} and P_{67} are collinear, the theorem follows at once. If not, the segment $P_{67}P_{62}$ contains points P_{60} on l_{045} and P_{61} on l_{145}. The segment

$P_{67}P_{73}$ contains points P_{70} on l_{045} and P_{71} on l_{145}. The lines $P_{60}P_{71}$ and $P_{70}P_{61}$ determine a point P_{01}. The points P_{67} and P_{01} determine a line containing points P_{80} on l_{045} and P_{81} on l_{145} in the order $P_{67}P_{80}P_{01}P_{81}$. The segment $P_{61}P_{73}$ meets the line $P_{67}P_{01}$ in a point P_{31}. Either $P_{71}P_{31}$ and $P_{67}P_{62}$ meet in a point P_{63} or they do not. If they do meet in a point, we have the triangles $P_{60}P_{71}P_{63}$ and $P_{70}P_{61}P_{73}$ on the lines l_{045}, l_{145}, and $P_{73}P_{63}$ whose corresponding sides meet

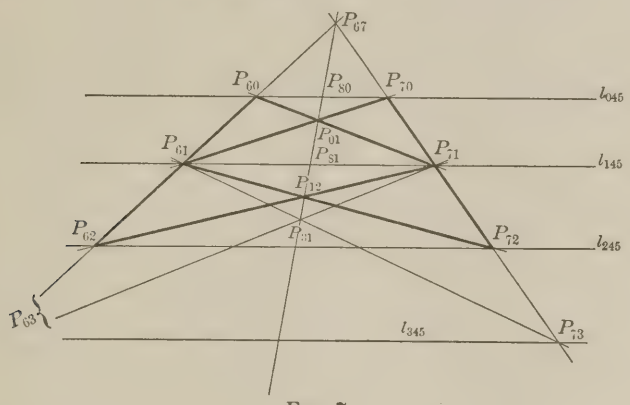

FIG. 7.

in three collinear points, P_{67}, P_{01}, P_{31}. Therefore l_{045}, l_{145}, and $P_{73}P_{63}$ are in pencil, and by Theorem IV, $P_{73}P_{63}$ is l_{345} and hence P_{63} is on l_{345}. Therefore the points P_{60}, P_{61}, P_{62}, and P_{63} are collinear, and are in one or the other of the orders $P_{60}P_{61}P_{62}P_{63}$ or $P_{60}P_{61}P_{63}P_{62}$. In either case we have the theorem. If $P_{71}P_{31}$ and $P_{67}P_{62}$ do not meet, the line $P_{71}P_{62}$ contains a point P_{12} of the line $P_{67}P_{01}$, which is in the order $P_{81}P_{12}P_{31}$; since if the points were in the order $P_{81}P_{34}P_{12}$, then $P_{71}P_{31}$ would cut the segment $P_{61}P_{62}$. Since P_{12} is on the segment $P_{81}P_{31}$, it is within the triangle $P_{71}P_{61}P_{73}$, and hence the line $P_{61}P_{12}$ must cut the segment $P_{71}P_{73}$ in a point P_{72}. The triangles $P_{62}P_{71}P_{60}$ and $P_{72}P_{61}P_{70}$ have their corresponding sides meeting in three collinear points, P_{67}, P_{01} and P_{12}. Therefore $P_{72}P_{62}$ is in pencil with l_{045} and l_{145}, and by Theorem IV, P_{72} is on l_{245}. Hence $P_{70}P_{73}$ contains P_{71} on l_{145} and P_{72} on l_{245}, and hence l_{145} and l_{245} are between l_{045} and l_{345}. This completes the proof.

COROLLARY. A triangle can always be so chosen that it will contain segments of each of the four lines l_{045}, l_{145}, l_{245}, l_{345} in its interior, and such that within it no two of the lines meet, and within such a triangle the order in which the four lines are cut by a transversal is invariant.

THEOREM VI. If l_{045}, l_{145}, l_{245} are three lines in pencil, and l_{045}, l_{145}, l_{345} are three lines in pencil, then l_{045}, l_{245}, l_{345} are in pencil, and l_{145}, l_{245}, l_{345} are in pencil.

Proof. If l_{045} and l_{145} meet in a point P_k, then l_{245} and l_{345} pass through P_k, and we have at once the conclusion, by the corollary to Theorem II.

If * l_{045} and l_{145} do not meet (Fig. 8), we will consider the interior of a triangle T chosen as in the corollary to Theorem V.

All of the points chosen below are to be taken in the interior of T. Let the notation l_w, l_x, l_y, l_z be so assigned to the four lines l_{045}, l_{145}, l_{245}, l_{345} that if they are cut by any transversal in T in the points P_w, P_x, P_y, P_z these points

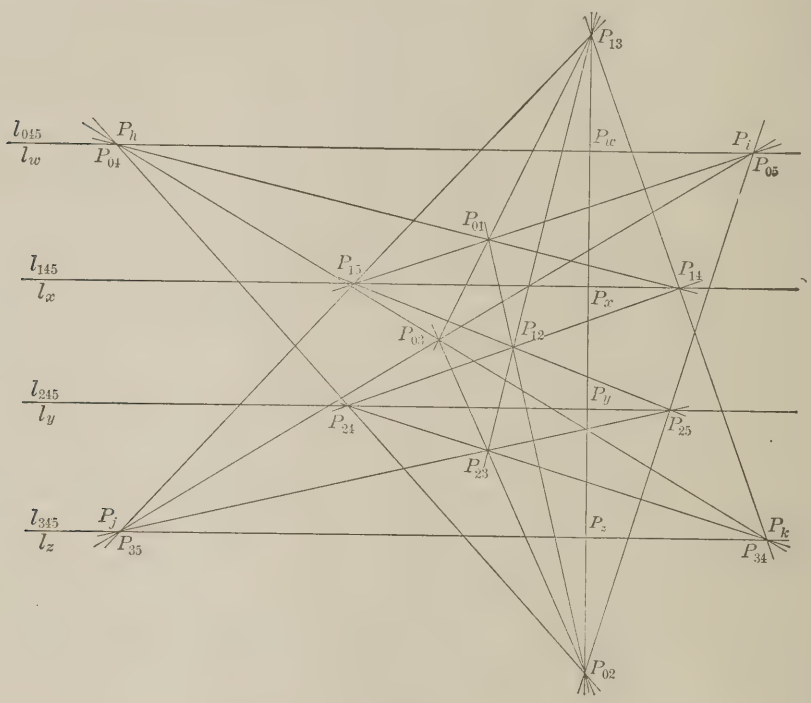

FIG. 8.

will be in the order $P_w P_x P_y P_z$. Choose P_{13} and P_{02} in the order $P_{13} P_w P_x P_y P_z P_{02}$. Then take two points P_j and P_k on l_z in the order $P_j P_z P_k$, and two points P_h and P_i on l_w, in the order $P_h P_w P_i$, and such that P_h and P_j are on the same side of the line $P_{13} P_{02}$. The line $P_{13} P_j$ meets l_{145} in a point P_{15} and l_{345} in a point P_{35}. The line $P_{13} P_k$ meets the line l_{145} in a point P_{14} and l_{345} in a point P_{34}. The line $P_{02} P_h$ meets l_{245} in a point P_{24} and l_{045} in a point P_{04}. The line $P_{02} P_i$ meets l_{245} in a point P_{25} and l_{045} in a point P_{05}. Then the points P_{14} and P_{04} are on opposite sides of the line $P_{15} P_{05}$, hence the lines $P_{14} P_{04}$ and $P_{15} P_{05}$ meet in a point P_{01}. Similarly, $P_{15} P_{25}$ and $P_{24} P_{14}$ meet in a point P_{12}; $P_{04} P_{34}$ and $P_{05} P_{35}$ meet in a point P_{03}; $P_{24} P_{34}$ and $P_{25} P_{35}$ meet in a point P_{23}.

* The configuration obtained in the proof of this theorem is a plane section of the three-space configuration containing six points, no three of them collinear. The configuration contains fifteen points and twenty lines.

Then the hypothesis l_{045}, l_{145}, l_{245} in pencil, and the hypothesis l_{045}, l_{145}, l_{345} in pencil give respectively the Desargues configurations

$$(A),\ (B)\qquad \begin{array}{ccc} & Q_{45} & \\ P_{04} & P_{14} & P_{24} \\ P_{05} & P_{15} & P_{25} \\ \hline P_{12} & P_{02} & P_{01} \end{array}\qquad\qquad \begin{array}{ccc} & Q_{45} & \\ P_{04} & P_{14} & P_{34} \\ P_{05} & P_{15} & P_{35} \\ \hline P_{13} & P_{03} & P_{01} \end{array}$$

Then by using the conclusions of (A) and (B), and remembering that any three lines through a point are in pencil, we have

$$(C),\ (D)\qquad \begin{array}{ccc} & Q_{14} & \\ P_{31} & P_{21} & P_{01} \\ P_{34} & P_{24} & P_{04} \\ \hline P_{02} & P_{03} & P_{23} \end{array}\qquad\qquad \begin{array}{ccc} & Q_{04} & \\ P_{30} & P_{20} & P_{10} \\ P_{34} & P_{24} & P_{14} \\ \hline P_{12} & P_{13} & P_{23} \end{array}$$

In view of (C) and (D) respectively, we have

$$(E),\ (F)\qquad \begin{array}{ccc} P_{23} & P_{03} & P_{02} \\ \hline P_{04} & P_{24} & P_{34} \\ P_{05} & P_{25} & P_{35} \\ & Q_{45} & \end{array}\qquad\qquad \begin{array}{ccc} P_{23} & P_{13} & P_{12} \\ \hline P_{14} & P_{24} & P_{34} \\ P_{15} & P_{25} & P_{35} \\ & Q_{45} & \end{array}$$

The conclusion of the theorem is expressed by (E) and (F) in the configurational notation.

COROLLARY. The set of all lines through a point constitutes a pencil which is determined by any pair of the lines.

THEOREM VII. If l_{034} and l_{134} are two lines and P_{32} is a point not on either l_{034} or l_{134}, then there is one and only one line l_{234} which passes through the point P_{32} and which is in pencil with l_{034} and l_{134}.

This is a generalization of Theorem IV, in which we have the proof for all cases except when l_{034} and l_{134} do not meet, and l_{234} is between l_{034} and l_{134}. We can now prove this last case and have the more general theorem by the aid of Theorem VI. We will first prove that there can not be two such lines. Suppose l_{234} and l_{534} are two distinct lines through P_{34} each in pencil with l_{034} and l_{134}. Then by Theorem VI, l_{034}, l_{234}, and l_{534} are in pencil, and by Theorem I, l_{034}, l_{134}, l_{534} must meet in P_{32}, but this contradicts the hypothesis that l_{034} and l_{134} do not meet, and hence there can not be two such lines.

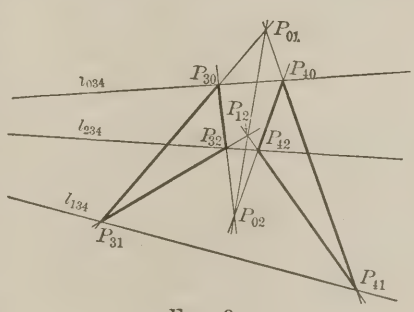

FIG. 9.

One line, and by this proof the only line l_{234}, through P_{32} and in pencil with l_{034} and l_{134} will now be exhibited. (Fig. 9.) Take any point P_{30} on l_{034}, and any point P_{02} not on l_{134}, in the order $P_{30}P_{32}P_{02}$, and such that the segment $P_{32}P_{02}$ contains no point of l_{134}. Then take P_{31} on the line l_{134}, but not on the line $P_{30}P_{32}$, and then any point P_{01} in the order $P_{31}P_{30}P_{01}$. Choose $P_{40} \neq P_{30}$ on l_{034} but not on the line $P_{01}P_{02}$, and any point $P_{41} \neq P_{31}$ on the line l_{134}, and such that these points are in the order $P_{01}P_{40}P_{41}$. Then $P_{31}P_{32}$ and $P_{01}P_{02}$ meet in a point P_{12}, and the segment $P_{40}P_{02}$ and the segment $P_{41}P_{12}$ meet in a point P_{42}. Hence we have the Desargues configuration:

$$\begin{array}{ccc} P_{12} & P_{02} & P_{01} \\ \hline P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ & Q_{34} & \end{array}$$

whence the line $P_{32}P_{42}$ is the line l_{234} of the theorem.

COROLLARY. Any two lines determine a pencil.

In view of Theorem VII we can now speak of P_xQ_y, where P_x is not the center of Q_y as a definite line, viz., the line of the pencil Q_y which passes through P_x. We will also speak of Q_x as collinear with P_y and P_z if the line P_xP_y belongs to the pencil Q_z; also if the line P_xQ_y is a line of the pencil Q_z, $P_xQ_yQ_z$ will be said to be collinear. Collinearity, when used in this sense, of course, does *not imply order relations.* A line can belong to any number of pencils, viz., the pencils determined by it and each of the lines through some point not on it. But two pencils can not have more than one *line* in common, by Theorem VII. Since a point Q_x always determines uniquely a pencil, viz., the pencil of which it is the center, we may use any point as a pencil, when we mean this pencil. Points will be so used, in speaking of ranges, as in the definition of "in range," points might have been used for the pencils, as indicating the pencils of which they are centers.

SECTION II.

Consequences of Axioms I–XI.

The theorems of Section 2 will have to do with ranges and their properties. It is evident from the definitions of pencils and "in range," that any three points of a line are the centers of three pencils which are in range. The two triangles whose sides are lines of three pencils in range, and the joins of whose vertices are in pencil, will be said to be *perspective triangles.*

Suppose Q_1, Q_2, Q_3 are any three distinct pencils, and T is any triangle not containing in its interior or on any of its sides any segment common to any two of these pencils. Let $P_aP''_bP_c$ be any three points in T such that $P_aP''_b$ is a

line of Q_1 and P_aP_c is a line of Q_2. (Fig. 10.) Then the lines P_cQ_1 and P''_bQ_2 meet in a point P_d of T, or do not. If not, let any point of P_cQ_1 on the same side of P_aP_c that P_b is, be called P_d; then P_dQ_2 will meet $P_aP''_b$ in a point P_b which may be used for P''_b.

Such a figure ($P_aP_bP_cP_d$) will be called a *quadrilateral* in T on Q_1 and Q_2.

Some one at least of the lines P_aQ_3, P_bQ_3, P_cQ_3, P_dQ_3 enters the quadrilateral $P_aP_bP_cP_d$, and by a proper assignment of notation it may be considered P_aQ_3. Then P_aQ_3 will meet either the segment P_cP_d or the segment P_bP_d in a point P''_d or P'_d respectively. In the first case, the line P''_dQ_2 meets the segment P_aP_b in a point P'_b, and P''_d and P'_b may be used as a new P_d and P_b, respectively. In the second case, the line P'_dQ_1 meets the segment P_aP_c in a point P'_c, and P'_d and P'_c may be used as P_d and P_c respectively, and hence the points $P_aP_bP_cP_d$ can be so chosen that they form a quadrilateral on Q_1 and Q_2, and so that P_aP_d is a line of Q_3. Such a figure will be called a quadrilateral on Q_1, Q_2 and Q_3, in T. Any line l_1 of Q_1 and any line l_2 of Q_2, which are within the quadrilateral, meet each other in a point of the quadrilateral.

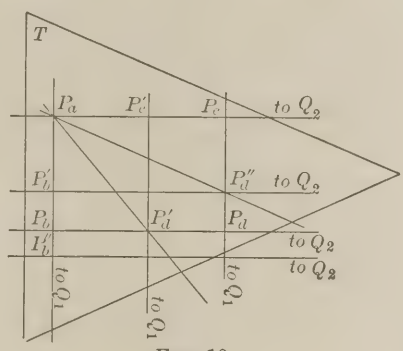

FIG. 10.

The notation for a Desargues configuration will sometimes be written, when the notation is in the form of Definition 7, D. C. (34)–(012), and when in the form of Axiom XI, D. C. (012)–(34), for shortness in writing as this may readily be expanded to the previous form.

THEOREM VIII. If Q_{01} and Q_{02} are any two distinct pencils, and l is any line not belonging to either Q_{01} or Q_{02}, then l is in one and only one pencil Q_{12} which is in range with Q_{01} and Q_{02}. (Fig. 11.)

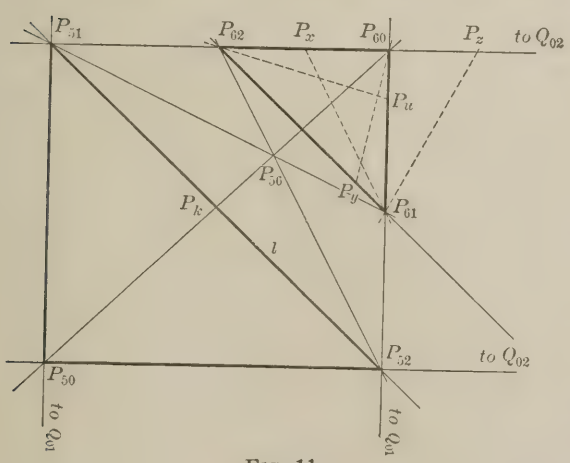

FIG. 11.

Proof. Let T be any triangle containing in its interior a segment of l, but no segment of a line common to Q_{01} and Q_{02}. Choose P_{51} and P_{52} on l, in T, and such that $P_{51}Q_{01}$ and $P_{52}Q_{02}$ meet in a point P_{50} of T, and such that $P_{50}Q_{01}$ and $P_{51}Q_{02}$ meet in a

point P_{60} of T. Then the segments $P_{51}P_{52}$ and $P_{60}P_{50}$ meet in a point P_k; choose P_{56} in the order $P_kP_{56}P_{60}$. Then $P_{52}P_{56}$ meets the segment $P_{51}P_{60}$ in a point P_{62}, and $P_{51}P_{56}$ meets the segment $P_{52}P_{60}$ in a point P_{61}. Then from the triangles $P_{50}P_{51}P_{52}$ and $P_{60}P_{61}P_{62}$ perspective in P_{56}, $P_{51}P_{52}$ and $P_{61}P_{62}$ determine a pencil Q_{12} which is in range with Q_{01} and Q_{02}.

Suppose there is a pencil Q'_{12}, distinct from Q_{12}, such that l is a line of Q'_{12}, and such that Q_{01}, Q_{02}, and Q'_{12} are in range. Then at least one of the lines $P_{60}Q'_{12}$, $P_{61}Q'_{12}$, $P_{62}Q'_{12}$ must enter the triangle $P_{60}P_{61}P_{62}$, unless $P_{60}Q'_{12}$ is identical with $P_{60}Q_{01}$ or $P_{60}Q_{02}$. But either of these was avoidable by a new choice of P_{52} and P_{51}, since P_{52} and P_{51} might have been any points on the segment $P_{51}P_{52}$.

Case I. Suppose $P_{60}Q'_{12}$ cuts the segment $P_{61}P_{62}$ in P_y. Then choose a point P_z in the order $P_{51}P_{60}P_z$ such that $P_zQ'_{12}$ passes through P_{61}, or a point P'_{61} on the segment $P_{61}P_{60}$. Then since P_{56} was chosen arbitrarily on the segment P_kP_{60}, it may be chosen so that for some point P_z in order $P_{51}P_{60}P_z$, $P_zQ'_{12}$ passes through P_{61}. Then since $Q_{01}Q_{02}Q'_{12}$ are in range, the triangles $P_{51}P_{52}P_{50}$ and $P_{61}P_zP_{60}$ are in perspective. But $P_{61}P_{51}$ and $P_{60}P_{50}$ meet in P_{56}, and $P_{52}P_z$ does not pass through P_{56}. Hence this case is impossible.

Case II. $P_{61}Q'_{12}$ meets the segment $P_{60}P_{62}$ in a point P_x. Then since Q_{01}, Q_{02}, and Q'_{12} are in range, by hypothesis, $P_{60}P_{61}P_x$ and $P_{50}P_{51}P_{52}$ are in perspective. But $P_{60}P_{50}$ and $P_{61}P_{51}$ meet in P_{56} and P_xP_{52} does not pass through P_{56}. Hence this case is impossible.

Case III. $P_{62}Q'_{12}$ meets the segment $P_{61}P_{62}$ in P_u. The proof in this case is identical with that of Case II, except for notation. Hence there can be no such pencil Q'_{12}, and the theorem is proved.

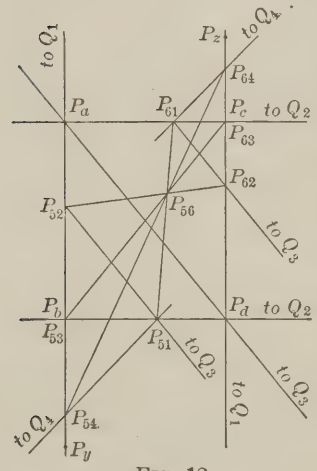

FIG. 12.

THEOREM IX. If Q_1, Q_2, Q_3 and Q_4 are four distinct pencils, and Q_1, Q_2, Q_3 are in range, and Q_1, Q_2, Q_4 are in range, then also Q_1, Q_3, Q_4 are in range. (Fig. 12.)

Proof. Let $P_aP_bP_cP_d$ be a quadrilateral on Q_1, Q_2, Q_3 in a region T not containing any point of a line common to any pair of the pencils Q_1, Q_2, Q_3, Q_4. Let P_y and P_z be any points in T in the orders $P_aP_bP_y$ and $P_dP_cP_z$ respectively. Let P_{51} be any point on the segment P_bP_d such that $P_{51}Q_4$ meets the segment P_aP_y in a point P_{54}. Let P_{61} be any point on the segment P_aP_c, such that $P_{61}Q_4$ meets the segment P_dP_z in a point P_{64}. $P_{51}Q_3$ meets the segment P_aP_b in a point P_{52}, and $P_{61}Q_3$ meets the segment P_cP_d in a point P_{62}. Since P_{61}, P_{51} are on one pair of opposite sides of the quadrilateral, and

P_{62}, P_{52} are on the other pair of opposite sides, $P_{61}P_{51}$ and $P_{62}P_{52}$ meet in a point P_{56}, of the interior of the quadrilateral. If P_b and P_c be called P_{53} and P_{63} respectively, we have the Desargues configuration (123)–(56), whence $P_{53}P_{63}$ also passes through the point P_{56}. But from the hypothesis that Q_1, Q_2 and Q_4 are in range, and the triangles $P_{51}P_{52}P_{54}$ and $P_{61}P_{62}P_{64}$, we have the Desargues configuration (124)–(56), whence $P_{54}P_{64}$ also passes through P_{56}. Hence, from triangles $P_{51}P_{53}P_{54}$ and $P_{61}P_{63}P_{64}$, perspective in P_{56}, we have the Desargues configuration (56)–(134), whence Q_1, Q_3, and Q_4 are in range, and the theorem is proved.

COROLLARY. Any two pencils of a range determine the range.

THEOREM X. If three points P_{01}, P_{02}, P_{12} are in range, they are collinear.

Proof. (See Fig. 13.) Let P_{30} be a point not on any of the lines $P_{01}P_{02}$, $P_{01}P_{12}$, $P_{02}P_{12}$. Take P_{31} on the line $P_{30}P_{01}$ such that the line $P_{31}P_{12}$ will

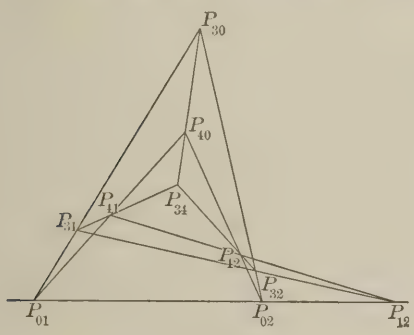

FIG. 13.

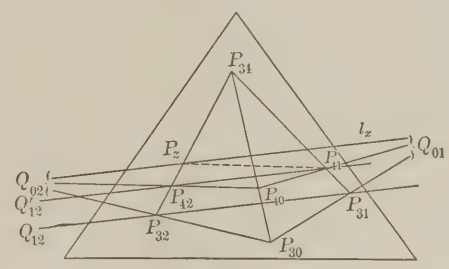

FIG. 14.

meet the segment $P_{30}P_{02}$ in a point P_{32}, and such that no point of any of the lines $P_{01}P_{02}$, $P_{01}P_{12}$, $P_{02}P_{12}$ is within the triangle $P_{30}P_{31}P_{32}$. Let P_{34} be any point within the triangle $P_{30}P_{31}P_{32}$. Take P_{40} in the order $P_{30}P_{40}P_{34}$. Then the line $P_{40}P_{01}$ will meet the segment $P_{31}P_{34}$ in a point P_{41}, and the line $P_{40}P_{02}$ will meet the segment $P_{34}P_{32}$ in a point P_{42}. But the lines $P_{42}P_{41}$ and $P_{32}P_{31}$ determine a pencil Q'_{12}, which, by definition, is in range with P_{01} and P_{02}. Hence by Theorem VIII Q_{12} and Q'_{12} are identical, and we have the theorem.

THEOREM XI. If Q_{01}, Q_{02}, Q_{12} are in range and l_x is a line belonging to Q_{01} and Q_{02}, then l_x belongs also to Q_{12}. (Fig. 14.)

Proof. Let T be a triangle containing in its interior a segment of the line l_x, but no center of any one of the three pencils. Let P_{30} be any point in T not on l_x, and P_{31} a point in T, on line $P_{30}Q_{01}$, and P_{32} a point in T on $Q_{02}P_{30}$, such that $P_{31}P_{32}$ is a line of Q_{12}, and P_{30}, P_{31}, P_{32} are all on the same side of l_x. Take P_{34} on opposite side of l_x from P_{30}, not on $P_{32}Q_{12}$. Suppose now l_x is *not* a line of Q_{12}; let P_z be the point in which $P_{32}P_{34}$ meets l_x. Then P_zQ_{12} will meet the segment $P_{34}P_{31}$ in a point P_{41}, not on l_x. $P_{41}Q_{01}$ meets $P_{30}P_{34}$ in a point P_{40}, not on l_x. $P_{40}Q_{02}$ meets $P_{32}P_{34}$ in a point P_{42}, not on l_x. Hence

$P_{31}P_{32}$ and $P_{41}P_{42}$ determine a pencil Q'_{12} which is in range with Q_{01} and Q_{02}. By Theorem VIII Q'_{12} is Q_{12}. But $P_{41}P_z$ was supposed to be a line of Q_{12}, and that would make P_{41} the center of Q_{12}. But Q_{12} does not have its center in T, and P_{41} is in T, hence l_x must be a line of Q_{12}.

THEOREM XII. If l_x is a line of a pencil Q_{12}, and P_{02} and P_{01} are any two distinct points of l_x not the center of Q_{12}, then Q_{12}, P_{02} and P_{01} are in range.

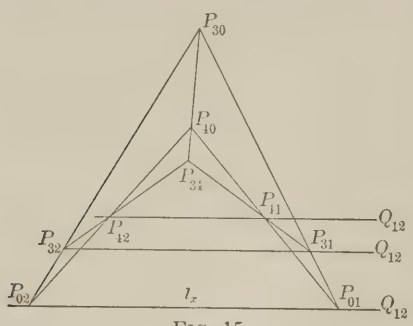

FIG. 15.

Proof. (See Fig. 15.) Let P_{30} be any point not on l_x, and P_{31} and P_{32} two points on the same side of l_x that P_{30} is, and such that $P_{30}P_{31}$ is a line of Q_{01}, $P_{30}P_{32}$ is a line of Q_{02}, and $P_{31}P_{32}$ is a line of Q_{12}. Let P_{34} be any point within the triangle $P_{30}P_{31}P_{32}$. Let P_{40} be any point in the order $P_{30}P_{40}P_{34}$. Then $P_{40}Q_1$ meets the segment $P_{31}P_{34}$ in a point P_{41}, and $P_{40}Q_{02}$ meets the segment $P_{34}P_{32}$ in a point P_{42}. Then the line $P_{42}P_{41}$ belongs to the pencil Q_{12}, from the Desargues configuration, (043)–(12), and hence the theorem is true.

THEOREM XIII. If l_x belongs to Q_{01} and Q_{02}, then any other pencil Q_{12} to which l_x belongs is in range with Q_{01} and Q_{02}. Let P_x and P_y be two points of l_x not centers of Q_{01} or Q_{02} or Q_{12}. Then $P_xP_yQ_{01}$, $P_xP_yQ_{02}$, $P_xP_yQ_{12}$ are in range, by Theorem XII. Then by Theorem IX, $P_xQ_{01}Q_{02}$ and $P_xQ_{01}Q_{12}$ and therefore Q_{01}, Q_{02}, Q_{12} are in range.

THEOREM XIV. Any two distinct ranges r_{023} and r_{123} have in common one and only one pencil, Q_{23}. (Fig. 16.)

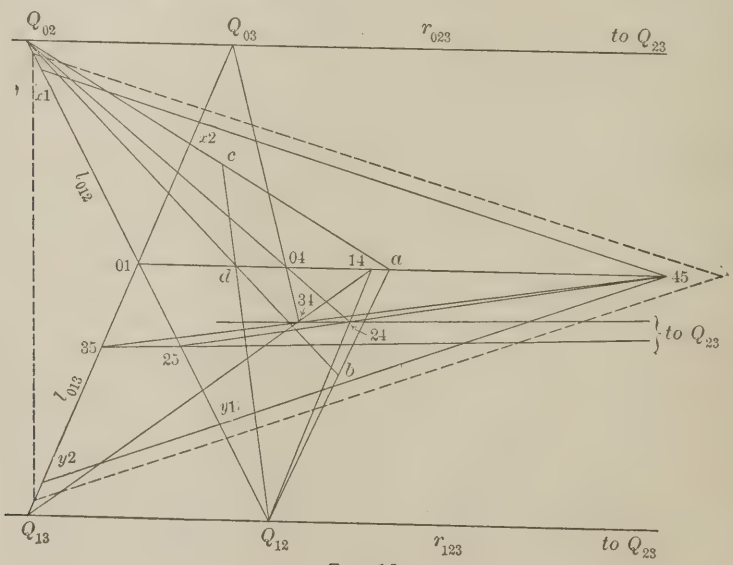

FIG. 16.

Proof. Through any point P_{01}, not a point of either r_{023} or r_{123}, draw lines l_{013} and l_{012}, which with r_{023} and r_{123} determine respectively the pencils Q_{03} and Q_{13}, Q_{02} and Q_{12}. Choose P_{45}, P_{x2} and P_{y1} such that P_{x2} is on the line l_{013} and P_{y1} is on the line l_{012}, and such that $P_{45}P_{x2}$ meets l_{012} in a point P_{x1} and $P_{45}P_{y1}$ meets l_{013} in a point P_{y2} in the orders $P_{x1}P_{01}P_{y1}$ and $P_{x2}P_{01}P_{y2}$, and such that all of these points lie within a triangle containing no points of r_{123} or r_{023}. Take a quadrilateral $P_aP_bP_cP_d$ on Q_{12} or Q_{02}, and P_{01}, and such that P_a and P_d are on the segment $P_{01}P_{45}$, and such that P_cP_{45} and P_bP_{45} are lines within the triangles $P_{45}P_{x1}P_{01}$ and $P_{45}P_{y2}P_{01}$ respectively. Take P_{04} and P_{14} on the segment P_aP_d, and such that $P_{04}P_{03}$ and $P_{14}Q_{13}$ meet at a point P_{34} within the quadrilateral $P_aP_bP_cP_d$; $P_{04}Q_{02}$ and $P_{14}Q_{12}$ meet at a point P_{24} within the quadrilateral. Then $P_{45}P_{24}$ meets l_{012} in a point P_{25} and $P_{45}P_{34}$ meets the line l_{013} in a point P_{35}. Then $P_{25}P_{35}$ and $P_{24}P_{34}$ determine a pencil Q_{23} which is in range with $Q_{02}Q_{03}$ from D. C. (45)–(023), and in range with $Q_{12}Q_{13}$ from D. C. (45)–(123).*

That there is only one such pencil is an immediate consequence of Theorem IX. For if there were two such pencils, then these would determine any range containing them, and hence the ranges r_{023} and r_{123} would not be distinct.

THEOREM XV. If any ten pencils Q_{ij} $(i, j = 1, \cdots, 5;\ i \neq j)$ are such that no four of them are in range, and $Q_{ij}Q_{ik}Q_{jk}$ $(i, j, k = 1, 2, 3, 4, 5;\ i \neq j \neq k$, and $i, j, k \neq 1, 2, 3$ in any order) are in range r_{ijk}, then also Q_{12}, Q_{13} and Q_{23} are in range r_{123}, and conversely.

The theorem is the general Desargues theorem, pencils and ranges replacing points and lines, and forming the configuration $\left|\begin{smallmatrix}10 & 3\\ 3 & 10\end{smallmatrix}\right|$. This configuration, for the purpose of the proof, is enlarged to a configuration $\left|\begin{smallmatrix}21 & 3\\ 5 & 35\end{smallmatrix}\right|$, P_{ij} or Q_{ij}, l_{ijk} or r_{ijk} $(i, j, k = 1, \cdots, 7)$, the new lines or points being within or crossing a region T, say the inside of a triangle, containing no point of the original ranges. The eleven points added form two perspective five-points $P_{1,2,3,4,5-6}$ and $P_{1,2,3,4,5-7}$, perspective from a point P_{67}.

Choose P_{36}, any point in T (Fig. 17), and draw $P_{36}Q_{3j}$ $(j = 1, 2, 4)$; in T, choose points P_{16} and P_{26}, on $P_{36}Q_{13}$ and $P_{36}Q_{23}$ respectively, and such that $P_{16}Q_{14}$ and $P_{26}Q_{24}$ meet in a point P_{46} of T, and $P_{26}Q_{25}$ and $P_{16}Q_{15}$ meet in a point P_{56} of T. Take P_{67} in T, but neither collinear with any two points, nor on any line yet chosen, and draw $P_{67}P_{6j}$ $(j = 1 \cdots 5)$. Take P_{17} on the segment $P_{67}P_{16}$, and draw $P_{17}Q_{12}$. This will cut the segment $P_{67}P_{26}$ in a point P_{27}. $P_{17}P_{14}$ meets segment $P_{67}P_{46}$ in a point P_{47}, which is also on the line $P_{27}Q_{24}$, from the D. C. (67)–(124). [That is, $P_{27}P_{47}$ and $P_{26}P_{46}$ determine a pencil which is in range with Q_{14} and Q_{12}, and since Q_{24} is a pencil of the line $P_{26}P_{46}$, it is this pencil.] Also $P_{17}Q_{15}$ meets the segment $P_{56}P_{67}$ in a point P_{57}, and from

* The configuration used in the proof of this theorem is a degenerate case of the configuration $\left|\begin{smallmatrix}15 & 4\\ 3 & 26\end{smallmatrix}\right|$, of notation P_{ij}, l_{ijk} $(i, j, k = 0, \cdots, 5)$ in which the points P_{01}, P_{05} and P_{15} are coincident.

the D. C. (67)–(125), $P_{26}P_{56}$ and $P_{27}P_{57}$ determine a pencil which is in range with Q_{15} and Q_{12}, and is therefore Q_{25}. From the D. C. (67)–(ijk), ($i=1, 2$; $j, k = 3, 4, 5, j \neq k$) $P_{j6}P_{k6}$ and $P_{j7}P_{k7}$ belong to Q_{jk}. Hence Q_{34}, Q_{35},

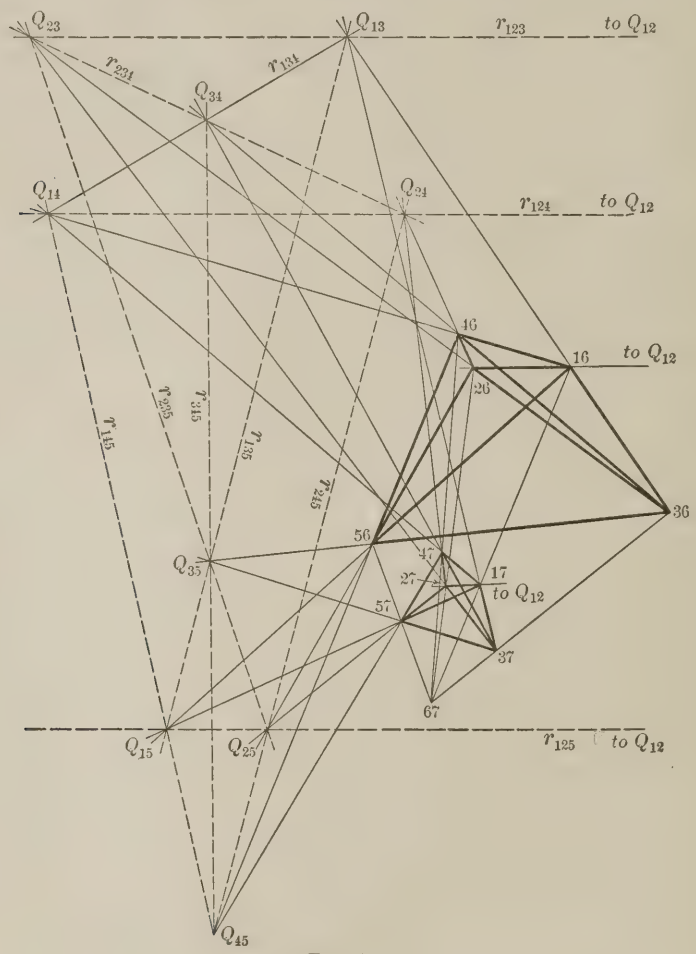

FIG. 17.

Q_{45} are in range, from the D. C. (67)–(345), and we have established the first part of the theorem.

The same configuration serves for the proof of the converse theorem, if the points are chosen in a slightly different order. Choose in T a point P_{36}, then P_{46} and P_{56} such that $P_{46}P_{56}$ is a line of Q_{56}, $P_{56}Q_{25}$ and $P_{46}Q_{24}$ meet in a point P_{26} of T, and $P_{56}Q_{15}$ and $P_{46}Q_{14}$ meet in a point P_{16} of T. Choose P_{67} in T, not on any line or range of the figure so far determined. On the segment $P_{67}P_{36}$ choose P_{37}. Then $P_{37}Q_{35}$ meets the segment $P_{56}P_{67}$ in a point P_{57}. The line $P_{57}Q_{45}$ meets the segment $P_{46}P_{67}$ in a point P_{47}. Then from

the D. C. (67)–(345), $P_{37}P_{47}$ is a line of Q_{34}. Then $Q_{14}P_{47}$ meets the segment $P_{16}P_{67}$ in a point P_{17}. From D. C. (67)–(145) and Theorem VIII, $P_{17}P_{57}Q_{15}$ are collinear. We have now hypotheses for the configuration (67)–(134) and (67)–(135), from which, and Theorem VIII, we have the collinearities $P_{16}P_{36}Q_{13}$, $P_{17}P_{37}Q_{13}$. The line $P_{47}Q_{24}$ meets the segment $P_{26}P_{67}$ in a point P_{27}, and from the D. C. (67)–(245) we have $P_{27}P_{57}Q_{25}$ collinear. From the D. C.'s (67)–(234) and (67)–(235) we have the collinearities $P_{26}P_{36}Q_{23}$ and $P_{27}P_{37}Q_{23}$.

But from the D. C.'s (67)–(123), (67)–(124), (67)–(125), the lines $P_{16}P_{26}$ and $P_{17}P_{27}$ determine a pencil Q_{12} which is in range with $Q_{13}Q_{23}$, $Q_{14}Q_{24}$, $Q_{15}Q_{25}$ respectively, and this proves the theorem.

Section III.

Generalization of Order.

In this section it is shown that if some one range of pencils is omitted from the set of all pencils, the remaining subset of pencils has the property, that, with a suitable definition of order, its elements satisfy the Axioms I–XI, a well-known property of the elements of a projective plane.

Before giving the definition of order it will be useful to prove certain lemmas.

Lemma I. If Q_h, Q_i, Q_j, Q_k are any four distinct pencils of a range r_{hijk}, and $T_x \equiv P_{xhi}P_{xhj}P_{xij}$ is a triangle whose sides belong to Q_h, Q_i, Q_j respectively (i. e., the line $P_{xhi}P_{xhj}$ belongs to the pencil Q_h, etc.), all of whose vertices are on the same side of r_{hijk}, then of the three statements

1) $P_{xhi}Q_k$ cuts the segment $P_{xjh}P_{xij}$ in a point P_{xjk};
2) $P_{xhj}Q_k$ cuts the segment $P_{xhi}P_{xij}$ in a point P_{xik};
3) $P_{xij}Q_k$ cuts the segment $P_{xhi}P_{xhj}$ in a point P_{xhk};

one and only one is true.

Proof. Since the vertices of the triangle are all on one side of the range r_{hijk}, no side of the triangle is r_{hijk}. Then the lines $P_{xhi}Q_k$, $P_{xhj}Q_k$, $P_{xij}Q_k$ are all distinct lines of a pencil, whose center, if it exists, is not within the triangle. If the pencil has a center, the lemma follows at once from theorem g of the introduction. If it has no center, it follows from Theorem III. It is evident that by a proper assignment of the notation Q_h, Q_i, Q_j the true statement may be made any desired one of the three.

Lemma II. If T_3 and T_4 are two triangles satisfying the hypothesis of Lemma I, and such that in the triangle T_3, the line $P_{3hj}Q_k$ meets the segment $P_{3hi}P_{3ij}$ in a point P_{3ik}, then also the line $P_{4hj}Q_k$ meets the segment $P_{4hi}P_{4ij}$ in a point P_{4ik}.

Proof. Case I. (Fig. 18.) When no pair of corresponding sides of the triangles T_3 and T_4 lie in the same line, and no vertices are in common. The lines

$P_{3hi}P_{4hi}$, $P_{3hj}P_{4hj}$, $P_{3ij}P_{4ij}$ are in pencil Q_{34}. The line $P_{3ik}Q_{34}$ meets the segment $P_{4hi}P_{4ij}$ in a point P_{4ik}, by theorems on division of a plane by lines and Theorem I, if Q_{34} has a center, and by Theorem IV, if Q_{34} has no center. But

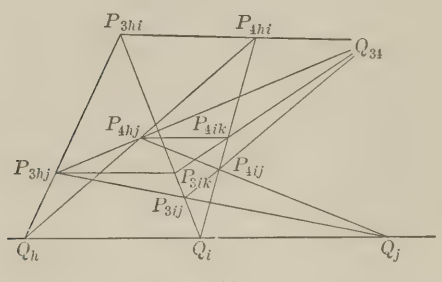

FIG. 18.

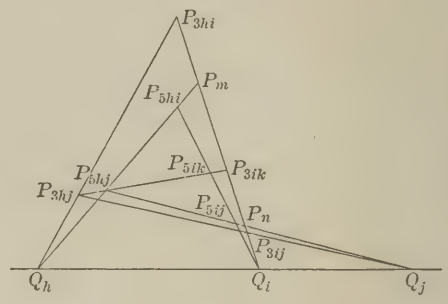

FIG. 19.

$P_{4hj}P_{4ik}$ and $P_{3hj}P_{3ik}$ determine a pencil in range with Q_h and Q_i, and hence Q_k.

Case II. (Fig. 19.) In case some parts of the triangles T_3 and T_4 are coincident, we can make the proof as follows. It is always possible to obtain a triangle T_5, satisfying hypothesis of Lemma II, all of whose parts are distinct from all the parts of T_3 and T_4, and such that $P_{5hj}Q_k$ meets the segment $P_{5hi}P_{5ij}$ in a point P_{5ik}, whence the theorem follows by Case I. Such a triangle T_5 is obtained as follows. Within the triangle T_3 and on the segment $P_{3hj}P_{3ik}$ but on no side of T_4, choose a point P_{5hj}. The line $P_{5hj}Q_h$ will meet the segment $P_{3hi}P_{3ik}$ in a point P_m, and the line $P_{5hj}Q_j$ will meet the segment $P_{3ik}P_{3ij}$ in a point P_n. Choose a point P_{5hi} on the segment $P_{5hj}P_m$ and not on any line of T_4. Then $P_{5hi}Q_i$ will meet the lines $P_{5hj}P_n$ and $P_{5hj}P_{3ik}$ in points P_{5ij} and P_{5ik} respectively, and the triangle $P_{5hj}P_{5hi}P_{5ij}$ is the triangle required.

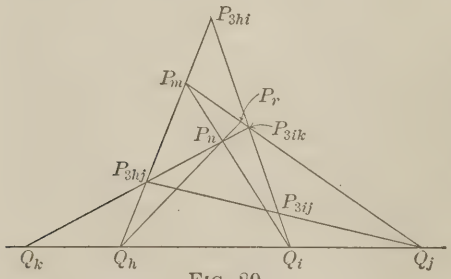

FIG. 20.

LEMMA III. (Fig. 20.) If Q_h, Q_i, Q_j, Q_k are four distinct pencils of a range, r_{hijk}, and T_3 is a triangle satisfying the hypotheses of Lemma II, then in any triangle

$$T_x \equiv \left\{ \begin{matrix} P_{xhi}P_{xhk}P_{xik} \\ P_{xhj}P_{xhk}P_{xjk} \\ P_{xij}P_{xik}P_{xjk} \end{matrix} \right\}$$

all of whose vertices are on the same side of r_{hijk}, and whose sides belong to the pencils

$$\left\{ \begin{matrix} Q_hQ_iQ_k \\ Q_hQ_jQ_k \\ Q_iQ_jQ_k \end{matrix} \right\},$$

respectively, the line and segment

$$\left\{\begin{matrix} P_{xik}\,Q_j \\ P_{xhj}\,Q_i \\ P_{xik}\,Q_h \end{matrix}\right\}, \qquad \left\{\begin{matrix} P_{xhi}\,P_{xhk} \\ P_{xhk}\,P_{xjk} \\ P_{xij}\,P_{xjk} \end{matrix}\right\},$$

meet in a point

$$\left\{\begin{matrix} P_{xhj} \\ P_{xki} \\ P_{xjh} \end{matrix}\right\}.$$

Proof. The line $Q_j P_{3ik}$ can not meet the segment $P_{3hj} P_{3ij}$, hence meets the segment $P_{3hj} P_{3hi}$ in a point P_m, and hence from the triangle $P_{3hi} P_{3hj} P_{3ik}$ and Lemma II we have the first part of the lemma. Similarly, $P_m Q_i$ must meet the segment $P_{3hj} P_{3ik}$ in a point P_n, and hence from the triangle $P_m P_{3hj} P_{3ik}$ and Lemma II we have the second part of the lemma. Also $P_n Q_h$ must meet the segment $P_m P_{3ik}$ in a point P_r, and from the triangle $P_m P_n P_{3ik}$ and Lemma II we have the third part of the lemma.

The range of pencils omitted from the set Q_k will be called r_∞, and pencils not on r_∞ will be denoted by the notation S_k.

Definition of Order. (Fig. 21.) Three pencils have order relations if and only if they are distinct, lie in a range, and no one of them lies on r_∞. Three such pencils S_i, S_j, S_k, of a range r_{ijk} meeting r_∞ in Q_∞, are in the order $S_i S_j S_k$ if and only if in a triangle $P_{xij} P_{xik} P_{xjk}$, whose sides belong to the pencils S_i, S_j, S_k respectively, and all of whose vertices are on the same side of r_∞ and r_{ijk}, the line $P_{xik} Q_\infty$ meets the segment $P_{xij} P_{xjk}$ in a point $P_{j\infty}$.

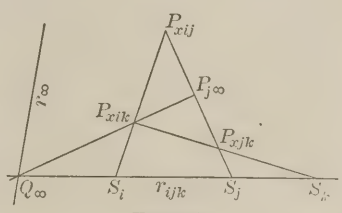

Fig. 21.

It will now be shown that the set of pencils S, subject to the above definition of order, satisfies the Axioms I–XI.

Axiom I is evident, since every point is the center of a pencil.

Axiom II is evident from the symmetry of the definition of order.

Axiom III follows from Lemma I.

Axiom IV is evident from definition.

Axiom V is evident from definition and Lemma III.

Axiom VI is a consequence of Theorem IX.

Axiom VII is obvious.

Axioms IX, X, XI are evident, since all points are centers of pencils, and they hold for the set of *all* pencils.

Axiom VIII. (Fig. 22.) This will be paraphrased as follows. If in the set S_k, S_{145}, S_{245}, and S_{345} are any three pencils not in range, and S_{235} is a pencil in the order $S_{245} S_{345} S_{235}$, and S_{135} is a pencil in the order $S_{145} S_{135} S_{345}$,

then there exists a pencil S_{125} which is in the order $S_{145}\,S_{125}\,S_{245}$, and which is in range with S_{135} and S_{235}.

Proof. The ranges $S_{135}\,S_{235}$ and $S_{145}\,S_{245}$ are distinct, and hence they determine a pencil Q_{125}. Select a triangle T containing in its interior no point of any of the ranges $S_{245}\,S_{125}$, $S_{245}\,S_{345}$, $S_{145}\,S_{345}$, r_∞. All points selected below are to be taken within the triangle T. Call the pencils common to r_∞ and $S_{145}\,S_{245}$,

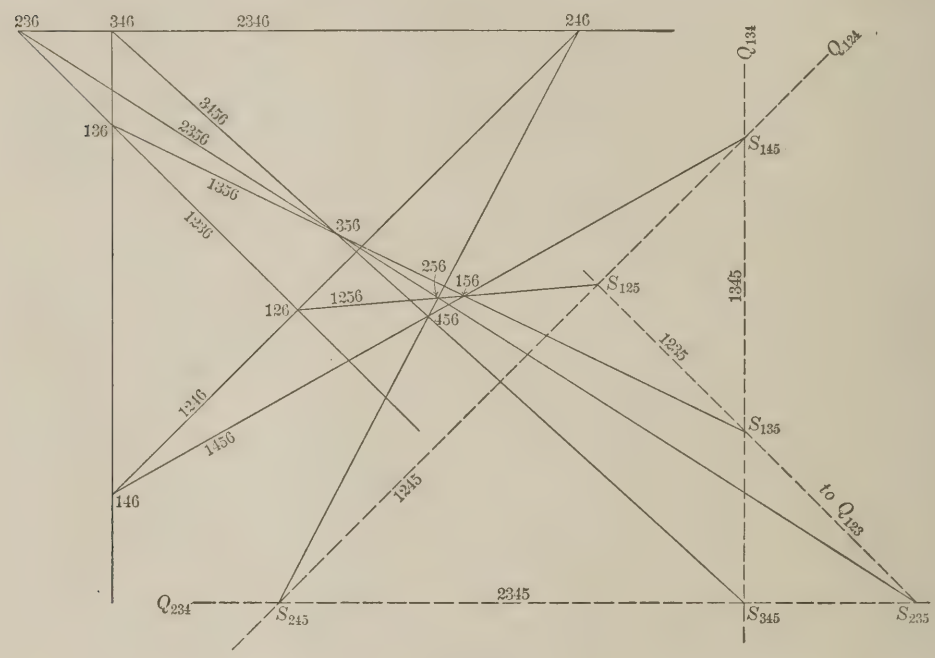

FIG. 22.

$S_{245}\,S_{345}$, $S_{145}\,S_{345}$, $S_{135}\,S_{235}$, respectively Q_{124}, Q_{234}, Q_{134}, Q_{123}. Choose in T a point P_{456}, and then points P_{146}, P_{246}, P_{346} on the lines $P_{456}\,S_{145}$, $P_{456}\,S_{245}$, $P_{456}\,S_{345}$, respectively, and such that the lines $P_{346}\,P_{246}$, $P_{346}\,P_{146}$, and $P_{146}\,P_{246}$ belong to the pencils Q_{234}, Q_{134}, and Q_{124}, respectively. From the order $S_{245}\,S_{345}\,S_{235}$ of the hypothesis, $P_{346}\,S_{235}$ meets the segment $P_{456}\,P_{246}$ of the triangle $P_{246}\,P_{346}\,P_{546}$. Hence it is possible to find in T points $P_{236}\,P_{356}\,P_{256}$ in the orders $P_{236}\,P_{346}\,P_{246}$, $P_{346}\,P_{356}\,P_{456}$, $P_{246}\,P_{256}\,P_{456}$, $P_{236}\,P_{356}\,P_{256}$. (Equally well might another set of orders have been secured.)

Then $P_{356}\,S_{135}$ meets $P_{456}\,S_{145}$ in a point P_{156} of T or does not. If it does, P_{156} is in the order $P_{146}\,P_{456}\,P_{156}$, from the order $S_{145}\,S_{135}\,S_{345}$ of the hypothesis and the triangle $P_{346}\,P_{146}\,P_{456}$. If not, choose in T a point P'_{156} in the order $P_{146}\,P_{456}\,P'_{156}$, and from the order $S_{145}\,S_{135}\,S_{345}$ of the hypothesis, and the triangle $P_{346}\,P_{146}\,P_{456}$, by Lemmas I, III, $P_{456}\,S_{135}$ meets the segment $P_{346}\,P_{146}$, and $P'_{156}\,S_{135}$ will meet the segment $P_{456}\,P_{356}$ in a point P'_{356}. The line

$P'_{356}S_{235}$ will meet the segment $P_{456}P_{256}$ in P'_{256}, and will or will not meet $P_{346}P_{246}$ in a point P_x of T. Take P'_{236} in the order $P'_{256}P'_{356}P'_{236}$, and in the order $P'_{356}P'_{236}P_x$ if P_x exists, and such that the line $P'_{236}Q_{234}$ meets the segment $P_{346}P_{146}$ in a point P'_{346}, and the segment $P_{146}P_{246}$ in P'_{246}. In this case use the points marked with primes in the further discussion, where points without the primes are used. The line $P_{356}S_{135}$ will meet the segment $P_{346}P_{146}$ in a point P_{136}. The sets of three pencils, $P_{256}P_{356}P_{456}$ and $S_{215}S_{315}S_{415}$ have corresponding ranges belonging to the pencils Q_{345}, Q_{245}, and Q_{235} of the range r_{2345}, and the lines $S_{456}S_{145}$ and $P_{356}S_{135}$ meet in P_{156}. Hence the line $P_{256}S_{125}$ also passes through the point P_{156} (by Theorem XIII), and from the order $P_{456}P_{356}P_{346}$, the point P_{136} is in the orders $P_{156}P_{356}P_{136}$ and $P_{346}P_{136}P_{146}$. Also, the line $P_{236}P_{136}$ meets the line $P_{146}P_{246}$ in a point P_{126} in the orders $P_{236}P_{136}P_{126}$ and $P_{146}P_{126}P_{246}$. From the sets of three pencils $P_{136}P_{236}P_{436}$ and $S_{135}S_{235}S_{435}$ "perspective" in P_{356}, the line $P_{136}P_{236}$ belongs to the pencil Q_{123}. From the sets of three pencils $S_{125}S_{325}S_{425}$ and $P_{126}P_{326}P_{426}$, whose sides belong in pairs to the three pencils Q_{124}, Q_{234}, Q_{123}, respectively, P_{126} is on the line $P_{256}P_{156}$. From the triangle $P_{156}P_{256}P_{456}$, the line $P_{456}Q_{124}$ cuts the segment $P_{156}P_{256}$, since it is not either side line of the triangle, and since it can not meet the segment $P_{146}P_{246}$. Hence if Q_{125} is not on r_∞, it is S_{125} and in the order $S_{145}S_{125}S_{245}$. That it is not on r_∞ may be seen as follows: $S_{245}S_{145}$ meets r_∞ in Q_{124}, of which $P_{146}P_{246}$ is a line. If this line were the same as $P_{126}P_{256}$, P_{156} would be on the line $P_{146}P_{246}$, contrary to the hypothesis. Hence the axiom holds.

The configuration used in the proof may be indicated schematically by $\left|\begin{smallmatrix} 20 & 4 \\ 3 & 15 \end{smallmatrix}\right|$, as may readily be verified from the figure.

Section IV.

In terms of the plane set of pencils S, already exhibited in Section III, a new set of elements will now be defined which will be shown to form a three dimensional space, as mentioned in the introduction. Upon the assumption of a plane satisfying the ordinary incidence and order relations, and, furthermore, the parallel axiom, Hilbert* demonstrates that the validity of the Desargues theorem is the necessary and sufficient condition that the plane constitutes a part of a three-dimensional geometry with analogous relations. That the Desargues theorem holds for a plane of such a three space is well known. On the other hand, by the aid of an algebra of segments based upon the Desargues theorem and the parallel axiom, and a resulting analytic geometry, Hilbert exhibited an analytic three-space containing the original plane, while Schor† later exhibited a geo-

* *Grundlagen der Geometrie, Festschrift,* § 29.

† Schor, D., *Neuer Beweis eines Satzes aus den Grundlagen der Geometrie von Hilbert.* Mathematische Annalen, vol. 58 (1904), pp. 427–433.

metric three-space without the algebra of segments. The method here used is analogous to that of Schor, but makes no use of the parallel axiom; this is possible in view of the use of the ideal elements already introduced on the basis of the Desargues theorem. The ideal elements of the resulting three-space, which were not considered by either Hilbert or Schor, are also developed in terms of the original plane system.

Let three special distinct pencils Q_u, Q_v, Q_w of r_∞ be chosen. Ranges S_k distinct from r_∞ and belonging to Q_u, Q_v, Q_w respectively, will be denoted by u_k, v_k, w_k, respectively. A 3-point will then be defined as follows:

Every triple of ranges u_k, v_k, w_k, not in pencil, is a 3-point. Also since every pencil S_k, with the pencils Q_u, Q_v, Q_w, respectively, determines three ranges u_k, v_k, w_k, respectively, which are in pencil, and conversely every three ranges u_k, v_k, w_k which are in pencil determine a pencil S_k, every pencil S_k will also be called a 3-point, and the notation $T_k \equiv u_k, v_k, w_k$ will be used for 3-points of either class. If T_x is a 3-point, u_x and v_x, u_x and w_x, v_x and w_x, determine three pencils W_x, V_x and U_x, respectively, which are coincident or distinct according as T_k is or is not a pencil.

Definition. Three 3-points T_x, T_y, T_z are in the order $T_x T_y T_z$ if and only if the pencils U_x, U_y, U_z are in the order $U_x U_y U_z$ as pencils.

In view of the Lemma to Theorem III, the order $U_x U_y U_z$ implies also the orders $V_x V_y V_z$ and $W_x W_y W_z$, and conversely. Also all three, or any one, of the three 3-points having order relations may be pencils, but if two are pencils the third is a pencil.

If three 3-points T_x, T_y, T_z are collinear, U_x, U_y, U_z are in range, V_x, V_y, V_z are in range, W_x, W_y, W_z are in range, and these three ranges are in pencil Q_{xyz}.

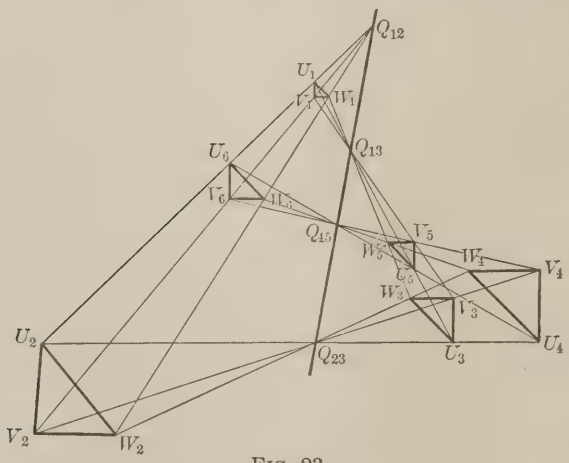

Fig. 23.

It will now be shown that the set of elements T_k satisfies the Axioms I–XI, except IX, and in place of IX possesses a property IX′, characteristic of three-dimensionality.

The validity of Axioms I–VII inclusive follows immediately from our definitions and their validity with respect to the elements S.

Axiom VIII. (Fig. 23.) *General Case.* If T_1, T_2, T_3 are any three non-collinear 3-points, forming a triangle, and T_4 is a 3-point in the order $T_2 T_3 T_4$, and T_5 is a 3-point in the order $T_3 T_5 T_1$, then there exists a 3-point T_6 in the order $T_1 T_6 T_2$, and on the line $T_4 T_5$.

Proof. Let Q_{12} be the pencil determined by the ranges U_1U_2, V_1V_2, W_1W_2; Q_{45} the pencil determined by the ranges U_4U_5, V_4V_5, W_4W_5; Q_{23} the pencil determined by U_2U_3, V_2V_3, W_2W_3; and Q_{13} the pencil determined by the ranges U_1U_3, V_1V_3, W_1W_3.

Then we have two Desargues configurations.

$$(A)\qquad \begin{array}{ccc} & Q_w & \\ V_1 & V_2 & V_3 \\ U_1 & U_2 & U_3 \\ \hline Q_{23} & Q_{13} & Q_{12} \end{array} \qquad\qquad (B)\qquad \begin{array}{ccc} & Q_w & \\ U_3 & U_4 & U_5 \\ V_3 & V_4 & V_5 \\ \hline Q_{45} & Q_{13} & Q_{23} \end{array}$$

Hence from (A) and (B) together we have Q_{23}, Q_{13}, Q_{12}, Q_{45} in range, and this gives us the hypotheses of the Desargues configuration (C), (D), (E).

$$(C)\qquad \begin{array}{ccc} & Q_{23} & \\ Q_{45} & W_4 & U_4 \\ Q_{12} & W_2 & U_2 \\ \hline Q_v & U_6 & W_6 \end{array} \qquad\qquad (D)\qquad \begin{array}{ccc} & Q_{23} & \\ Q_{45} & W_4 & V_4 \\ Q_{12} & W_2 & V_2 \\ \hline Q_u & V_6 & W_6 \end{array} \qquad\qquad (E)\qquad \begin{array}{ccc} & Q_{23} & \\ Q_{45} & U_4 & V_4 \\ Q_{12} & U_2 & V_2 \\ \hline Q_w & V_6 & U_6 \end{array}$$

From (C), (D), and (E), we know that the set of ranges U_6W_6, U_6V_6, V_6W_6 is a 3-point. It still remains to show that this 3-point is in the order $T_1T_6T_2$. But the figure formed by the pencils Q_u, Q_v, Q_w, U_1, U_2, U_3, U_4, U_5, U_6 is precisely the figure of Axiom VIII, Section III, whence U_6 is in the order $U_1U_6U_2$, and hence T_6 is in the order $T_1T_6T_2$. This also shows that no one of the three pencils U_6, V_6, W_6 could be on the range r_∞.

It is obvious that one or more of the 3-points T_1, T_2, T_3, T_4, T_5, T_6 could be pencils, but the above general proof is easily modified to fit all the special cases, and the details of these proofs are omitted.

Axiom IX is evidently *not* satisfied.

Axiom X. The hypothesis of Axiom X gives at once

$$\begin{array}{ccc} X_{12} & X_{02} & X_{01} \\ \hline X_{40} & X_{41} & X_{42} \\ X_{30} & X_{31} & X_{32} \\ & X^*_{34} & \end{array} \qquad (X = U,\ V,\ W),$$

the asterisk indicating that U^*_{34}, V^*_{34}, W^*_{34} may or may not be a 3-point, but simply that U^*_{34}, V^*_{34}, W^*_{34} exist as pencils, whether or not on r_∞.

We have then at once the configurations,

$$\begin{array}{ccc} & Q_z & \\ X_{m4} & X_{m3} & X_{mn} \\ Y_{m4} & Y_{m3} & Y_{mn} \\ \hline Q_{mn3} & Q_{mn4} & Q_{m34} \end{array} \qquad\qquad \begin{array}{ccc} & Q_z & \\ X_{n4} & X_{n3} & X_{nm} \\ Y_{n4} & Y_{n3} & Y_{nm} \\ \hline Q_{nm3} & Q_{nm4} & Q_{n34} \end{array}$$

for $z, X, Y = u, V, W; v, W, U; w, U, V$; and $m, n = 1, 2; 0, 1; 0, 2$.

From these configurations we have at once the collinearities $Q_{124}Q_{134}Q_{123}Q_{234}$, $Q_{013}Q_{014}Q_{034}Q_{134}$ and $Q_{023}Q_{024}Q_{034}Q_{234}$. These give the hypotheses for the configurations

$$\begin{array}{ccc} & Q_{jmn} & \\ X_{jm} & Y_{jm} & Q_{34m} \\ X_{jn} & Y_{jn} & Q_{34n} \\ \hline Y^*_{34} & X^*_{34} & Q_z \end{array}$$

for $j = 3, 4$; $m, n = 0, 1$; $0, 2$; $1, 2$; and $X, Y, z = U, V, w; V, W, u; W, U, v$.

From this it follows that $U^*_{34}, V^*_{34}, W^*_{34}$ is a 3-point if these pencils do not lie on r_∞, and if it is not a 3-point they do lie on r_∞.

Now let $T_{31}, T_{41}; T_{30}, T_{40}; T_{32}, T_{42}$ be any pairs whatever of 3-points on the same 3-lines, such that T_{01}, T_{02}, T_{12} exist, and it must be shown that T_{01}, T_{02}, T_{12} are collinear.

We have at once the Desargues configurations

$$\begin{array}{ccc} & X^*_{34} & \\ X_{31} & X_{30} & X_{32} \\ X_{41} & X_{40} & X_{42} \\ \hline X_{02} & X_{12} & X_{01} \end{array} \qquad (X = U, V, W),$$

and then the configurations

$$\begin{array}{ccc} & Q_z & \\ X_{ik} & X_{il} & X_{ij} \\ Y_{ik} & Y_{il} & Y_{ij} \\ \hline Q_{ijl} & Q_{ijk} & Q_{ikl} \end{array}$$

where $i, j, k, l = 0, 1, 2, 3, 4$ but all different and $z, X, Y = u, V, W$; v, W, U; w, U, V. These give the collinearities $Q_{ijk}, Q_{ijl}, Q_{ikl}, Q_{jkl}$ ($i, j, k, l = 0, 1, 2, 3, 4$ but all different).

These give the hypotheses for the configurations

$$\begin{array}{ccc} & Q_i & \\ Q_{i4j} & X_{i4} & Y_{i4} \\ Q_{i3j} & X_{i3} & Y_{i3} \\ \hline Q_z & Y_{ij} & X_{ij} \end{array} \quad \left(\begin{array}{l} i, j = 0, 1; 0, 2; 1, 2 \\ z, X, Y = \begin{cases} u, V, W \\ v, W, U \\ w, U, V \end{cases} \end{array}\right), \qquad \begin{array}{ccc} Q_u & Q_v & Q_w \\ \hline U_{ij} & V_{ij} & W_{ij} \\ U_{ik} & V_{ik} & W_{ik} \\ & Q_{012} & \end{array} \quad \left(\begin{array}{l} i, j, k = 0, 1, 2, \\ i \neq j \neq k \neq i \end{array}\right),$$

whence T_{01}, T_{02}, and T_{12} are collinear, and we have the conclusion of the theorem.

Axiom XI. The hypothesis of three 3-points in a range implies the existence of a pair of triangles the joins of whose corresponding vertices are in a pencil.

But, conversely, by Axiom X, this implies that the three 3-points are in a 3-line. But any pair of triangles with sides through these three collinear points are in pencil, by definition. Hence Axiom XI holds.

DEFINITION. A tetrahedron is a set of four planes, determined by four non-coplanar points taken in triples. The existence of such points follows from the fact that Axiom IX does not hold.

DEFINITION. A 3-space is the set of all points collinear with two distinct points of a tetrahedron.

THEOREM. (Fig. 24.) The set of 3-points T constitutes a 3-space.

Proof. Let $T_3 \equiv U_3, V_3, W_3$ (distinct) be any 3-point. Then, since U_3, V_3, W_3 as pencils of the set S, are themselves 3-points, we have a tetrahedron, T_3, U_3, V_3, W_3. Let $T_4 \equiv U_4, V_4, W_4$ be any other 3-point. Then U_3U_4, V_3V_4, W_3W_4, belong to a pencil Q_{34}. If Q_{34} is not a pencil of the range r_∞, it is a 3-point of the plane determined by U_3, V_3, W_3, and T_4 is collinear with T_3 and Q_{34}, and hence is a point of the 3-space $T_3U_3V_3W_3$.

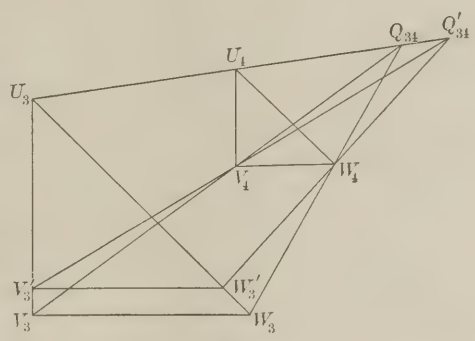

FIG. 24.

If Q_{34} is on r_∞, choose a new pencil V'_3 in the order $V_3V'_3U_3$. Then V'_3Q_u meets U_3Q_v in W'_3, and we have a 3-point T'_3. T'_3 is in the plane determined by $V_3U_3T_3$, and T'_3 and T_4 belong to a pencil $Q'_{34} \not\equiv Q_{34}$, hence not on r_∞. Hence T_4 is collinear with two distinct points of the tetrahedron, and the points T constitute a 3-space.

Ideal elements of the 3-*space* T. We have now a three-dimensional set of points satisfying order relations, and the ideal 3-space elements might be introduced in it directly, by the methods of Bonola, Schur, or Veblen.* It is interesting, from the point of view of the present paper, to do this in terms of the original plane elements.

Let $T_3 \equiv U_3, V_3, W_3$ and $T_4 \equiv U_4, V_4, W_4$ be any two 3-points. The ranges U_3U_4, V_3V_4, W_3W_4 meet the range r_∞ in three pencils $Q_{u'}, Q_{v'}, Q_{w'}$, which are in general distinct. The set of three pencils $Q_{u'}, Q_{v'}, Q_{w'}$, whether or not distinct, will be defined as an *ideal element* of the 3-space T, and will be said to lie on the 3-line $(U_3U_4,\ V_3V_4,\ W_3W_4)$. If these ideal 3-points are adjoined to the set T, giving a set T', the points of T' will not satisfy the order relations as defined, since the set Q did not, but will, evidently, have the following three properties:

Any two planes determine one and only one line.

* Cf. the previous citations.

Any two lines of a plane determine a point.

The general Desargues theorem.

In short, the set of elements T' has all the intersectional properties of a projective 3-space, except those which are dependent on Pascal's theorem. These must be excepted, since Pascal's theorem is independent of the Desargues theorem (cf. Hilbert).

Section V.

Extension to n-Space.

The method of Section IV may now be extended to cover the definition of n-space in terms of the plane set of elements already considered. We choose on the line r_∞ of the set of elements Q, n pencils, $Q_1, Q_2, \cdots, Q_n$. Any set of n ranges r_i $(i = 1, 2, \cdots, n)$, one belonging to each of the pencils $Q_1, \cdots, Q_n$, respectively, and each distinct from the range r_∞, will be defined as an n-point, and denoted by the notation N_x. These n ranges, taken in pairs, determine $n(n-1)/2$ pencils (not necessarily distinct), $S_{x,ij}$ $(i, j = 1, \cdots, n;\ i \neq j)$. Let two such n-points be N_x and N_y. The pencils $S_{x,ij}$ and $S_{y,ij}$ $(i, j = 1, \cdots, n;\ i \neq j)$ determine a range $r_{xy,ij}$ such that $r_{xy,ij}$, $r_{xy,ik}$, $r_{xy,jk}$ $(i, j, k = 1, \cdots, n;\ i \neq j \neq k \neq i)$ are in pencil.

The set of $\frac{1}{2}n(n-1)$ ranges $r_{xy,ij}$ will be called an n-line. An n-point N_z will be said to be on the n-line n_{xy}, if and only if, the pencil $S_{z,ij}$ belongs to the range $r_{xy,ij}$ $(i, j = 1, \cdots, n;\ i \neq j)$. Its *order* will be said to be the same as the order of the pencils $S_{x,ij}$, $S_{y,ij}$, $S_{z,ij}$. That this definition of order is unique is easily seen as follows. Each i, j, k may be thought of as fixing a 3-space geometry in which the order for ij, ik, jk is unique, and by a sequential process the result holds for all of the ranges, and hence we have the general result for the n-space. Hence we may write the order $N_x N_y N_z$ as implying the orders $S_{x,ij} S_{y,ij} S_{z,ij}$ for each triple of values for i, j, k, and conversely. The n-point N_x will be said to be the same as the pencil S_x if $S_{x,ij}$ is S_x for every ij, since S_x will in this case completely determine the n-point.

It can now be shown that this n-point geometry satisfies all the axioms except that of closure. Axioms I–VII inclusive follow immediately from the definitions. The validity of Axiom VIII may be shown as follows. The axiom states that N_1, N_2, N_3 are the vertices of a triangle, and if N_4 and N_5 are in the orders $N_2 N_3 N_4$ and $N_1 N_5 N_3$ respectively, then there exists an n-point N_6 in the order $N_1 N_6 N_2$ and on the line $N_4 N_5$. Consider any triple of the pencils $Q_1, \cdots, Q_n$, of r_∞, say Q_i, Q_j, Q_k. Then we have the existence of the three ranges $r_{6,i}$, $r_{6,j}$, $r_{6,k}$, of the n-point, N_6, from the corresponding theorem of the 3-space theory. Similarly, if i, j, k be allowed to take each set of values (distinct and of the set $1 \cdots n$), we obtain all of the ranges of the set forming the n-point N_6, and each of the pencils $S_{6,ij}$ will be in the order required from the corresponding 3-space theorem. Hence Axiom VIII is valid.

That Axiom IX does not hold is evident, since each pencil S_x is an n-point, but there are other n-points which do not lie in the plane of the pencils Q.

Axioms X and XI may be easily shown to hold by considering the pencils $Q_1, \cdots, Q_n$ of the range r_∞ in sets of triples, and then applying the theorem already proved for each of the resulting 3-spaces, thus obtaining the theorem for n-space.

That the space thus obtained is actually an n-space may be shown as follows: By the same method of proof as in Section IV, if for $Q_1, \cdots, Q_n$, we have an n-space, then by increasing the number of pencils by adding Q_{n+1} to the set, we introduce new elements into the geometry. But these may all be seen to be collinear with two points of $n + 1$ n-spaces, and hence constitute an $(n + 1)$-space. But the theorem is true for 3-space, and hence by induction, for any value of n.

The ideal elements which must be adjoined to this n-space to make it projective are the pencils (n in number, but not necessarily distinct) belonging to the range r_∞, in which any other n-line (i. e., its component ranges) meet this range r_∞. If we call each such set an ideal n-point, and adjoin all such ideal points to the set already obtained, we have a set possessing the properties that any two n-points determine an n-line, any two n-lines determine an n-point, and the general Desargues Theorem. The first two of these statements are evident, and the last may readily be seen by considering the parts of the figure separately, as in the proof of Axiom VIII.

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THE GROUP OF CLASSES OF CONGRUENT QUADRATIC INTEGERS WITH RESPECT TO A COMPOSITE IDEAL MODULUS*

BY

ARTHUR RANUM

Introduction.

If in the ordinary theory of rational numbers we consider a composite integer m as modulus, and if from among the classes of congruent integers with respect to that modulus we select those which are prime to the modulus, they form a well-known multiplicative group, which has been called by WEBER (*Algebra*, vol. 2, 2d edition, p. 60), the most important example of a finite abelian group. In the more general theory of numbers in an algebraic field we may in a corresponding manner take as modulus a composite ideal, which includes as a special case a composite principal ideal, that is, an integer in the field, and if we regard all those integers of the field which are congruent to one another with respect to the modulus as forming a class, and if we select those classes whose integers are prime to the modulus, they also will form a finite abelian group † under multiplication.

The investigation of the nature of this group is the object of the present paper. I shall confine my attention, however, to a quadratic number-field, and shall determine the structure of the group of classes of congruent quadratic integers with respect to any composite ideal modulus whatever. Several distinct cases arise depending on the nature of the prime ideal factors of the modulus; for every case I shall find a complete system of independent generators of the group.

Exactly as in the simpler theory of rational numbers it will appear that the solution of the problem depends essentially on the case in which the modulus is a prime-power ideal, that is, a power of a prime ideal. The most important case, however, is probably that in which the modulus is a rational principal ideal or in other words a rational integer; therefore a separate discussion will be given of this case. Another interesting case is that in which the group is

* Presented to the Society (Chicago), April 9, 1909.

† This group must not be confused with the group of classes of equivalent ideals in an algebraic field, which is also finite and abelian; the latter has been the subject of numerous investigations, in which, however, the *language* of the theory of groups has not usually been employed.

cyclic and primitive roots exist; this case arises in a variety of ways, all of which will be enumerated, and the corresponding primitive roots given.

Preliminary notions.

1. Let m be any rational integer whose prime factors are all distinct, and let $\omega = \sqrt{m}$, if $m \equiv 2$ or $3 \pmod 4$, and $\omega = \frac{1}{2}(1 + \sqrt{m})$, if $m \equiv 1 \pmod 4$; then in the quadratic number-field defined by $\sqrt{m}$ the numbers $a + b\omega$, where a and b are rational integers, are said to be the *integers of the field*, or simply *quadratic integers*. The factorization of quadratic integers, although apparently capricious and utterly lawless, has been shown to be really amenable to reason, when considered from the standpoint of, and in connexion with, certain artificial quantities called *ideals*. For an elementary and detailed discussion of ideals in a quadratic number-field, illustrated by concrete examples, see SOMMER's recent book *Vorlesungen über Zahlentheorie*,* especially pages 36–59.

Unless otherwise stated, the notation and nomenclature used by SOMMER will be followed in this paper. Thus rational integers will be denoted by Roman letters a, b, etc., quadratic integers by Greek letters α, β, etc., and *principal ideals* (Hauptideale) by the symbols (a), (α), etc. But ideals in general will be denoted by capitals A, P, etc., except that the particular letters G, H, J will be reserved for groups.

2. Let A be any ideal of the number-field (Zahlkörper) $\mathrm{k}(\sqrt{m})$; the number of integers of the field that form a complete system of residues with respect to the modulus A, that is, the number of classes of congruent integers with respect to A, is equal to the norm of A, $n(A)$. Among these classes those which are prime to A, $\Phi(A)$ in number, where the Φ-function is a generalization of Euler's ϕ-function,† evidently form an *abelian multiplicative group* G, of order $\Phi(A)$. However, it will be more convenient to let the group be made up, not of the classes themselves, but of $\Phi(A)$ quadratic integers, one chosen from each class. In other words we shall select from a complete system of residues those which are prime to the modulus, thus forming what will be called a *reduced system of residues*, and regard them as the elements of the group. This is legitimate, provided we agree that the product of two elements α and β of the group is congruent to a third element γ of the group, $\alpha\beta \equiv \gamma \pmod A$, and not necessarily equal to it. The rational integer 1 can always be taken to be the identical element of the group.

* The reader of this book should be on the lookout for a number of minor errors and some rather misleading statements. Reference will also be made to HILBERT's *Bericht* in the Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 4 (1897), pages 181–194, for the source of much of the material in SOMMER's book, and to BACHMANN's *Neuere Zahlentheorie*, Sammlung Schubert, for an introduction to the theory of algebraic numbers, in which the author, like SOMMER, confines himself to the quadratic field.

† SOMMER, l. c., pp. 78–81.

Modulus a Power of a Prime Ideal.

3. In studying this group we shall first take up the case in which the modulus A is a power of a prime ideal P, say $A = P^k$. If A is itself prime ($k = 1$), it is known that the group is *cyclic*, and therefore that *primitive roots* exist with respect to a prime ideal P (Sommer, l. c., pp. 84–86).

In order to investigate the case $k > 1$, it will be necessary to consider separately the three kinds of prime ideals, those of the first grade, both unambiguous and ambiguous, and those of the second grade (Sommer, l. c., p. 48 and p. 53). Let p be a rational prime such that the principal ideal (p) is divisible by the prime ideal P, and let P' be the conjugate of P; then the three kinds of prime ideals may be briefly characterized as follows:

First grade; unambiguous, $PP' = (p)$, $P' \neq P$,

ambiguous, $P^2 = (p)$.

Second grade; $P = (p)$.

Again, if P is of the first grade, $n(P^k) = p^k$ and $\Phi(P^k) = p^{k-1}(p-1)$, and if P is of the second grade, $n(P^k) = p^{2k}$ and $\Phi(P^k) = p^{2k-2}(p^2-1)$.

It will also be necessary to distinguish between different values of the rational prime p, according as $p = 2$, $p = 3$, or $p > 3$, and again between different forms of m, according as $m \equiv 1, 2$, or $3 \pmod 4$.

From the canonical forms of the bases of the different kinds of prime ideals, as given by Hilbert (*Bericht*, pp. 282–283) and Sommer (pp. 59–63), the corresponding canonical forms of the bases of their kth powers are easily derived and will be assumed in what follows.

Case I. *P an Unambiguous Prime of the First Grade.*

4. In this case $P = (p, a + \omega)$ and $P^k = (p^k, a_k + \omega)$; since the coefficient of ω in the basis of the modulus is 1, every integer in the field is congruent to some rational integer, and we may choose as a complete system of residues with respect to the modulus the rational integers $0, 1, \cdots, p^k - 1$. Now those which are prime to P^k are precisely those which are prime to p; so if we strike out from this complete system of residues those integers which are divisible by p, we shall have left $p^{k-1}(p-1)$ residues, one from each of the classes forming the group in question. In this first case, therefore, the group is simply isomorphic with the corresponding group in the rational number-field, and its generators can be written down immediately. There are two types, depending on the value of p.

Type 1. $p > 2$.

5. Thus, if $p > 2$, G is a cyclic group and can be generated by a single element of period $p^{k-1}(p-1)$. But it can also be generated by two indepen-

dent elements, one of period $p-1$ and the other of period p^{k-1}; and the latter choice of generators will be found more convenient for our purpose. Throughout this paper, similarly, the periods of the generators will always be chosen in such a way that no period shall be divisible both by p and by another prime factor.

Let us take $1+p$ as generator of period p^{k-1}, if $k>1$; if $k=1$, it reduces to the identical element and no longer figures as a generator. To find a generator of period $p-1$, let f_1 be a primitive root of the rational prime p; with respect to the modulus P^k, f_1 will be of period $p^i(p-1)$, where $i<k$, and $f=f_1^{p^i}$ will be of period $p-1$. *Therefore we shall take f and $1+p$ as the two required independent generators, of periods $p-1$ and p^{k-1}, respectively.*

Type 2. $p=2$.

6. If $p=2$, we obtain the second type, which is a non-cyclic group of order 2^{k-1}, having the invariants $2, 2^{k-2}(k>2)$. *Its generators will be taken to be -1 of period 2, and 5 of period 2^{k-2}. If $k=2$, there is only one generator, -1 of period 2; and if $k=1$, the group is of order 1.*

Case II. *P an Ambiguous Prime Ideal.*

7. In this case P is necessarily of the first grade and its norm p is a factor of the discriminant d of the number-field. Since $P^2=(p)$, every even power of P is a rational principal ideal, $P^{2s}=(p^s)$, and every odd power is equal to P multiplied by a rational principal ideal, $P^{2s+1}=(p^s)\cdot P$. This case will be found to give rise to five distinct types of groups, in two of which $p>2$ and in the other three $p=2$.

Subcase II_a. $p>2$.

8. By reference to Sommer's text, pages 59–63, we see that if $m\equiv 2$ or $3 \pmod 4$, $P=(p, \sqrt{m})$, and that if $m\equiv 1 \pmod 4$, then

$$\omega=\frac{1+\sqrt{m}}{2}, \qquad P=\left(p, \frac{p-1}{2}+\omega\right)=\left(p, \frac{p+\sqrt{m}}{2}\right).$$

It will be convenient to combine these two cases into one by introducing the symbol ω_1, defined by the equations

$$\omega_1=\omega=\sqrt{m}, \qquad \text{if } m\equiv 2 \text{ or } 3 \pmod 4,$$

$$\omega_1=\frac{p-1}{2}+\omega=\frac{p+\sqrt{m}}{2}, \qquad \text{if } m\equiv 1 \pmod 4.$$

In both cases, therefore, we may write

$$P=(p, \omega_1).$$

9. Further, raising both sides of this equation to even and to odd powe s, respectively, we have

$$(1) \qquad P^{2s} = (p^s, p^s\omega_1), \qquad P^{2s+1} = (p^{s+1}, p^s\omega_1).$$

In order to obtain a notation by means of which the even and the odd powers of P can be considered together, we again introduce new symbols, r and s, defined by the equations

$$\left[\frac{k}{2}\right] = s, \qquad \left[\frac{k+1}{2}\right] = r,$$

where k is a positive rational integer, and $[x]$ means the largest rational integer $\leqq x$. It follows that if $k = 2s$, $r = s$, and if $k = 2s+1$, $r = s+1$, and in either case $k = r + s$. By means of these symbols the two equations (1) can be replaced by the single equation

$$P^k = (p^r, p^s\omega_1).$$

10. Since $m \equiv 0 \pmod{p}$ and (ω_1) is not prime to (p), we can select as a reduced system of residues with respect to the modulus P^k the quadratic integers $u + v\omega_1$, in which v takes all rational integral values from 0 to $p^s - 1$, inclusive, and u takes all rational integral values from 1 to $p^r - 1$, inclusive, except those which are divisible by p. This gives p^s values to v and $p^{r-1}(p-1)$ values to u, and therefore furnishes $p^{r+s-1}(p-1)$ residues for the reduced system. This agrees with the formula $\Phi(P^k) = p^{k-1}(p-1)$ for the order of the group, since $r + s = k$.

A more convenient choice of residues for our purpose, however, is $u + v\sqrt{m}$, where u and v have the same values as before. If $m \equiv 2$ or $3 \pmod 4$, $\omega_1 = \sqrt{m}$ and the residues are also the same as before; if $m \equiv 1 \pmod 4$, $\omega_1 = \frac{1}{2}(p + \sqrt{m})$ and the residues are different, but still form a reduced system, as is evident from the formulæ

$$\sqrt{m} = -p + 2\omega_1,$$

$$\omega_1 \equiv -p\frac{p^{r-1}-1}{2} - \frac{p^s-1}{2}\sqrt{m} \pmod{P^k}.$$

11. Under this subcase all the groups are, for $p > 3$, of a single type, the third; while if $p = 3$, another type, the fourth, also occurs. In either type, however, it is immediately evident that the quadratic integers having rational residues form a cyclic subgroup analogous to type 1, and generated, therefore, by two rational integers, viz., g, of period $p - 1$, and (if $r > 1$) $1 + p$, of period p^{r-1}. If $p = 3$, $g = -1$.

12. Among the irrational elements of the group (if $k > 1$) the integer $1 + \sqrt{m}$ suggests itself as probably an available independent generator of period p^s. This conjecture will be found to be correct, if $p > 3$, and sometimes, but not always,

correct, if $p = 3$. Here the word *independent* means *independent of the rational residues.*

THEOREM. *If $k > 1$, $1 + \sqrt{m}$ is an independent generator of period p^s with respect to the modulus $P^k = (p^r, p^s\omega_1)$ in every case except that in which $p = 3$ and $m \equiv -3 \pmod{9}$.*

Proof. By means of the binomial theorem we obtain the formula

$$(2)\qquad (1+\sqrt{m})^{p^i} \equiv 1 + p^{i+1}\cdot\frac{p^i-1}{2}\cdot\frac{m}{p} + p^i\left[1 + \frac{p(p^i-1)(p^i-2)}{2\cdot 3}\cdot\frac{m}{p}\right]\sqrt{m} \qquad (\text{modd } p^{i+2},\ p^{i+2}\sqrt{m}),$$

where the coefficients are apparently rational fractions, but really integers. Putting $i = s$ in this congruence, and noting that $i + 1 \geqq r$, we see that $(1+\sqrt{m})^{p^s} \equiv 1 \pmod{P^k}$ and therefore that $1 + \sqrt{m}$ has a period which is a divisor of p^s and will have a period exactly equal to p^s and will be independent of the rational residues, if its (p^{s-1})th power is irrational. Putting $i = s - 1$ in formula (2), we see that

$$(1+\sqrt{m})^{p^{s-1}} \equiv 1 + p^s\cdot\frac{p^{s-1}-1}{2}\cdot\frac{m}{p} + p^{s-1}\cdot y\cdot\sqrt{m} \quad (\text{modd } p^{s+1},\ p^{s+1}\sqrt{m}),$$

where

$$y = 1 + \frac{p(p^{s-1}-1)(p^{s-1}-2)}{2\cdot 3}\cdot\frac{m}{p}.$$

Hence $(1+\sqrt{m})^{p^{s-1}}$ will have an irrational residue with respect to the modulus $(p^r, p^s\omega_1)$, if $y \not\equiv 0 \pmod{p}$. This is the case when $p > 3$; for then $y \equiv 1 \pmod{p}$: it is also the case, when $p = 3$ and $m \equiv 3 \pmod{9}$; for then $y \equiv 1 + \frac{1}{3}m \equiv 2 \pmod{3}$. The only other possibility, since $p > 2$, is $p = 3$ and $m \equiv -3 \pmod{9}$. But this supposition makes $y \equiv 0 \pmod{3}$, and the argument fails. The theorem is therefore proved.

Type 3. $p > 3$, *or* $p = 3$ *and* $m \equiv 3 \pmod{9}$.

13. This theorem enables us to complete the determination of the third type of group. *Thus if P is an ambiguous prime ideal and if either $p > 3$ or $p = 3$ and $m \equiv 3 \pmod{9}$, then the group of reduced residues with respect to the modulus $P^k = (p^r, p^s\omega_1)$ is generated by the three independent elements g, $1 + \sqrt{m}$* (if $k > 1$), and $1 + p$ (if $k > 2$), whose periods are $p - 1$, p^s, and p^{r-1}, respectively.*

In other words every quadratic integer α, prime to the modulus, is expressible in one and only one way as a product of the form

$$\alpha \equiv g^{n_1}(1+\sqrt{m})^{n_2}(1+p)^{n_3} \pmod{P^k},$$

* It is to be noticed that if $m \equiv 1 \pmod{4}$, $1 + \sqrt{m} = 2\omega$.

where the exponents n_1, n_2, and n_3 are residues with respect to the corresponding periods $p-1$, p^s, and p^{r-1}, respectively, and will be called the *indices* of α. If these indices range independently over complete systems of residues as to their respective moduli, the quadratic integer α will range over the entire group G. A similar definition of indices applies, of course, to every type of group.

14. It is evident that the elements whose periods are powers of p all have residues of the form $1+pu+v\sqrt{m}$ and constitute a Sylow subgroup H of order p^{k-1}, generated by $1+\sqrt{m}$ and $1+p$. It may easily be verified that the system of residues of the form $1+p^iu+p^jv\sqrt{m}$, where i and j are fixed exponents such that $1 \leqq i \leqq r$ and $0 \leqq j \leqq s$, constitute a group (a subgroup of H of order p^{k-i-j}), if, and only if, $i \leqq 2j+1$. Again, an element α is of period $p^i\,(i \leqq s)$, if, and only if, it is expressible in the form

$$\alpha \equiv 1+p^{r-i}u+p^{s-i}v\sqrt{m} \pmod{P^k},$$

where u and v are not both divisible by p.

Type 4. $p=3$, $m \equiv -3 \pmod 9$.

15. We now come to the exceptional case in which $p=3$, $m \equiv -3 \pmod 9$, and therefore $P^k=(3^r,\ 3^s\omega_1)$; it gives rise to another type of group, the fourth; $1+\sqrt{m}$ is no longer available as a generator, because it is not, in general, independent of the rational residues.

Let H, as before, be the subgroup of G whose elements are of the form $1+3u+v\sqrt{m}$ and whose order is 3^{k-1}, and let $J\,(k>1)$ be the subgroup of H whose elements are of the form $1+3u+3v\sqrt{m}$ and whose order is 3^{k-2}. From the formula

$$(1+3\sqrt{m})^{3^i} \equiv 1+3^{i+1}\sqrt{m} \quad (\text{modd } 3^{i+3},\ 3^{i+3}\sqrt{m})$$

it is immediately clear that $1+3\sqrt{m}$ is of period 3^{s-1} and independent of the rational residues, and consequently that if $k>3$, J has two independent generators, which can be taken to be 4 and $1+3\sqrt{m}$, of periods 3^{r-1} and 3^{s-1}, respectively.

16. We wish to show that, if $k>3$, H has the invariants $(3^{r-1},\ 3^{s-1},\ 3)$ and three independent generators, including those of J and one additional generator λ of period 3. This conclusion can be drawn the moment it is proved that H contains an element of period 3 not contained in J, that is, an element

$$\lambda = x+y\sqrt{m}, \tag{3}$$

such that

$$\lambda^3 \equiv 1 \pmod{P^k} \tag{4}$$

and

$$y \not\equiv 0 \pmod 3. \tag{5}$$

It is unnecessary to write $1+3u$ for x; the conditions (4) and (5) are sufficient to show that λ occurs in H. From the abstract properties of the group H we know that if there exists one such element λ, *there must exist exactly eighteen*, any one of which will suffice as an independent generator of H and therefore also of G.

17. THEOREM. *If x and y are rational integers satisfying the congruences*

$$2x \equiv -1 \quad (mod\ 3^{r-1}), \qquad r>1, \tag{6}$$

$$4\cdot\frac{m}{3}\cdot y^2 \equiv -1 \quad (mod\ 3^{s-1}), \qquad s>1, \tag{7}$$

then $\lambda = x + y\sqrt{m}$ will satisfy conditions (4) *and* (5).

Proof. (5) is an immediate consequence of (7). To show that (4) is also satisfied, we use the formula

$$8\lambda^3 = (2x)^3 + 9\cdot 2x\cdot 4\cdot\frac{m}{3}\cdot y^2 + 6y\left(4x^2 + 4\frac{m}{3}y^2\right)\sqrt{m},$$

from which, in view of (6) and (7) and the congruence $(2x)^3 \equiv -1 \pmod{3^r}$, derived from (6), we obtain the result

$$8\lambda^3 \equiv -1 + 9(-1)(-1) + 6y(1-1)\sqrt{m} \equiv 8 \quad (\text{modd } 3^r,\ 3^s\sqrt{m}),$$

and therefore also (4).

18. Since in (6) and (7) the coefficients of x and y are prime to 3, and since in (7) $4m/3$ and -1 are both non-squares, mod 3, therefore these two congruences can always be solved for x and y, and λ exists. Moreover, they have one and two solutions, respectively, which give three and six incongruent solutions, with respect to the moduli 3^r and 3^s, and determine eighteen distinct values of λ of the form $(x \pm y\sqrt{m}) + (3^{r-1}u + 3^{s-1}v\sqrt{m})$, where $u, v = 0, 1, 2$. These are precisely the eighteen residues of period 3 not contained in J. Hence (6) and (7) are necessary as well as sufficient conditions that $\lambda = x + y\sqrt{m}$ be an independent generator of period 3.

19. In order to extend the application of the theorem to the cases $k=2$ and $k=3$, in which $s=1$, we shall restrict x and y, in those cases, to the value 1. This will agree with the obvious fact that $1+\sqrt{m}$ is an independent generator of period 3 with respect to the moduli $(3, 3\omega_1)$ and $(9, 3\omega_1)$. Moreover, when the modulus is $(9, 9\omega_1)$, congruences (6) and (7) show that $1+\sqrt{m}$ is again a generator of period 3; but when the modulus is $(27, 9\omega_1)$, $4+\sqrt{m}$ is the corresponding generator. It is to be noticed that the solution of (6) can be written $x \equiv 1 + 3 + \cdots + 3^{r-2} \pmod{3^{r-1}}$.

There is one case in which the generator of period 3 is a rational multiple of $1+\sqrt{m}$ for all values of k, and that is when $m=-3$; in that case (7) takes the form $(2y)^2 \equiv 1 \pmod{3^{s-1}}$ and is satisfied by $y=x$; therefore $\lambda = x(1+\sqrt{m})$, where x is a root of the congruence (6).

20. The results obtained for type 4 may be summarized as follows: *If* $m \equiv -3 \pmod 9$ *and if* $P^k = (3^r, 3^s\omega_1)$ *is a power of an ambiguous prime ideal whose norm is* 3, *then the group of reduced residues, mod* P^k, *is generated by* -1 *of period* 2, *if* $k = 1$; *if* $k > 1$, *the quadratic integer* λ *determined by* (3), (6), *and* (7), *is an independent generator of period* 3; *if* $k > 2$, 4 *is another independent generator of period* 3^{r-1}; *and if* $k > 3$, $1 + 3\sqrt{m}$ *is still another, and the last, independent generator of period* 3^{s-1}.

The third power of every element of H is obviously equal to the third power of some element of J and is therefore of the form $1 + 9(u + v\sqrt{m})$. More generally, the 3^i-th power of every element of H is of the form $1 + 3^{i+1}(u + v\sqrt{m})$.

21. *Examples.* To illustrate the difference between the third and fourth types, take first the case where $m = -6$, $p = 3$, and $A = P^4 = (9, 9\sqrt{-6})$; here G is of type 3 and is generated by -1 of period 2, 4 of period 3, and $\alpha = 1 + \sqrt{-6}$ of period 9. The cyclic subgroup of order 9 generated by α may be written out as follows:

$$\begin{array}{lll} 1 = 1, & \alpha^3 \equiv 1 + 6\sqrt{-6}, & \alpha^6 \equiv 1 + 3\sqrt{-6}, \\ \alpha = 1 + \sqrt{-6}, & \alpha^4 \equiv 1 + 7\sqrt{-6}, & \alpha^7 \equiv 1 + 4\sqrt{-6}, \quad (\bmod P^4). \\ \alpha^2 \equiv 4 + 2\sqrt{-6}, & \alpha^5 \equiv 4 + 8\sqrt{-6}, & \alpha^8 \equiv 4 + 5\sqrt{-6}, \end{array}$$

Then take the case where $m = 6$, $p = 3$, and $A = P^4 = (9, 9\sqrt{6})$; G is of type 4 and is generated by -1 of period 2, 4 of period 3, $\lambda = 1 + \sqrt{6}$ of period 3, and $\beta = 1 + 3\sqrt{6}$ of period 3. The non-cyclic subgroup of order 9 generated by λ and β may be written out as follows:

$$\begin{array}{lll} 1 = 1, & \beta = 1 + 3\sqrt{6}, & \beta^2 \equiv 1 + 6\sqrt{6}, \\ \lambda = 1 + \sqrt{6}, & \beta\lambda \equiv 1 + 4\sqrt{6}, & \beta^2\lambda \equiv 1 + 7\sqrt{6}, \quad (\bmod P^4). \\ \lambda^2 \equiv 7 + 2\sqrt{6}, & \beta\lambda^2 \equiv 7 + 5\sqrt{6}, & \beta^2\lambda^2 \equiv 7 + 8\sqrt{6}, \end{array}$$

Subcase II_b. $p = 2$.

22. Here we shall find three new types of groups, the fifth, sixth, and seventh, depending on the value of m. If $m \equiv 2 \pmod 4$, $P = (2, \sqrt{m})$, $A = P^k = (2^r, 2^s\sqrt{m})$, and G is of the fifth type; while if $m \equiv 3 \pmod 4$, $P = (2, 1 + \sqrt{m})$, $A = P^k = (2^r, 2^s + 2^s\sqrt{m})$, and G is of the sixth or seventh type according as $m \equiv 3$ or $7 \pmod 8$; m cannot be $\equiv 1 \pmod 4$, since in that case p would not be a factor of the discriminant d. In all three types the rational residues form a subgroup J of order 2^{r-1} generated by -1, of period 2, and (if $r > 2$) 5, of period 2^{r-2}. In all three types, also, the case $k = 1$ is trivial and will be left out of account, since G is then of order 1.

Type 5. $m \equiv 2 \pmod 4$.

23. In this case the modulus is $P^k = (2^r, 2^s\sqrt{m})$, and since m is even, the elements of G are the residues $u + v\sqrt{m}$, where u takes the odd values $1, 3, \cdots, 2^r - 1$, and v ranges over *all* the integral values from 0 to $2^s - 1$. The group is easily disposed of, because only one irrational generator, $1 + \sqrt{m}$, is needed. The latter is of period 2^s and independent of the rational residues, exactly as in type 3 (§ 12, Theorem). For in the formula

$$(1 + \sqrt{m})^{2^i} \equiv 1 + 2^i\sqrt{m} \pmod{2^{i+1}, 2^{i+1}\sqrt{m}},$$

which is easily verified, we see, by putting $i = s$ and $s - 1$ in turn, that the 2^s-th power of $1 + \sqrt{m}$ is $\equiv 1 \pmod{P^k}$ and that the 2^{s-1}-th power is irrational.

Therefore the independent generators of G can be taken to be $1 + \sqrt{m}$ (*if* $k > 1$), -1 (*if* $k > 2$), *and* 5 (*if* $k > 4$), *and their periods are* 2^s, 2, *and* 2^{r-2}, *respectively.*

24. On the other hand, if $m \equiv 3 \pmod 4$, the modulus is $P^k = (2^r, 2^s + 2^s\sqrt{m})$. In both the resulting types of groups the elements are of the form $u + v(1 + \sqrt{m})$ in which u is odd, or in other words they are of the form $u + v\sqrt{m}$ in which one of the coefficients u, v is odd and the other even. In both types there is the same subgroup J formed by the rational residues, and also the same independent irrational generator of period 2^{s-1}, namely $1 + 2\sqrt{m}$. For by expanding in powers of $1 + \sqrt{m}$ and noting that

$$(1 + \sqrt{m})^2 = 2\left(\frac{m-1}{2} + (1 + \sqrt{m})\right),$$

we easily derive the formula

$$(1 + 2\sqrt{m})^{2^i} = [-1 + 2(1 + \sqrt{m})]^{2^i} \equiv 1 - 2^{i+1}(1 + \sqrt{m})$$
$$[\text{modd } 2^{i+2}, 2^{i+2}(1 + \sqrt{m})];$$

and by putting $i = s - 1$ and $s - 2$ in turn, we infer that the 2^{s-1}-th power of $1 + 2\sqrt{m}$ is $\equiv 1 \pmod A$ and that the 2^{s-2}-th power is irrational.

So far we have accounted for just half the group G; for J and $1 + 2\sqrt{m}$ generate a subgroup H of index 2 formed by those residues $u + v\sqrt{m}$ for which u is odd and v even. The other half of the group consists of the residues for which u is even and v odd. It will now be necessary to distinguish between the two cases $m \equiv 3$ and $m \equiv 7 \pmod 8$.

Type 6. $m \equiv 3 \pmod 8$.

25. Recalling the well-known fact that in the rational subgroup J the elements which are $\equiv \pm 3 \pmod 8$ are all of period 2^{r-2} $(r > 2)$, while those which are $\equiv \pm 1 \pmod 8$ are all of lower period $(r > 3)$, we see that in this case m is of

period $2^{r-2}(r > 2)$ and that J is generated by -1 and m. Now $\sqrt{m}$ is an element of G, but not of H, whose square is m. It is therefore of period $2^{r-1}(r > 2)$ and independent of -1; since its even powers are rational, it is also independent of $1 + 2\sqrt{m}$. Consequently, if $r > 2$, we may choose as three independent generators of the group G, $\sqrt{m}$, $1 + 2\sqrt{m}$, and -1, whose periods are 2^{r-1}, 2^{s-1}, and 2, respectively.

This conclusion requires a slight modification to cover the cases $r = 1$ and $r = 2$. It is easy to verify the following: if $k = 2$ $(r = 1, s = 1)$, G is generated by $\sqrt{m}$, of period 2; if $k = 3$ $(r = 2, s = 1)$, G is generated by $\sqrt{m}$, of period 4; if $k = 4$ $(r = 2, s = 2)$, G is generated by $\sqrt{m}$, of period 4, and $1 + 2\sqrt{m}$, of period 2. Every case will be included in the statement that $\sqrt{m}$ *is an independent generator of* G, *whose period is* 2^r, *if* $k = 2, 3,$ *or* 4, *and* 2^{r-1}, *if* $k > 4$; *if* $k > 3$, $1 + 2\sqrt{m}$ *is another independent generator, of period* 2^{s-1}; *and if* $k > 4$, -1 *is still another independent generator, of period* 2.

Type 7. $m \equiv 7 \pmod 8$.

26. In this case the period of m is $< 2^{r-2}$ $(r > 3)$ and that of $\sqrt{m}$ is $< 2^{r-1}$. But it is easy to find an element μ of period 4, not contained in H, which, together with 5 and $1 + 2\sqrt{m}$, will generate G. If one such element exists, *there must be exactly thirty-two*, if $r > 2$, since the invariants of G are 2^{r-2}, 2^{s-1}, and 4.

It will be sufficient to put

$$\mu = x\sqrt{m} \tag{8}$$

and determine x as a rational integer satisfying the congruence

$$mx^2 \equiv -1 \pmod{2^{r-1}}. \tag{9}$$

For since $\mu^2 \equiv -1 \pmod{2^{r-1}}$, therefore $\mu^4 \equiv 1 \pmod{2^r}$, and μ is of period 4; since the powers of 5 are all congruent to 1 (mod 8), μ is independent of 5; and since μ^2 is rational, μ is independent of $1 + 2\sqrt{m}$. Moreover, since $m \equiv -1 \pmod 8$, the congruence $mx^2 \equiv -1 \pmod 8$ and therefore also the congruence (9), can always be solved, and the element μ can always be found.

27. If $r = 2$, the congruence (9) evidently still holds and determines the generator $\sqrt{m}$, of period 4. But if $r = 1$ and $k = 2$, it will be necessary to define $x = 1$ as the corresponding root of (9) in order to obtain the generator $\sqrt{m}$; and moreover, its period in this case is 2 instead of 4.

Our final result is, therefore, that *the quadratic integer* μ, *defined by* (8) *and* (9), *is an independent generator whose period is* 2, *if* $k = 2$, *and* 4, *if* $k > 2$; *if* $k > 3$, $1 + 2\sqrt{m}$ *is another independent generator, of period* 2^{s-1}; *and if* $k > 4$, 5 *is still another independent generator, of period* 2^{r-2}.

If $k < 9$, and therefore $r < 5$, the simplest solution of (9) is $x = 1$ and the

simplest choice of the generator μ is $\sqrt{m}$. If $k \geqq 9$, this simple form is no longer in general available; but there is one case in which it is available for every value of k, and that is when $m = -1$. In that case the elements $\sqrt{-1}$, $1 + 2\sqrt{-1}$, and 5 always generate the group.

28. *Examples.* To illustrate types 5, 6, and 7, let m be chosen to be 2, 3, and -1, respectively. If $m = 2$ and $k = 5$, so that $P^5 = (8, 4\sqrt{2})$, then G is of type 5 and order 16 and is generated by $\alpha = 1 + \sqrt{2}$, of period 4, $b = -1$, of period 2, and $c = 5$, of period 2. Thus:

$$\begin{array}{llll} 1 = 1, & \alpha = 1 + \sqrt{2}, & \alpha^2 \equiv 3 + 2\sqrt{2}, & \alpha^3 \equiv 7 + \sqrt{2}, \\ c = 5, & \alpha c \equiv +5\sqrt{2}, & \alpha^2 c \equiv 7 + 2\sqrt{2}, & \alpha^3 c \equiv 3 + \sqrt{2}, \\ b = 7, & \alpha b \equiv 7 + 3\sqrt{2}, & \alpha^2 b \equiv 5 + 2\sqrt{2}, & \alpha^3 b \equiv 1 + 3\sqrt{2}, \\ bc \equiv 3, & \alpha bc \equiv 3 + 3\sqrt{2}, & \alpha^2 bc \equiv 1 + 2\sqrt{2}, & \alpha^3 bc \equiv 5 + 2\sqrt{2}, \end{array} \pmod{P^5}.$$

By way of contrast to this consider a corresponding group of type 7. E. g., take $m = -1$, $k = 5$, and $P^5 = (8, 4 + 4i)$, where $i = \sqrt{-1}$; then G is of order 16 and is generated by $\alpha = i$, of period 4, $\beta = 1 + 2i$, of period 2, and $c = 5$, of period 2. Thus:

$$\begin{array}{llll} 1 = 1, & \alpha = i, & \alpha^2 \equiv 7, & \alpha^3 \equiv 4 + 3i, \\ \beta = 1 + 2i, & \alpha\beta \equiv 6 + i, & \alpha^2\beta \equiv 3 + 2i, & \alpha^3\beta \equiv 6 + 3i, \\ c = 5, & \alpha c \equiv 4 + i, & \alpha^2 c \equiv 3, & \alpha^3 c \equiv 3i, \\ \beta c \equiv 5 + 2i, & \alpha\beta c \equiv 2 + i, & \alpha^2\beta c \equiv 7 + 2i, & \alpha^3\beta c \equiv 2 + 3i, \end{array} \pmod{P^5}.$$

To illustrate the difference between types 6 and 7 it will be better to take groups of higher order. E. g., put $m = 3$, $k = 8$, and $P^8 = (16, 16 + 16\sqrt{3})$; then G is of type 6 and order 128 and is generated by $\alpha = \sqrt{3}$ of period 8, $\beta = 1 + 2\sqrt{3}$ of period 8, and $c = -1$ of period 2. The subgroup $\{\alpha, c\}$ of order 16 may be written out as follows:

$$\begin{array}{llll} 1 = 1, & \alpha = \sqrt{3}, & c \equiv 15, & \alpha c \equiv 15\sqrt{3}, \\ \alpha^2 \equiv 3, & \alpha^3 \equiv 3\sqrt{3}, & \alpha^2 c \equiv 13, & \alpha^3 c \equiv 13\sqrt{3}, \\ \alpha^4 \equiv 9, & \alpha^5 \equiv 9\sqrt{3}, & \alpha^4 c \equiv 7, & \alpha^5 c \equiv 7\sqrt{3}, \\ \alpha^6 \equiv 11, & \alpha^7 \equiv 11\sqrt{3}, & \alpha^6 c \equiv 5, & \alpha^7 c \equiv 5\sqrt{3}, \end{array} \pmod{P^8}.$$

On the other hand, if $m = -1$, $k = 8$, and $P^8 = (16, 16 + 16i)$, then G is of the same order 128, but of type 7, and is generated by $\alpha = i$ of period 4, $\beta = 1 + 2i$ of period 8, and $c = 5$ of period 4. The subgroup $\{\alpha, c\}$ of

order 16 may be written out as follows:

$$\begin{array}{llll} 1 = 1, & \alpha = i, & \alpha^2 \equiv 15, & \alpha^3 \equiv 15i, \\ c = 5, & \alpha c \equiv 5i, & \alpha^2 c \equiv 11, & \alpha^3 c \equiv 11i, \\ c^2 \equiv 9, & \alpha c^2 \equiv 9i, & \alpha^2 c^2 \equiv 7, & \alpha^3 c^2 \equiv 7i, \\ c^3 \equiv 13, & \alpha c^3 \equiv 13i, & \alpha^2 c^3 \equiv 3, & \alpha^3 c^3 \equiv 3i, \end{array} \pmod{P^3}.$$

Case III. *P a prime ideal of the second grade.*

29. In this case $P = (p) = (p, p\omega)$, that is, the prime ideal is a rational principal ideal, and its norm is p^2. Further, $P^k = (p^k) = (p^k, p^k\omega)$. Since ω and p are relatively prime, every quadratic integer $u + v\omega$ is prime to P^k unless u and v are both divisible by p. The number of residues in G is therefore $p^{2k-2}(p^2 - 1)$. If $k > 1$, the residues $1 + p(u + v\omega)$ obviously form a Sylow subgroup H of order p^{2k-2} comprising all the elements whose periods are powers of p.

Since d, the discriminant of the field, is a non-square modulo p, the congruence $x^2 - d \equiv 0 \pmod{p}$ is irreducible and defines a *Galois field* of order p^2 whose elements may therefore be concretely represented by a complete system of residues $u + v\omega$ with respect to the modulus $(p, p\omega)$. But these residues, excepting 0, are precisely the elements of the group G in the case $k = 1$; hence the latter is a cyclic group of order $p^2 - 1$.

30. *Therefore any complete system of residues with respect to a prime ideal P of the second grade constitutes a Galois field F of order p^2.* It goes without saying that a complete system of residues with respect to a prime ideal of the first grade may also be looked upon as a Galois field of order p, and that with respect to a composite ideal the residues do not form a field at all.

Let δ_1 be a primitive root of the Galois field F, or in other words a generator of the group G, mod P. With respect to the modulus P^k, $\delta_1^{p^2-1}$ will be of period a power of p, say p^i, and $\delta = \delta_1^{p^i}$ will be of period $p^2 - 1$. We shall therefore choose δ to be a generator of the group G, mod P^k. In order to generate the entire group, it will only be necessary in addition to find generators of the subgroup H. At this point two cases must be distinguished, the case $p > 2$ in which G is of type 8, and the case $p = 2$ in which G is of type 9.

Type 8. $p > 2$.

31. In this case H is evidently generated by the two independent elements $1 + p$, and $1 + p\omega$, both of period p^{k-1}; hence *the entire group G, if $k > 1$, has the three independent generators δ, $1 + p$, and $1 + p\omega$, whose periods are $p^2 - 1$, p^{k-1}, and p^{k-1}, respectively.*

The elements having rational residues form a subgroup of order $p^{k-1}(p-1)$ generated by δ^{p+1} of period $p-1$, and $1+p$ of period p^{k-1}. It is easy to see that the system of residues of the form $1+p^i u+p^j v\omega$, where i and j are fixed exponents such that $1 \leqq i \leqq k$ and $1 \leqq j \leqq k$, constitute a group (a subgroup of H of order p^{2k-i-j}), if and only if $i \leqq 2j$. An element α is of period p^i $(i < k)$, if and only if it is expressible in the form

$$\alpha \equiv 1 + p^{k-i}(u + v\omega) \pmod{P^k},$$

where u and v are not both divisible by p.

Type 9. $p = 2$.

32. In this case $m \equiv 5 \pmod 8$, $\omega = \frac{1}{2}(1+\sqrt{m})$, $P^k = (2^k, 2^k\omega)$, and $\Phi(P^k) = 2^{2k-2}\cdot 3$. In the subgroup H whose elements are of the form $1+2(u+v\omega)$, the rational elements $1, 3, \cdots, 2^k-1$, which are now the only rational elements in the entire group G, are generated as usual by -1 and 5; but the irrational element $1+2\omega$ will no longer serve as the additional generator of H, as it is not independent of the rational elements. Now if $k > 2$, $1+4\omega$ is plainly of period 2^{k-2} and independent of the rational elements, so that -1, 5, and $1+4\omega$ generate just half of H.

Since m, being a non-square (mod 8), is of period 2^{k-2} $(k > 2)$, $\sqrt{m} = -1+2\omega$ is an irrational element of period 2^{k-1} independent of -1 and $1+4\omega$. So H is generated by $\sqrt{m}$, -1, and $1+4\omega$, if $k > 2$, and evidently by $\sqrt{m}$ and -1, if $k = 2$.

Summing up, we may say that *if P is a prime ideal of the second grade whose norm is* 4, *then in the group of reduced residues, mod P^k, the element δ defined in* § 30 *is an independent generator of period* 3; *if $k > 1$ the elements $\sqrt{m}$ and -1 are independent generators of periods 2^{k-1} and* 2, *respectively; and if $k > 2$, $1+4\omega$ is an independent generator of period 2^{k-2}.*

33. It may easily be proved that every residue of the form $1+2^{k-i}(u+v\omega)$, where $i < k$ and where u and v are not both even, is of period 2^i, except in the case where $i = k-1$, u is odd, and v is even.

While the generator δ must in general be determined separately for each value of k, there is one simple case in which the same determination of δ will hold for all values of k. Namely, if $m = -3$, $-\omega = -\frac{1}{2}(1+\sqrt{-3})$ is of period 3 irrespective of k, and can be taken to be the generator δ.

34. *Examples.* To illustrate type 8, we take $m = -1$, $p = 3$, and therefore $\omega = \sqrt{-1}$, $P^k = (3^k, 3^k\omega)$. When $k = 1$, G is cyclic, of order 8, and generated by $\delta = 1+\omega$. When $k = 2$, $\delta_1 = 1+\omega$ is of period 24, $\delta = \delta_1^3 \equiv -2+2\omega$ is of period 8, and G, of order 72, is generated by δ, 4, and $1+3\omega$, the last two being of period 3; the rational residues are generated

by $\delta^4 \equiv -1$ and 4; the powers of δ are

$$\begin{aligned} 1 &= 1, & \delta &= -2+2\omega, \\ \delta^2 &\equiv \omega, & \delta^3 &\equiv -2-2\omega, \\ \delta^4 &\equiv -1, & \delta^5 &\equiv 2-2\omega, \\ \delta^6 &\equiv -\omega, & \delta^7 &\equiv 2+2\omega, \end{aligned} \pmod{P^2}.$$

To illustrate type 9, we take $m = -3$ and therefore $\omega = \frac{1}{2}(1+\sqrt{-3})$ and $P^k = (2^k, 2^k\omega)$. When $k = 2$, G is of order 12 and is generated by $\delta = -\omega$, $\epsilon = \sqrt{-3}$, and $b = -1$, of periods 3, 2, and 2, respectively; written out in full, it is

$$\begin{aligned} &1 = 1, & &\epsilon = -1+2\omega, & &b = -1, & &\epsilon b \equiv 1+2\omega, \\ &\delta = -\omega, & &\delta\epsilon \equiv 2-\omega, & &\delta b \equiv \omega, & &\delta\epsilon b \equiv 2+\omega, \quad \pmod{P^2}. \\ &\delta^2 \equiv -1+\omega, & &\delta^2\epsilon \equiv -1-\omega, & &\delta^2 b \equiv 1-\omega, & &\delta^2\epsilon b \equiv 1+\omega, \end{aligned}$$

When $k = 3$, G is of order 48 and is generated by $\delta = -\omega$ of period 3, $\sqrt{-3}$ of period 4, -1 of period 2, and $1+4\omega$ of period 2.

Table of Moduli, Elements, and Generators.

35. For convenience of reference we shall now exhibit in tabular form the principal results already obtained, namely those relating to the nine types of groups, mod P^k, and in addition, for the sake of completeness, those relating to two other types, the tenth and eleventh, to be treated in §§ 46–51.

Modulus $A = P^k$.

I. P unambiguous, of the first grade; $PP' = (p)$, $(d/p) = 1$.

Type 1. $p > 2$, $m \equiv 1, 2$, or $3 \pmod 4$. $P = (p, a_1+\omega)$, $P^k = (p^k, a_k+\omega)$, $\Phi(P^k) = p^{k-1}(p-1)$. Elements $u = 1, 2, \cdots, p^k - 1$; $u \not\equiv 0 \pmod p$. Generators f, defined in § 5, and $1+p$; periods $p-1$ and p^{k-1}.

Type 2. $p = 2$, $m \equiv 1 \pmod 8$. $P = (2, \omega)$, $P^k = (2^k, a_k+\omega)$, $\Phi(P^k) = 2^{k-1}$. Elements $u = 1, 3, \cdots, 2^k - 1$. Generators -1 and 5; periods 2 and 2^{k-2}.

II. P ambiguous, of the first grade; $P^2 = (p)$, $(d/p) = 0$.

II_a. $p > 2$, $m \equiv 1, 2$, or $3 \pmod 4$.

Type 3. Either $p > 3$ and $m \equiv 0 \pmod p$, or $p = 3$ and $m \equiv 3 \pmod 9$. $P = (p, \omega_1)$, $P^k = (p^r, p^s\omega_1)$, $\Phi(P^k) = p^{k-1}(p-1)$. Elements $\alpha = u + v\sqrt{m}$; $u = 1, 2, \cdots, p^r - 1$; $u \not\equiv 0 \pmod p$; $v = 0, 1, \cdots, p^s - 1$. Generators g, defined in § 11, $1+\sqrt{m}$, and $1+p$; periods $p-1$, p^s, and $1+p^{r-1}$.

Type 4. $p=3$ and $m\equiv -3 \pmod 9$. $P=(3, \omega_1)$, $P^k=(3^r, 3^s\omega_1)$, $\Phi(P^k)=3^{k-1}\cdot 2$. Elements $\alpha=u+v\sqrt{m}$; $u=1, 2, \cdots, 3^r-1$; $u \not\equiv 0 \pmod 3$; $v=0, 1, \cdots, 3^s-1$. Generators -1, λ, defined in § 17, 4, and $1+3\sqrt{m}$; periods 2, 3, 3^{r-1}, and 3^{s-1}, if $k>1$; 2, 1, 1, and 1, if $k=1$.

II_b. $p=2$.

Type 5. $m\equiv 2 \pmod 4$. $P=(2, \sqrt{m})$, $P^k=(2^r, 2^s\sqrt{m})$, $\Phi(P^k)=2^{k-1}$. Elements $\alpha=u+v\sqrt{m}$; $u=1, 3, \cdots, 2^r-1$; $v=0, 1, \cdots, 2^s-1$. Generators $1+\sqrt{m}$, -1, and 5; periods 2^s, 2, and 2^{r-2}, if $k>2$; 2, 1, and 1, if $k=2$.

Type 6. $m\equiv 3 \pmod 8$. $P=(2, 1+\sqrt{m})$, $P^k=(2^r, 2^s+2^s\sqrt{m})$, $\Phi(P^k)=2^{k-1}$. Elements $\alpha=u+v\sqrt{m}$; $u=0, 1, \cdots, 2^r-1$; $v=0, 1, \cdots, 2^s-1$; $u+v\equiv 1 \pmod 2$. Generators $\sqrt{m}$, $1+2\sqrt{m}$, and -1; periods 2^{r-1}, 2^{s-1}, and 2, if $k>4$; 2^r, 2^{s-1}, and 1, if $k=2$, 3, or 4.

Type 7. $m\equiv 7 \pmod 8$. $P=(2, 1+\sqrt{m})$, $P^k=(2^r, 2^s+2^s\sqrt{m})$, $\Phi(P^k)=2^{k-1}$. Elements $\alpha=u+v\sqrt{m}$, as in type 6. Generators μ, defined in § 26, $1+2\sqrt{m}$, and 5; periods 4, 2^{s-1}, and 2^{r-2}, if $k>2$; 2, 1, and 1, if $k=2$.

III. P of the second grade; $P=(p)$, $(d/p)=-1$.

Type 8. $p>2$, $m\equiv 1, 2$, or $3 \pmod 4$. $P=(p, p\omega)$, $P^k=(p^k, p^k\omega)$, $\Phi(P^k)=p^{2k-2}(p^2-1)$. Elements $\alpha=u+v\omega$; $u, v=0, 1, \cdots, p^k-1$; u and v not both $\equiv 0 \pmod p$. Generators δ, defined in § 30, $1+p$, and $1+p\omega$; periods p^2-1, p^{k-1}, and p^{k-1}.

Type 9. $p=2$, $m\equiv 5 \pmod 8$. $P=(2, 2\omega)$, $P^k=(2^k, 2^k\omega)$, $\Phi(P^k)=2^{2k-2}\cdot 3$. Elements $\alpha=u+v\omega$; $u, v=0, 1, \cdots, 2^k-1$; u and v not both even. Generators δ, defined in § 30, $\sqrt{m}=-1+2\omega$, -1, and $1+4\omega$; periods 3, 2^{k-1}, 2, and 2^{k-2}, if $k>1$; 3, 1, 1, and 1, if $k=1$.

Modulus $A=P^kP'^k$, $P\neq P'$.

P unambiguous, of the first grade; $PP'=(p)$, $(d/p)=1$.

Type 10. $p>2$, $m\equiv 1, 2$, or $3 \pmod 4$. $P=(p, a+\sqrt{m})$, if $m\equiv 2$ or $3 \pmod 4$; $P=(p, \frac{1}{2}a+\frac{1}{2}\sqrt{m})$, if $m\equiv 1 \pmod 4$. $P^kP'^k=(p^k)$, $\Phi(A)=p^{2k-2}(p-1)^2$. Elements $\alpha=u+v\sqrt{m}$; $u, v=0, 1, \cdots, p^k-1$; $u\not\equiv \pm av \pmod p$. Generators h, defined in § 48, ζ, defined in § 48, $1+p$, and $1+p\sqrt{m}$; periods $p-1$, $p-1$, p^{k-1}, and p^{k-1}.

Type 11. $p=2$, $m\equiv 1 \pmod 8$. $P=(2, \frac{1}{2}+\frac{1}{2}\sqrt{m})$, $P^kP'^k=(2^k)$, $\Phi(A)=2^{2k-2}$. Elements $\alpha=u+v\sqrt{m}$; $u, v=0, 1, \cdots, 2^k-1$; $u+v\equiv 1 \pmod 2$. Generators -1, θ, defined in § 50, 5, and $1+2\sqrt{m}$; periods 2, 2, 2^{k-2}, and 2^{k-2}.

Modulus any Composite Ideal.

36. Having found the structure of the group G in every case in which the modulus is a power of a prime ideal, we now have sufficient material at hand for constructing the group in the general case in which the modulus is any composite ideal whatever. This is easily accomplished in a similar manner to that used in the rational number-field.*

Let $A = P_1^{k_1} \cdots P_n^{k_n}$ be any ideal whose distinct prime factors are $P_1, \cdots, P_n$. Then the order of the group G formed by a reduced system of residues with respect to the modulus A is $\Phi(A) = \Phi(P_1^{k_1}) \cdots \Phi(P_n^{k_n})$ and the group itself is the direct product of n subgroups $G_i\,(i = 1, \cdots, n)$, each of which is simply isomorphic with a corresponding group $\bar{G}_i$, whose elements form a reduced system of residues, mod $P_i^{k_i}$. Every quadratic integer α, prime to A, uniquely determines n residues $\alpha_1, \cdots, \alpha_n$, belonging respectively to $\bar{G}_1, \cdots \bar{G}_n$, which satisfy the congruences

$$(10) \qquad \alpha \equiv \alpha_1 \pmod{P_1^{k_1}}, \ \cdots, \ \alpha \equiv \alpha_n \pmod{P_n^{k_n}}.$$

When every residue α_i ranges over the elements of its group $\bar{G}_i$, α, determined uniquely by congruences (10), will range once and only once over all the elements of the group G. When every $\alpha_j = 1\ (j \neq i)$, and α_i ranges over the elements of its group $\bar{G}_i$, α will range over the elements of the subgroup G_i $(i = 1, \cdots, n)$.

37. Let $\bar{\lambda}_i$, μ_i, etc., be a *complete set of independent generators* of the group $\bar{G}_i\,(i = 1, \cdots, n)$, and let their periods be l'_i, m'_i, etc., respectively. Then every element α_i of $\bar{G}_i$ is expressible in the form $\alpha_i \equiv \bar{\lambda}_i^{l_i} \bar{\mu}_i^{m_i} \cdots \pmod{P_i^{k_i}}$, where the exponents l_i, m_i, etc., are residues with respect to the corresponding periods, l'_i, m'_i, etc. The congruences (10) now become

$$\alpha \equiv \bar{\lambda}_1^{l_1} \bar{\mu}_1^{m_1} \cdots \pmod{P_1^{k_1}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\alpha \equiv \bar{\lambda}_n^{l_n} \bar{\mu}_n^{m_n} \cdots \pmod{P_n^{k_n}}.$$

Corresponding to the generators $\bar{\lambda}_i$, $\bar{\mu}_i$, etc., of the groups $\bar{G}_i\,(i = 1, \cdots, n)$, we now define generators λ_i, μ_i, etc., of the subgroups $G_i\,(i = 1, \cdots, n)$, and therefore of the entire group G, by means of the congruences

$$\left.\begin{aligned} \lambda_i &\equiv \bar{\lambda}_i \pmod{P_i^{k_i}} \\ &\equiv 1 \pmod{P_j^{k_j}} \quad (j = 1, \cdots, n;\ j \neq i), \\ \mu_i &\equiv \bar{\mu}_i \pmod{P_i^{k_i}} \\ &\equiv 1 \pmod{P_j^{k_j}} \quad (j = 1, \cdots, n;\ j \neq i), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}\right\} \quad (i = 1, \cdots, n).$$

* Cf. Weber, *Algebra*, vol. 2, 2nd edition, pp. 60–61.

The generators $\lambda_1, \mu_1, \cdots, \lambda_n, \mu_n$, etc., so defined, are of periods $l_1', m_1', \cdots, l_n', m_n'$, etc., and form a complete set of independent generators of G, in terms of which every element α of the group can be written as follows

$$\alpha \equiv (\lambda_1^{l_1} \mu_1^{m_1} \cdots), \cdots, \alpha \equiv (\lambda_n^{l_n} \mu_n^{m_n} \cdots), \quad \bmod A.$$

The exponents $l_1, m_1, \cdots, l_n, m_n$, will be called the *indices* of the quadratic integer α with respect to the modulus A.

This completes the solution of the general problem of determining the group of classes of congruent quadratic integers with respect to any composite modulus. But there are certain special cases that merit separate treatment.

Cyclic Groups.

38. If the group G, whose modulus is the ideal A, can be generated by a single element η and is therefore *cyclic*, η is said to be a *primitive root* of A. The number of primitive roots of A, if there is one such, is $\phi[\Phi(A)]$, where ϕ is Euler's ϕ-function. If G is not cyclic, A does not possess primitive roots.

The object of this part of the paper, §§ 38–45 inclusive, is to enumerate all the cases in which G is cyclic and to give at least one primitive root in every case. G will be cyclic, (1) if there is only one independent generator, and (2) if the periods of each pair of independent generators are relatively prime. In examining for cyclic groups, therefore, it will be well to note (1) that p and $p-1$ are relatively prime, (2) if p is an odd prime, $p-1$ and p^2-1 are even, and (3) if $p > 3$, p^2-1 is divisible by 3.

Case I. *Modulus a Power of a Prime Ideal;* $A = P^k$.

39. (a) Ideal P unambiguous, of the first grade.

(a_1) $p > 2$, $m \equiv 1, 2$, or $3 \pmod 4$. Modulus $P^k = (p^k, a_k + \omega)$. In this case G is of type 1 and is cyclic for all values of k. We choose as primitive root of A the rational integer $f(1+p)$, of period $p^{k-1}(p-1)$, where f is defined in § 5.

(a_2) $p = 2$, $m \equiv 1 \pmod 8$. Here G is of type 2 and is cyclic, only if $k = 2$ or 1.

Modulus $P^2 = (4, 2a + \omega)$. Primitive root -1, of period 2.

Modulus $P = (2, \omega)$. Primitive root 1, of period 1. This case would be trivial except for its application to Case II.

(b) Ideal P ambiguous, of the first grade.

(b_1) $p > 2$, $m \equiv 1, 2$, or $3 \pmod 4$, $m \equiv 0 \pmod p$. G is of type 3 or 4, and can only be cyclic when $k = 2$ or 1.

Modulus $P^2 = (p, p\omega)$. Primitive root $g(1+\sqrt{m})$ (§ 11), of period $p(p-1)$.

Modulus $P = (p, \omega_1)$. Primitive root g, of period $p-1$.

(b_2) $p = 2$, $m \equiv 2 \pmod 4$. G is of type 5, and is cyclic when $k = 2$ or 1.

Modulus $P^2 = (2, 2\sqrt{m})$. Primitive root $1 + \sqrt{m}$, of period 2.

Modulus $P = (2, \sqrt{m})$. Primitive root 1, of period 1.

(b_3) $p = 2$, $m \equiv 3 \pmod 4$. G is of type 6 or 7 and is cyclic, when $k = 3$, 2, or 1.

Modulus $P^3 = (4, 2 + 2\sqrt{m})$. Primitive root $\sqrt{m}$, of period 4.

Modulus $P^2 = (2, 2 + 2\sqrt{m})$. Primitive root $\sqrt{m}$, of period 2.

Modulus $P = (2, 1 + \sqrt{m})$. Primitive root 1, of period 1.

(c) Ideal P of the second grade.

G is of type 8 or 9 and is cyclic only when $k = 1$.

Modulus $P = (p, p\omega)$. Primitive root δ, defined in § 30, of period $p^2 - 1$. In the two simplest cases in which $p = 3$ and $p = 2$, δ can be found easily by means of a formula. Namely, if $p = 3$, $1 + \sqrt{m}$ is always a primitive root of P, of period 8, and if $m \equiv 1 \pmod 4$, $\omega = \frac{1}{2}(1 + \sqrt{m})$ is also a primitive root; again, if $p = 2$ and therefore $m \equiv 5 \pmod 8$, $\omega = \frac{1}{2}(1 + \sqrt{m})$ is a primitive root of P, of period 3.

40. Summarizing these results, we observe first that G is always cyclic when the modulus is the first power of P whether P is of the first or second grade. This agrees with the well-known fact that every prime ideal possesses primitive roots. Next we observe that *when P is a prime ideal of the first grade, whether ambiguous or not, P^2 as well as P possesses primitive roots; and when P is an ambiguous prime whose norm is 2, and $m \equiv 3$ (mod 4), then P, P^2, and P^3 all have primitive roots. Finally, the only case in which P^k has primitive roots for all values of k is that in which P is an unambiguous prime of the first grade whose norm p exceeds 2.*

Case II. *Cyclic Groups with Modulus Divisible by Two Distinct Prime Ideals;* $A = P^k Q^l$.

41. To obtain the moduli that come under this case, we multiply together any two (more than two are evidently impossible) prime-power moduli coming under Case I, which occur for the same value of m and which are so related that the periods of their primitive roots are relatively prime. If P^k and Q^l are two such moduli, and if ρ, of period r, and σ, of period s, are a pair of their respective primitive roots, then a primitive root η of the modulus $A = P^k Q^l$ is defined by the congruences

$$\eta \equiv \rho \pmod{P^k}, \qquad \eta \equiv \sigma \pmod{Q^l}.$$

For, since r and s are relatively prime, η is of period rs.

42. Examining the moduli of case 1, we find that they fall into three classes: first, those whose primitive roots are of period 3, namely those for which P is of the second grade, $p = 2$, $k = 1$, and therefore $m \equiv 5 \pmod 8$; second, those

whose primitive roots are of period 1, namely those for which P is of the first grade, $p = 2$, $k = 1$, and therefore $m \equiv 1, 2, 3, 6$, or $7 \pmod 8$; and third, those whose primitive roots are of even period, namely all the remainder of the moduli.

Now it is clear that for a given value of m there exists just one modulus (or at most two, in a certain exceptional case) of either the first or second class, and an infinite number of moduli of the third class. But the product of two moduli of the third class will not have primitive roots. Therefore, to obtain a modulus having primitive roots we are restricted (with one exception) to multiplying a modulus of the third class by the particular modulus of the first or second class that happens to exist for the given value of m.

Subcase II_a. $m \equiv 5 \pmod 8$.

43. In this case there exists a modulus of the first class, namely a prime ideal $Q = (2, 2\omega)$ of the second grade having a primitive root $\omega = \frac{1}{2}(1 + \sqrt{m})$ of period 3. We proceed to construct moduli of the form $P^k Q$ by selecting from the third class an ideal P^k such that $p > 2$ and the period of its primitive roots is not divisible by 3. If ρ is a primitive root of P^k, then the congruences

$$\eta \equiv \rho \pmod{P^k}, \qquad \eta \equiv \omega \pmod{Q}$$

determine a primitive root η of $P^k Q$.

(a) Ideal P unambiguous, of the first grade, $(m/p) = 1$.

(a_1) $p > 3$, $p \equiv 2 \pmod 3$, $P^k = (p^k, a_k + \omega)$.

Modulus $P^k Q = (2p^k, 2a_k + 2\omega)$. Primitive root $\eta = f(1 + p) + ep^k + a_k + \omega$ of period $3p^{k-1}(p - 1)$, where f is defined in § 5, and $e = 0$ or 1 according as a_k is even or odd.

(a_2) $p = 3$, $m \equiv 13 \pmod{24}$, $P = (3, \omega)$.

Modulus $PQ = (6, 2\omega)$. Primitive root $\eta = 2 + \omega$ of period 6.

(b) Ideal P ambiguous, of the first grade, $m \equiv 0 \pmod p$.

(b_1) $p > 3$, $p \equiv 2 \pmod 3$, $P = (p, \frac{1}{2}p - \frac{1}{2} + \omega)$, $P^2 = (p, p\omega)$.

Modulus $P^2 Q = (2p, 2p\omega)$. Primitive root $\eta = (2g + p)\omega$ (see § 11) of period $3p(p - 1)$.

Modulus $PQ = (2p, p - 1 + 2\omega)$. Primitive root $\eta = g + ep + \frac{1}{2}(p - 1) + \omega$ of period $3(p - 1)$, where $e = 0$ or 1 according as $g + \frac{1}{2}(p - 1)$ is even or odd.

(b_2) $p = 3$, $m \equiv -3 \pmod{24}$, $P = (3, 1 + \omega)$.

Modulus $PQ = (6, 2 + 2\omega)$. Primitive root $\eta = \omega$, of period 6.

(c) Ideal P of the second grade.

$p = 3$, $m \equiv 5 \pmod{24}$, $P = (3, 3\omega)$.

Modulus $PQ = (6, 6\omega)$. Primitive root ω of period 24.

Subcase II_b. $m \equiv 1, 2, 3, 6$, or $7 \pmod 8$.

44. In this case there exists a modulus of the second class, namely a prime ideal $Q = (2, b + \omega)$ of the first grade, having a primitive root 1 of period 1. Any modulus of the form $P^k Q$ for which P^k belongs to the third class and $p > 2$ will obviously have a primitive root η whose period is equal to that of the primitive roots of P^k. If ρ is one of the latter, then η can be defined by the congruences

$$\eta \equiv \rho \pmod{P^k}, \qquad \eta \equiv 1 \pmod{Q}.$$

(a) Ideal P unambiguous, of the first grade, $p > 2$, $P^k = (p^k, a_k + \omega)$.

Modulus $P^k Q = (2p^k, b_k + \omega)$. Primitive root $\eta = f(1 + p) + p^k$ (see § 5), of period $p^{k-1}(p - 1)$.

In the two remaining cases, in which P is ambiguous or of the second grade, it is evident that η will be equal to ρ or $\rho + p$; therefore in every case $\eta = \rho + ep$, where $e = 0$ or 1.

(b) Ideal P ambiguous, $p > 2$, $m \equiv 0 \pmod p$, $P = (p, a + \omega)$, $P^2 = (p, p\omega)$.

Modulus $P^2 Q = (2p, bp + p\omega)$. Primitive root $\eta = g(1 + \sqrt{m}) + ep$ (see § 11), of period $p(p - 1)$.

Modulus $PQ = (2p, c + \omega)$. Primitive root $\eta = g + ep$, of period $p - 1$.

(c) Ideal P of the second grade, $p > 2$, $P = (p, p\omega)$.

Modulus $PQ = (2p, bp + p\omega)$. Primitive root $\eta = \delta + ep$ (see § 30), of period $p^2 - 1$.

Subcase II_c.

45. This is the exceptional case mentioned in § 42, in which the modulus is divisible by two distinct conjugate primes P and P'. P is therefore an unambiguous prime of the first grade, $p = 2$, and $m \equiv 1 \pmod 8$. $P = (2, \omega)$ and $P' = (2, \omega')$.

Modulus $P^2 P' = (4, 2\omega)$. Primitive root -1, of period 2.

Modulus $PP' = (2, 2\omega)$. Primitive root 1, of period 1.

We have now exhausted all the possible cases of cyclic groups.

Modulus $(p^k) = P^k P'^k$.

46. A rational prime p, considered as a principal ideal (p), may be either a prime ideal of the second grade, or the square of an ambiguous prime of the first grade, or finally the product of an unambiguous prime of the first grade and its conjugate. The corresponding group of residues with respect to the modulus (p^k) has been considered for the first two cases in §§ 29–34 and 7–28, respectively. For the last case, in which the modulus contains two distinct conjugate primes, the general procedure of §§ 36, 37 is applicable, but a special direct method will now be developed, giving simpler results.

Let P be any unambiguous prime ideal of the first grade and P' its conjugate; then $(d/p) = 1$, and $PP' = (p) = (p, p\omega)$. Let

$$A = P^k P'^k = (p^k) = (p^k, p^k \omega),$$

and let G be the group of residues prime to the modulus A. G will be of two distinct types, depending on the value of p. Continuing the former numbering, we shall call them types 10 and 11. The results to be obtained have already been tabulated in § 35.

Type 10. $p > 2$.

47. In this case m may be $\equiv 1, 2$, or $3 \pmod 4$, and $P = (p, a_1 + \omega)$, $P' = (p, a_1 + \omega')$. If $m \equiv 2$ or $3 \pmod 4$, $\omega = \sqrt{m}$, $\omega' = -\sqrt{m}$, and, putting $a_1 = a$, we have

$$(11) \qquad P = (p, a + \sqrt{m}), \qquad P' = (p, a - \sqrt{m}).$$

If $m \equiv 1 \pmod 4$,

$$a_1 + \omega = \frac{2a_1 + 1 + \sqrt{m}}{2}, \qquad a_1 + \omega' = \frac{2a_1 + 1 - \sqrt{m}}{2};$$

putting $2a_1 + 1 = a$, we have

$$(12) \qquad P = \left(p, \frac{a + \sqrt{m}}{2}\right), \qquad P' = \left(p, \frac{a - \sqrt{m}}{2}\right).$$

Instead of writing the elements of G in the form $u + v\omega$, it will be allowable and convenient, exactly as in the similar case of § 10, to write them in the form $u + v\sqrt{m}$, where u and v are rational integers, residues with respect to the modulus p^k.

Since every element $u + v\sqrt{m}$ must be prime to P and also to P', therefore in view of (11) and (12) u and v must satisfy the condition $u \not\equiv \pm av \pmod p$. This gives $p^{2k-2}(p-1)^2$ as the order of the group, in accord with the value of $\Phi(A)$ and also with the fact (see § 36) that G is the direct product of two cyclic subgroups of order $p^{k-1}(p-1)$, which are simply isomorphic with groups whose moduli are P^k and P'^k, respectively.

48. As generators we could choose two independent elements of period $p^{k-1}(p-1)$, but we shall rather, in accordance with our former rule, choose four independent generators, two of period $p-1$ and, if $k > 1$, two others of period p^{k-1}. It is obvious that the two latter can be chosen to be $1 + p$ and $1 + p\sqrt{m}$, and that they generate a subgroup H, of order p^{2k-2}, comprising the residues $1 + pu + pv\sqrt{m}$, where u and v range from 0 to $p^{k-1} - 1$.

Let h be a rational residue of period $p - 1$, like the generator f in type 1 (§ 5), and let ζ be a residue, necessarily irrational, satisfying the congruences

$$(13) \qquad \zeta \equiv h \pmod{P^k}, \qquad \zeta \equiv 1 \pmod{P'^k}.$$

Then ζ is also of period $p - 1$ and is evidently independent of h (cf. § 37).

Hence G is generated by the following four elements: h and ζ, both of period $p-1$, and if $k>1$, $1+p$ and $1+p\sqrt{m}$, both of period p^{k-1}. The rational residues are generated by $1+p$ and h; if we add to the rational residues the rational multiples of $\sqrt{m}$, we obtain a subgroup of order $2p^{k-1}(p-1)$, which is generated by $1+p$, h, and $\zeta^{\frac{1}{2}(p-1)}$. Every residue of period p^i $(0<i<k)$ is of the form $1+p^{k-i}(u+v\sqrt{m})$, where u and v are not both divisible by p.

49. *Example.* Let $m=-1$, $p=5$, and $k=1$; then $P=(5, 2+i)$, $P'=(5, 2-i)$, $A=PP'=(5, 5i)$, and $\Phi(A)=16$. The elements of G are $u+vi$, where $u, v=0, \pm 1, \pm 2$, and $u \not\equiv \pm 2v \pmod 5$. Its generators are $h=2$ and $\zeta=-1+i$, defined by (13), both of period 4. In terms of h and ζ, G may be written out, as follows:

$$\begin{array}{llll} 1=1, & \zeta=-1+i, & \zeta^2\equiv -2i, & \zeta^3\equiv 2+2i, \\ h=2, & h\zeta\equiv -2+2i, & h\zeta^2\equiv i, & h\zeta^3\equiv -1-i, \\ h^2\equiv -1, & h^2\zeta\equiv 1-i, & h^2\zeta^2\equiv 2i, & h^2\zeta^3\equiv -2-2i, \\ h^3\equiv -2, & h^3\zeta\equiv 2-2i, & h^3\zeta^2\equiv -i, & h^3\zeta^3\equiv 1+i, \end{array} \pmod{PP'}.$$

If we had used the general method of § 37, the generators would have been $-1+i$ and $-1-i$.

Type 11. $p=2$.

50. In this case $m\equiv 1 \pmod 8$, $P=(2, \omega)=(2, \frac{1}{2}+\frac{1}{2}\sqrt{m})$, and $P'=(2, \omega')=(2, \frac{1}{2}-\frac{1}{2}\sqrt{m})$; also $A=(2^k, 2^k\omega)=(2^k, 2^{k-1}+2^{k-1}\sqrt{m})$. Since the residues $u+v\sqrt{m}$ are subject to the condition $u\not\equiv v \pmod 2$, it follows that either u is odd and v even, or u is even and v odd. Except for this restriction u ranges from 0 to 2^k-1 and v ranges from 0 to $2^{k-1}-1$. G is therefore of order 2^{2k-2}, which agrees with the value of $\Phi(A)$. If $k>1$, the residues for which u is odd and v even, namely those of the form $1+2u+2v\sqrt{m}$, obviously constitute a subgroup H of index 2 and order 2^{2k-3}. H is generated by the rational residues and the irrational residue $1+2\sqrt{m}=-1+4\omega$, of period 2^{k-2}; and the former are generated, as usual, by -1 and 5, of periods 2 and 2^{k-2}, respectively. Finally, G is generated by H and a single element θ of period 2 of the form $2u+(2v+1)\sqrt{m}$; if $k>2$, *θ can be chosen in eight different ways*, the simplest of which is the following:

$$\theta = x\sqrt{m}, \tag{14}$$

where x is a root of the congruence

$$mx^2 \equiv 1 \pmod{2^k}. \tag{15}$$

This congruence can always be solved for x, since m, being congruent to

$1 \pmod 8$, is necessarily a square, mod $2^k (k > 2)$; if $k = 2$, the same is true. Therefore, if $k > 1$, θ is a generator of period 2 of the required form.

Summing up, we may say that *the group G of reduced residues with respect to the modulus $(2^k) = P^k P'^k$ is of order* 1, *if* $k = 1$ (cf. § 45, cyclic groups); *if $k > 1$, it has the two independent generators* -1, *of period* 2, *and θ defined by* (14) *and* (15), *also of period* 2; *if $k > 2$, it has the two additional independent generators* 5 *and* $1 + 2\sqrt{m}$, *both of period* 2^{k-2}.

The residues $1 + 4u + 2v\sqrt{m}$ evidently form a subgroup of order 2^{2k-4} generated by 5 and $1 + 2\sqrt{m}$.

51. *Example.* If $m = -7$ and $k = 3$, then $P = (2, \frac{1}{2} + \frac{1}{2}\sqrt{-7})$ and $A = P^3 P'^3 = (8, 4 + 4\sqrt{-7})$. G is of order 16 and is generated by the four independent elements $a = -1$, $b = 5$, $\theta = \sqrt{-7}$, and $\lambda = 1 + 2\sqrt{-7}$, all of period 2. Thus:

$$\left.\begin{array}{llll}
1=1, & \theta=\sqrt{-7}, & \lambda=1+2\sqrt{-7}, & \theta\lambda\equiv 2+\sqrt{-7}, \\
a\equiv 7, & a\theta\equiv 4+3\sqrt{-7}, & a\lambda\equiv 3+2\sqrt{-7}, & a\theta\lambda\equiv 2+3\sqrt{-7}, \\
b\equiv 5, & b\theta\equiv 4+\sqrt{-7}, & b\lambda\equiv 5+2\sqrt{-7}, & b\theta\lambda\equiv 6+\sqrt{-7}, \\
ab\equiv 3, & ab\theta\equiv 3\sqrt{-7}, & ab\lambda\equiv 7+2\sqrt{-7}, & ab\theta\lambda\equiv 6+3\sqrt{-7},
\end{array}\right\} \pmod{P^3 P'^3}.$$

Considerations of Rationality.

52. We shall now take up certain considerations of rationality that naturally give rise to two special kinds of moduli. Let $A = (gl, hl + l\omega)$ be any ideal in the number-field $\mathrm{k}(\sqrt{m})$; then $A = A_1 A_2$, where $A_1 = (l)$ and $A_2 = (g, h + \omega)$. Thus A_1 is a rational principal ideal and A_2 will be called a *completely irrational ideal*, because it contains no rational factors. If the unit ideal (1) is regarded as belonging to both classes, then every ideal whatever can be factored in one and essentially only one way into two factors, one of which is a rational principal ideal and the other a completely irrational ideal. Evidently the norm of A is $n(A) = gl^2$, and $n(A_1) = l^2$, $n(A_2) = g$.

Let us examine the character of the prime factors of A_1 and of A_2. Since a completely irrational ideal A_2 cannot be divisible by a prime of the second grade, or by a power of an ambiguous prime of the first grade higher than the first, or finally by the conjugate of any of its unambiguous prime factors of the first grade, and since the converse evidently holds, therefore A_2 is characterized by being expressible in the form

$$A_2 = P_1^{s_1} \cdots P_i^{s_i} \cdot Q_1 \cdots Q_j, \tag{16}$$

where $P_1, \cdots, P_i$ are distinct unambiguous primes of the first grade, no two of which are conjugate, and $Q_1, \cdots, Q_j$ are distinct ambiguous primes.

53. On the other hand, if a rational principal ideal A_1 contains an unambig-

uous prime factor of the first grade, it must contain the same power of its conjugate, and if it contains an ambiguous prime factor Q, the highest power of Q which it contains must be an even power. Therefore A_1 is characterized by being expressible in the form

$$(17) \qquad A_1 = (P_1 P_1')^{u_1} \cdots (P_i P_i')^{u_i} \cdot Q_1^{2v_1} \cdots Q_j^{2v_j} \cdot R_1^{w_1} \cdots R_k^{w_k},$$

where P_1 and P_1' are conjugate unambiguous primes of the first grade, $Q_1, \cdots, Q_j$ are ambiguous primes, and $R_1, \cdots, R_k$ are primes of the second grade.

In other words, *an ideal A_1 is a rational principal ideal if, and only if, its unambiguous prime factors of the first grade enter in pairs of conjugates and its ambiguous prime factors are raised to even powers.*

We now turn to the consideration of the groups whose moduli are A_1 and A_2, respectively, and their special peculiarities. It is obvious that the group whose modulus is $A_1 A_2$ is not, in general, the direct product of two subgroups simply isomorphic with the groups whose moduli are A_1 and A_2, respectively.

Modulus a completely irrational ideal.

54. First, let G be the group of reduced residues with respect to a completely irrational modulus $A_2 = (g, h + \omega)$ whose prime factors are given by formula (16). Evidently $g = n(A_2) = p_1^{s_1} \cdots p_i^{s_i} \cdot q_1 \cdots q_j$, and the order of G is $\Phi(A_2) = \phi(g) = \phi[n(A_2)]$. Now since the coefficient of ω in the canonical form of A_2 is 1, the elements of G all have rational residues and G is simply isomorphic with the group of rational residues, mod g. Conversely, if A_2 is not completely irrational, the elements of G cannot all have rational residues. For by reference to the groups of types 8–11, and to those groups of types 3–7 in which $k > 1$, we see that they all contain elements whose residues are necessarily irrational. We have therefore proved the following theorem:

A necessary and sufficient condition that the elements of a group of quadratic integers with respect to an ideal modulus all have rational residues is that the modulus be a completely irrational ideal.

Modulus a rational principal ideal.

55. Again, consider the group G of reduced residues with respect to a rational principal ideal $A_1 = (l, l\omega)$ whose prime factors are given by formula (17). In this case the group is of peculiar interest because *its elements can be defined without using the language of ideals at all.* Thus the congruence $a + b\omega \equiv a' + b'\omega \pmod{A_1}$ is equivalent to the pair of rational congruences $a \equiv a'$, $b \equiv b' \pmod{l}$, and the quadratic integer $a + b\omega$ will be prime to the ideal A_1 if, and only if, it is prime to the rational integer l. Therefore *G is really the group of classes of congruent quadratic integers with respect to a rational modulus l.*

56. In order to determine its structure when the field $k(\sqrt{m})$ and the rational integer l are given, we proceed as follows. First factor l and classify its prime factors into p's, q's, and r's in such a way that d, the discriminant of the field, is a square with respect to every p, a multiple of every q, and a non-square with respect to every r (if $p = 2$ or $r = 2$, d is a square or non-square, respectively, with respect to 8, instead of 2), or, briefly, so that $(d/p) = 1$, $(d/q) = 0$, and $(d/r) = -1$ for every p, q, and r, respectively. Suppose the result to be

$$l = p_1^{u_1} \cdots p_i^{u_i} \cdot q_1^{v_1} \cdots q_j^{v_j} \cdot r_1^{w_1} \cdots r_k^{w_k}.$$

Then the prime factors of the ideal A_1 will be precisely those given by (17), since l^2 is the norm of A_1, and since the unambiguous primes of the first grade enter in pairs of conjugates.

Now determine the groups whose moduli are the prime-power factors of A_1, except that simplicity will be gained by substituting for the pair of groups of types 1 or 2, whose moduli are $P_1^{u_1}$ and $(P_1')^{u_1}$, respectively, the single group of type 10 or 11, whose modulus is $(P_1 P_1')^{u_1}$, and by doing the same in the case of the other P's. Find the generators, therefore, of the groups of types 10 and 11, whose moduli are $(P_1 P_1')^{u_1}$, etc., of the groups of types 3–7, whose moduli are $Q_1^{2v_1}$, etc., and of the groups of types 8 and 9, whose moduli are $R_1^{w_1}$, etc. Finally, find the corresponding generators of the group G, whose modulus is (1), by using the method of § 37. In this way we obtain *a complete set of independent generators of any group of quadratic integers, whose modulus is a rational integer.*

57. *Example* 1. In the field $k(\sqrt{-35})$ find the group G whose modulus is the rational principal ideal $A_1 = (30)$. Here $\omega = \frac{1}{2}(1 + \sqrt{-35})$, $l = 30$, $d = -35$, and the elements of G are residues $a + b\omega$, mod 30, namely those which are prime to 30. Since $30 = 3 \cdot 5 \cdot 2$, and $(d/3) = 1$, $(d/5) = 0$, $(d/2) = -1$, therefore $p = 3$, $q = 5$, and $r = 2$; consequently $(3) = PP'$, $(5) = Q^2$, $(2) = R$, and $(30) = PP' \cdot Q^2 \cdot R$. The group whose modulus is PP' is of type 10 and is generated by -1 and $-1-\omega$, both of period 2; the group whose modulus is Q^2 is of type 3 and is generated by 2, of period 4, and 2ω, of period 5; the group whose modulus is R is of type 9 and is generated by ω, of period 3. The corresponding generators, λ_1, μ_1, λ_2, μ_2, λ_3, of G itself are therefore defined by the congruences

$$\begin{array}{llll}
\lambda_1 \equiv -1 & \pmod 3 \equiv 1 & \pmod 5 \equiv 1 & \pmod 2, \\
\mu_1 \equiv -1-\omega & \text{“} \equiv 1 & \text{“} \equiv 1 & \text{“}, \\
\lambda_2 \equiv 1 & \text{“} \equiv 2 & \text{“} \equiv 1 & \text{“}, \\
\mu_2 \equiv 1 & \text{“} \equiv 2\omega & \text{“} \equiv 1 & \text{“}, \\
\lambda_3 \equiv 1 & \text{“} \equiv 1 & \text{“} \equiv \omega & \text{“}.
\end{array}$$

The simplest solutions of these congruences are $\lambda_1 = 11$, $\mu_1 = 11 - 10\omega$, $\lambda_2 = 7$, $\mu_2 = -5 + 12\omega$, $\lambda_3 = 16 + 15\omega$. These are therefore independent generators of G and their periods are 2, 2, 4, 5, and 3, respectively; G is of order 240.

Example 2. By way of strong contrast to the first example take the same modulus (30), but a different field $k(\sqrt{-1})$. In this case $\omega = \sqrt{-1}$, $d = -4$, $l = 30$, $(d/5) = 1$, $(d/2) = 0$, $(d/3) = -1$, $(5) = PP'$, $(2) = Q^2$, $(3) = R$. Mod PP', the group is of type 10 and is generated by 2 and $1 + \omega$; mod Q^2, the group is of type 7 and is generated by ω; mod R, the group is of type 8 and is generated by $1 + \omega$. G is of order 256, and is generated by $\lambda_1 = 7$, $\mu_1 = 1 + 6\omega$, $\lambda_2 = 16 + 15\omega$, $\lambda_3 = 1 + 10\omega$, whose periods are 4, 4, 2, and 8, respectively.

It is to be noticed that out of the 900 residues $a + b\omega$, mod 30, those which are prime to 30 form two entirely different groups in these two examples, groups which differ in their elements, their orders, their generators, and their invariants. Even residues that are common to the two groups are not necessarily of the same period; e. g., the generator $16 + 15\omega$, which is of period 3 in the first example, is of period 2 in the second.

58. Since completing the above paper I have found in §§ 96–98 of the third volume of Weber's *Algebra*, which was published last year, some results that are rather closely connected with §§ 52–57 of this paper. The immense advantage gained by the theory of numbers from a liberal use of the group concept has been clearly shown by Weber throughout his Algebra (as well as by Bachmann in his books and by G. A. Miller in his memoirs); and although the particular kind of group that I have here investigated is, I believe, entirely new, still Weber approaches very close to it in § 98 of the third volume. His groups O, O', and O_0 are of infinite order, but the quotient-group O/O_0 is of finite order and is a special case of my group G, namely the case in which the modulus is a rational principal ideal; and the quotient-group O'/O_0 is the subgroup of G whose elements have rational residues. Moreover, his "primary ideals," of § 97, are my "completely irrational ideals" of § 52, and his "4. Satz," page 354, is practically the same as my theorem of § 54.

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A SIMPLIFIED TREATMENT OF THE REGULAR SINGULAR POINT*

BY

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The usual treatment of regular singular points involves the substitution of an infinite series in the differential equation; the series thus formally determined is then proved to be convergent. It is possible to replace this part of the work by a simpler and more direct argument, and this is given below in IV, along with an outline of the entire discussion.† The idea employed is quite similar to that used by LIAPOUNOFF in another connection (see PICARD, *Traité d'Analyse*, vol. 3, pp. 362–363). The analysis is made in full only for a single equation of the second order, and afterward the generalization is indicated.

I. We will assume that it has been proved, say by the method of successive approximations, that the general solution of

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

at a point of the x-plane where $p(x)$ and $q(x)$ are single-valued and analytic, has the form

$$y = c_1 y_1 + c_2 y_2,$$

in which y_1 and y_2 are linearly independent and analytic at the point.

II. Now let $x = a$ be an isolated singular point of one or both of the single-valued analytic functions $p(x)$ and $q(x)$, so that by definition $x = a$ is a *singular point of the differential equation.* Consider the solutions y of (1) in a certain vicinity of $x = a$ which is taken so small as to contain no further singular points; these solutions are analytic except perhaps at $x = a$, by I. If we make a positive circuit of $x = a$ the solutions will undergo a linear substitution, which we will write in the form

$$\bar{y}_1 = c_1 y_1 + c_2 y_2,$$
$$\bar{y}_2 = d_1 y_1 + d_2 y_2.$$

There will exist multiplicative solutions, $\bar{y} = \rho y$, for values of ρ such that

$$\begin{vmatrix} c_1 - \rho & d_1 \\ c_2 & d_2 - \rho \end{vmatrix} = 0.$$

* Presented to the Society, February 26, 1910.

† SCHLESINGER has avoided this substitution (*Journal für die reine und angewandte Mathematik*, vol. 132 (1907), pp. 247–254) but his method is not nearly so simple as the one here presented.

Thus we are led to the conclusion that there are two linearly independent solutions of the form

$$(2)\qquad \begin{aligned} y_1 &= (x-a)^{\kappa_1} f_1(x), \qquad \kappa_1 = \frac{\log \rho_1}{2i\pi}, \\ y_2 &= (x-a)^{\kappa_2} f_2(x), \qquad \kappa_2 = \frac{\log \rho_2}{2i\pi}; \end{aligned}$$

or in a degenerate case, $\rho_1 = \rho_2$, of the form

$$(3)\qquad y_1 = (x-a)^{\kappa_1} f_1(x), \qquad y_2 = y_1 \log(x-a) + (x-a)^{\kappa_1} f_2(x);$$

here f_1 and f_2 are *single-valued* and analytic in the neighborhood of $x = a$, and can therefore be expanded in Laurent series at this point.

III. *Regular solutions* are now defined as solutions of the form (2), (3) in which f_1, f_2 are either analytic or have a pole at $x = a$. A necessary condition that the solutions of (1) be regular at $x = a$ is found at once to be that $p(x)$ and $q(x)$ cannot become infinite to higher than the first and second orders respectively at $x = a$, by writing (1) in the form

$$\frac{1}{\begin{vmatrix} y_1' & y_1 \\ y_2' & y_2 \end{vmatrix}} \begin{vmatrix} y'' & y' & y \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix} = 0.$$

If the coefficients $p(x)$ and $q(x)$ have this character at $x = a$, that point is called a *regular singular point* of (1).

IV. *Conversely, at a regular singular point the solutions must be regular.* For write

$$z = (x-a)y'.$$

The differential equation (1) has as its equivalent the equations

$$(4)\qquad \begin{aligned} y' &= \frac{z}{x-a}, \\ z' &= (x-a)y'' + y' = -(x-a)[p(x)y' + q(x)y] + y' \\ &= -(x-a)q(x)y + \left(\frac{1}{x-a} - p(x)\right)z. \end{aligned}$$

Consider now the real positive function

$$U = |y^2| + |z^2|$$

for values of x on any ray emanating from $x = a$, and denote the distance $|x-a|$ by r. Since, if $y = \alpha + i\beta$, one has $|y^2| = \alpha^2 + \beta^2$, it is apparent that both terms of U, and therefore U also, are continuous functions of r with continuous r-derivatives. Moreover the inequality

$$\left|\frac{\partial U}{\partial r}\right| \leqq |2yy'| + |2zz'|$$

holds, inasmuch as, for any function w which is analytic, one has

$$w = Re^{\Theta i}, \qquad R = |w|,$$

$$\frac{\partial w}{\partial r} = e^{\Theta i}\left[\frac{\partial R}{\partial r} + iR\frac{\partial \Theta}{\partial r}\right], \qquad w' = \frac{dw}{dx} = \frac{1}{e^{\theta i}}\frac{\partial w}{\partial r},$$

and hence

$$\left|\frac{\partial |w|}{\partial r}\right| \leqq \left|\frac{\partial w}{\partial r}\right| = |w'|.$$

But in the two equations (4) the coefficients of y and z have poles of at most the first order at $x = a$, so that clearly one has

$$|y'|,\ |z'| \leqq \frac{M}{r}\left\{|y| + |z|\right\},$$

where M is independent of the particular ray under consideration. Accordingly there is obtained

$$\left|\frac{\partial U}{\partial r}\right| \leqq \frac{2M}{r}\left\{|y| + |z|\right\}^2 \leqq \frac{4MU}{r}.$$

This may be written

$$-\frac{4M}{r} \leqq \frac{\partial \log U}{\partial r} \leqq \frac{4M}{r}.$$

Integrating between r and r_0 $(r \leqq r_0)$ one obtains

$$-4M\log\frac{r_0}{r} \leqq \log\frac{U_0}{U} \leqq 4M\log\frac{r_0}{r},$$

so that

$$\left(\frac{r_0}{r}\right)^{-4M} \leqq \frac{U_0}{U} \leqq \left(\frac{r_0}{r}\right)^{4M}.$$

This gives the inequality

$$U \leqq U_0\left(\frac{r}{r_0}\right)^{-4M}.$$

Now U_0 is the value of U at a distance r_0 from $x = a$. Let G denote the upper limit of such values for $0 \leqq \arg(x - a) < 2\pi$. Then we have

$$U \leqq G\left(\frac{r}{r_0}\right)^{-4M}, \qquad [0 \leqq \arg(x - a) < 2\pi].$$

By definition of U it is clear then that

$$|y| \leqq \sqrt{G}\left(\frac{r}{r_0}\right)^{-2M} \qquad [0 \leqq \arg(x - a) < 2\pi],$$

i. e., *the branch of y corresponding to the condition* $0 \leqq \arg(x-a) < 2\pi$ *becomes infinite only to a finite order at* $x = a$.

From equations (2) and (3) it is now concluded that the single-valued functions f_1 and f_2, analytic except perhaps at $x = a$, have also this property. Hence, a positive integer m can be chosen so large that $(x-a)^m f_1$ and $(x-a)^m f_2$ are finite in the neighborhood of $x = a$. By Riemann's theorem these functions are analytic at $x = a$. *Therefore f_1 and f_2 are analytic or have poles at* $x = a$, which we desired to prove.

V. The formal determination of the series (2) and (3) still remains to be made, but for many purposes this expansion is not necessary. The usual discussion shows that there exist essentially two and only two expansions (2) or (3) which formally satisfy (1), and these must, of course, coincide with the desired series.

Part IV of the above treatment (as well as the remainder) admits of extension to a single equation of the nth order in y_1, or n equations of the first order in $y_1, y_2, \cdots, y_n$; in fact the first case is always reduced to the second by a substitution

$$y_2 = (x-a)y_1', \qquad y_3 = (x-a)^2 y_1'', \ \cdots.$$

If we write the linear system as follows:

$$\frac{dy_i}{dx} = \sum_{j=1}^{n} a_{ij}(x) y_j \qquad (i = 1, 2, \cdots, n), \tag{5}$$

the coefficients $a_{ij}(x)$ have then at most poles of the first order at $x = a$ if $x = a$ is a regular singular point. Hence one finds in a certain vicinity of $x = a$

$$|a_{ij}(x)| \leqq \frac{M}{r}.$$

If one considers the positive real function

$$U = \sum_{i=1}^{n} |y_i^2|$$

one finds

$$\begin{aligned} \left|\frac{\partial U}{\partial r}\right| &\leqq \sum_{i=1}^{n} |2y_i y_i'| \\ &\leqq \frac{2M}{r}\left(\sum_{i=1}^{n} |y_i|\right)^2. \qquad \text{by (5)}, \\ &\leqq \frac{2nM}{r} U. \end{aligned}$$

The remainder of the argument is now made as before.

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THE STRAIN OF A GRAVITATING, COMPRESSIBLE ELASTIC SPHERE*

BY

L. M. HOSKINS

In the solution of the problem of the strain of a homogeneous elastic sphere, as given by LAMÉ and by KELVIN, the bodily forces are supposed expressible as known functions of the coördinates of position. When self-gravitation is among the forces considered, however, the bodily forces depend in part upon the state of strain, and it is only in the case of assumed incompressibility that the usual method of solution is applicable.

The present paper has for its main object the solution of the problem of the strain of an isotropic elastic sphere, initially homogeneous, due to disturbing forces of a certain type, taking into account the changes in the gravitational forces which result from the strain.

The contents of the paper fall under the following four heads:

I. Strain of a gravitating, compressible elastic sphere under the action of tidal or centrifugal forces, with numerical computations for the case in which the surface is free from stress.

II. Strain of a gravitating, compressible elastic sphere covered by a shallow ocean, under the action of tidal or centrifugal forces.

III. Effect of compressibility upon estimates of the rigidity of the earth.

IV. Strain of a gravitating, compressible elastic sphere under the action of small disturbing forces derivable from a potential which is any spherical harmonic of degree not less than 2.

I. STRAIN OF A COMPRESSIBLE, GRAVITATING ELASTIC SPHERE UNDER THE ACTION OF TIDAL OR CENTRIFUGAL FORCES.

§ 1. *Physical analysis of the problem.*

The object of the analysis is to determine the strain produced in an elastic sphere by the action of disturbing forces of the type of tidal or centrifugal forces — i. e., forces derivable from a potential proportional to $P_2 r^2$, in which P_2 is the zonal surface harmonic $\frac{3}{2}\cos^2\theta - \frac{1}{2}$. The disturbing forces are assumed small in comparison with the actual forces of gravity, but the changes in the gravita-

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tional forces due to the strain are assumed to be of the same order as the disturbing forces. This accords with the facts in the case of the actual tidal and centrifugal forces acting upon the earth.

Before the action of the disturbing forces the body is assumed to be in equilibrium under the action of the mutual gravitation of its parts without surface stress. The stress-condition at any interior point need not be specified more definitely; it may be hydrostatic equilibrium, or any departure from that condition which is consistent with the forces of self-attraction. The stresses and strains considered in the following analysis are to be regarded as superposed upon the supposed initial condition, it being assumed that small departures from this initial condition follow the laws of elasticity as ordinarily given for a homogeneous isotropic solid.*

When the external disturbing forces come into action, the body assumes a new configuration of equilibrium. The gravitational force which now acts upon a given element of volume is different from that which originally acted upon the same element, for two reasons, — because the density of the attracted element is changed, and because the density-distribution of the attracting mass is changed. In computing the new value of the gravitational force the new configuration of the body may be treated as made up of two parts, — the original configuration, and a certain mass-distribution superposed upon it. The total attraction upon a volume-element may then be expressed as the resultant of the following parts: (1) The attraction of the original mass-configuration upon the original mass of the element; (2) the attraction of the original mass-configuration upon the increment of mass of the element; (3) the attraction of the increment of mass-distribution upon the original mass of the element; and (4) the attraction of the increment of mass-distribution upon the increment of mass of the element. Of these four forces (1) is balanced by the initial stresses and may be omitted from the equations of equilibrium of the element; (2) and (3) are of the same order of magnitude as the disturbing forces; while (4) is negligible in comparison with (2) and (3). Hence (2) and (3) are the only bodily forces to be considered in addition to the external disturbance.

* It has been held by certain writers that the existence of initial stresses of great magnitude vitiates the application of the theory of elasticity to a gravitating body comparable in size and mass with the earth. As to applications which are sometimes made of the theory, the objection appears to be undoubtedly valid. For example, it would be absurd to compute the actual stresses within the earth as if they were due to gravitation acting upon a body initially free from stress; even if such an assumption were in harmony with any reasonable supposition as to the history of the earth, the strains thus computed would far surpass the limits of elasticity. This objection does not, however, appear to hold against such an application of the laws of elasticity as is made in the present paper. To assume that a body under great stress, whether this is everywhere purely a normal pressure or whether involving also tangential stresses, is elastic for small departures from the initial configuration, is not inconsistent with what is known regarding bodies under ordinary experimental conditions. It can hardly be doubted that the great body of the earth is elastic as regards slight changes of configuration.

The original density being called ρ, the increment of density $\delta\rho$ is equal to $-\rho\Delta$, if Δ is the cubical expansion. Hence the force (2) is to be expressed as the attraction of the undisturbed mass upon a density $-\rho\Delta$, while (3) is the attraction of a mass-distribution of density $-\rho\Delta$ upon a density ρ, together with the attraction due to the surface inequality produced by the strain.

Besides the bodily forces it is necessary to consider the stress-condition at the surface. Upon the actual bounding surface it is assumed that the stress remains zero; but if (as is convenient) the equations of interior equilibrium be assumed to apply throughout the original spherical volume, the surface of this volume must be regarded as sustaining a stress whose value must be properly expressed. It is shown below that, to the first order of small quantities, this stress is radial and equal to the weight of the surface inequality produced by the strain.

§ 2. *Formation of equations of equilibrium for an element of volume.*

(*a*) *General form of equations in terms of polar coördinates.* Let the axis of symmetry of the disturbing forces be taken as initial line of a system of polar coördinates, the displacements in the three primary directions at a point (r, θ, ϕ) being called

$$u_r, \qquad u_\theta, \qquad u_\phi,$$

while Δ is the cubical expansion. In the case now under consideration

$$u_\phi = 0.$$

Let the total bodily force per unit volume have components

$$\rho F_r, \qquad \rho F_\theta,$$

(the ϕ-component being 0).

In accordance with a common notation, let μ denote the modulus of distortion and $\lambda + \frac{2}{3}\mu$ the bulk-modulus of the material. The equations of equilibrium for an elementary volume then assume the following forms:*

$$(1) \qquad (\lambda + 2\mu)\frac{\partial\Delta}{\partial r} - \frac{\mu}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left[\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right)\sin\theta\right] + \rho F_r = 0,$$

$$(2) \qquad (\lambda + 2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta} + \frac{\mu}{r\sin\theta}\frac{\partial}{\partial r}\left[\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right)\sin\theta\right] + \rho F_\theta = 0;$$

in which

$$(3) \qquad \Delta = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2 u_r \sin\theta) + \frac{\partial}{\partial\theta}(ru_\theta \sin\theta)\right].$$

(*b*) *Assumptions.* It is found that by making certain assumptions the equations are reduced to a form admitting of solution. It will be well to state these

*See A. E. H. Love's *A Treatise on the Mathematical Theory of Elasticity*, 2d edition (1906), pp. 138, 56. The terms representing bodily forces are omitted from the equations as given on p. 138.

assumptions explicitly, at the same time remarking that their justification is the fact that they lead to a solution satisfying all the conditions of the problem.

(1) The potential of the disturbing forces being proportional to $r^2 P_2$, in which $P_2 = \frac{3}{2}\cos^2\theta - \frac{1}{2}$, it is assumed that the radial displacement and the cubical expansion depend upon θ in the same way as this potential does; i. e., each is assumed to be the product of P_2 by a function of r.

(2) It is assumed that the mean radius of an originally spherical surface is unchanged by the strain.

(c) *Linear displacements replaced by ellipticity and maximum angular displacement.* It is found convenient to replace u_r and u_θ in the foregoing equations by two quantities e, α, defined as follows:

The former represents the ellipticity of the surface originally spherical and of radius r; that is,

$$u_r = -\tfrac{2}{3} e r P_2 \tag{4}$$

(it being assumed that a positive value of e corresponds to oblateness).

The quantity α represents the angular displacement of a radius vector for which $\theta = 45°$. It is then found that, for any point (r, θ),

$$\frac{u_\theta}{r} = \alpha \sin 2\theta. \tag{5}$$

This follows from the assumption that Δ involves θ only in a factor P_2.

To make the last statement clear, notice that the introduction of e in place of u_r reduces (3) to the form

$$\Delta = -\left(2e + \tfrac{2}{3} r \frac{de}{dr}\right) P_2 + \frac{1}{r \sin\theta} \frac{\partial(u_\theta \sin\theta)}{\partial\theta}.$$

In order that the last term may be equivalent to P_2 multiplied into a function of r, it is easy to show that we must have

$$u_\theta = \alpha \sin 2\theta + \frac{\beta}{\sin\theta},$$

in which α and β are independent of θ. Infinite values of u_θ being excluded, β must be 0.*

If, now, u_r and u_θ be replaced by e and α by means of (4) and (5), equations (1) and (2) take the following forms:

* It is otherwise evident that, for a given value of r, u_θ will have equal values for complementary values of θ. This is seen by considering the effect of superposing strains of the same type about different axes. Thus the θ-displacements in a given meridian plane (say that of xz) will be annulled by superposing upon the given strain either (a) an exactly similar strain symmetrical about the x-axis, or (b) a strain symmetrical about the z-axis which is just the reverse of the given strain.

$$(6)\quad \left[4(\lambda+\mu)\frac{d\alpha}{dr}-8\mu\frac{\alpha}{r}-\tfrac{2}{3}(\lambda+2\mu)r\frac{d^2e}{dr^2}-\tfrac{8}{3}(\lambda+2\mu)\frac{de}{dr}+4\mu\frac{e}{r}\right]P_2+\rho F_r=0,$$

$$(7)\quad \left[\mu r\frac{d^2\alpha}{dr^2}+4\mu\frac{d\alpha}{dr}-2(3\lambda+5\mu)\frac{\alpha}{r}+(\lambda+\mu)\frac{de}{dr}+(3\lambda+5\mu)\frac{e}{r}\right]\sin 2\theta+\rho F_\theta=0,$$

while the value of Δ becomes

$$(8)\qquad \Delta=\left(4\alpha-2e-\tfrac{2}{3}r\frac{de}{dr}\right)P_2.$$

(*d*) *Expression of bodily forces.* The bodily forces acting upon an elementary volume are (1) the disturbing force (here supposed to be the centrifugal force due to a uniform rotation about a fixed axis), (2) the gravitational force due to the change of density of the attracted element, (3) the attraction due to the change of configuration of the attracting mass. We proceed to get expressions for these in terms of r, θ, e and α.

(1) The disturbing force.

Representing the angular velocity of the rotation by ω, the centrifugal forces have a potential

$$(9)\qquad W=-\tfrac{1}{3}\omega^2r^2P_2$$

(neglecting a term independent of θ). If a is the radius of the sphere, g the value of gravity at the surface, and c the ratio of equatorial centrifugal force to gravity, i. e.,

$$(10)\qquad c=\frac{\omega^2a}{g},$$

we may write

$$(11)\qquad W=-\frac{cg}{3a}r^2P_2.$$

The parts of F_r and F_θ corresponding to this potential are

$$(12)\qquad \frac{\partial W}{\partial r}=-\frac{2cg}{3a}rP_2,$$

$$(13)\qquad \frac{1}{r}\frac{\partial W}{\partial\theta}=\frac{cg}{2a}r\sin 2\theta.$$

(2) Force due to change of density of attracted element.

The attraction per unit mass at any point, due to the original homogeneous sphere, is a radial force equal to

$$-g\frac{r}{a}.$$

The increment of density of the attracted element being $-\rho\Delta$, the force per unit volume is

$$\left(-g\frac{r}{a}\right)(-\rho\Delta)=g\frac{\rho r\Delta}{a};$$

hence this force is accounted for by adding to the value of ρF_r a term

$$\frac{\rho g r \Delta}{a} = \rho g \frac{r}{a}\left(4\alpha - 2e - \tfrac{2}{3} r \frac{de}{dr}\right) P_2. \tag{14}$$

(3) Force due to change of configuration of the attracting mass.

We have now to express the value of the potential at any point due to a mass-distribution of density $-\rho\Delta$ throughout the original spherical volume, together with that due to the surface layer superposed upon this sphere by reason of the strain.

Consider a shell of radius r', thickness dr', and density

$$-\rho\Delta' = -\rho\left(4\alpha' - 2e' - \tfrac{2}{3} r' \frac{de'}{dr'}\right) P_2.$$

The potential at a point (r, θ) due to this shell is

$$-\tfrac{4}{5}\pi\gamma\rho\left(4\alpha' - 2e' - \tfrac{2}{3} r' \frac{de'}{dr'}\right)\frac{r^2}{r'}\, dr' \cdot P_2 \qquad (\text{if } r < r'), \tag{15}$$

$$-\tfrac{4}{5}\pi\gamma\rho\left(4\alpha' - 2e' - \tfrac{2}{3} r' \frac{de'}{dr'}\right)\frac{r'^4}{r^3}\, dr' \cdot P_2 \qquad (\text{if } r > r'). \tag{16}$$

Hence the potential at any point (r, θ) within the spherical volume of radius a, due to the density-distribution $-\rho\Delta$, is

$$-\frac{4\pi\gamma\rho P_2}{5}\left[\frac{1}{r^3}\int_0^r \left(4\alpha' - 2e' - \tfrac{2}{3} r' \frac{de'}{dr'}\right) r'^4 dr' + r^2 \int_r^a \left(4\alpha' - 2e' - \tfrac{2}{3} r' \frac{de'}{dr'}\right)\frac{dr'}{r'}\right]. \tag{17}$$

To this must be added the potential due to the surface inequality. If e_1 is the surface value of e, the thickness of the layer is $-\tfrac{2}{3} e_1 a P_2$, and the potential due to it at a point (r, θ) within the body is

$$-\tfrac{8}{15}\pi\gamma\rho e_1 r^2 P_2. \tag{18}$$

The total potential due to the change of mass-distribution may therefore be written in the following form, in which g replaces $\tfrac{4}{3}\pi\gamma\rho a$, the surface value of gravity:

$$V = -\frac{3gP_2}{5a}\left[\frac{1}{r^3}\int_0^r \left(4\alpha' - 2e' - \tfrac{2}{3} r' \frac{de'}{dr'}\right) r'^4 dr' + r^2 \int_r^a \left(4\alpha' - 2e' - \tfrac{2}{3} r' \frac{de'}{dr'}\right)\frac{dr'}{r'} + \tfrac{2}{3} e_1 r^2\right]. \tag{19}$$

By partial integration we find

$$\frac{1}{r^3}\int_0^r \left(4\alpha' - 2e' - \tfrac{2}{3} r' \frac{de'}{dr'}\right) r'^4 dr' = \frac{4}{3r^3}\int_0^r (3\alpha' + e') r'^4 dr' - \tfrac{2}{3} r^2 e,$$

$$r^2\int_r^a\left(4\alpha'-2e'-\tfrac{2}{3}r'\frac{de'}{dr'}\right)\frac{dr'}{r'}=2r^2\int_r^a(2\alpha'-e')\frac{dr'}{r'}+\tfrac{2}{3}r^2(e-e_1),$$

so that the expression for V reduces to

$$V=-\frac{2gP_2}{5a}\left[\frac{2}{r^3}\int_0^r(3\alpha'+e')r'^4dr'+3r^2\int_r^a(2\alpha'-e')\frac{dr'}{r'}\right]. \tag{20}$$

The forces derived from the potential V are (per unit mass)

$$\frac{\partial V}{\partial r}=\frac{12gP_2}{5a}\left[\frac{1}{r^4}\int_0^r(3\alpha'+e')r'^4dr'-r\int_r^a(2\alpha'-e')\frac{dr'}{r'}-\tfrac{5}{6}er\right], \tag{21}$$

$$\frac{1}{r}\frac{\partial V}{\partial\theta}=\frac{3g\sin 2\theta}{5a}\left[\frac{2}{r^4}\int_0^r(3\alpha'+e')r'^4dr'+3r\int_r^a(2\alpha'-e')\frac{dr'}{r'}\right]. \tag{22}$$

(4) Components of total bodily force.

Combining the foregoing results, we have for the components of the bodily force per unit volume at the point (r,θ),

$$\begin{aligned}\rho F_r=\frac{\rho gP_2}{a}\Bigg[&-\tfrac{2}{3}cr+\left(4\alpha-4e-\tfrac{2}{3}r\frac{de}{dr}\right)r\\&+\frac{12}{5r^4}\int_0^r(3\alpha'+e')r'^4dr'-\frac{12r}{5}\int_r^a(2\alpha'-e')\frac{dr'}{r'}\Bigg],\end{aligned} \tag{23}$$

$$\rho F_\theta=\frac{\rho g\sin 2\theta}{a}\left[\tfrac{1}{2}cr+\frac{6}{5r^4}\int_0^r(3\alpha'+e')r'^4dr'+\frac{9r}{5}\int_r^a(2\alpha'-e')\frac{dr'}{r'}\right]. \tag{24}$$

(e) *Final form of equations of equilibrium for elementary volume.* In writing the two equations of equilibrium for a volume element in their final form, it is convenient to introduce the factor r throughout each equation, and then to replace r as independent variable by $x=r/a$. The equations then take the following forms:

$$\begin{aligned}&4(\lambda+\mu)x\frac{d\alpha}{dx}-8\mu\alpha-\tfrac{2}{3}(\lambda+2\mu)x^2\frac{d^2e}{dx^2}-\tfrac{8}{3}(\lambda+2\mu)x\frac{de}{dx}+4\mu e\\&\quad+\rho ga\Bigg[-\tfrac{2}{3}cx^2+\left(4\alpha-4e-\tfrac{2}{3}x\frac{de}{dx}\right)x^2\\&\qquad+\frac{12}{5x^3}\int_0^x(3\alpha'+e')x'^4dx'-\frac{12x^2}{5}\int_x^1(2\alpha'-e')\frac{dx'}{x'}\Bigg]=0,\end{aligned} \tag{25}$$

$$\begin{aligned}&\mu x^2\frac{d^2\alpha}{dx^2}+4\mu x\frac{d\alpha}{dx}-2(3\lambda+5\mu)\alpha+(\lambda+\mu)x\frac{de}{dx}+(3\lambda+5\mu)e\\&+\rho ga\left[\tfrac{1}{2}cx^2+\frac{6}{5x^3}\int_0^x(3\alpha'+e')x'^4dx'+\frac{9x^2}{5}\int_x^1(2\alpha'-e')\frac{dx'}{x'}\right]=0.\end{aligned} \tag{26}$$

§ 3. *Surface conditions.*

Upon the bounding surface of the strained body the stress is zero. It seems most convenient, however, to treat the strained body as extending to or beyond the original spherical boundary $r = a$ in every direction, and to determine what stress must exist upon this surface in order that the surface $r = a(1 - \frac{2}{3}e_1 P_2)$ may be free from stress. For this purpose consider the layer of matter of density ρ included between the surfaces $r = a$, $r = a(1 - \frac{2}{3}e_1 P_2)$. Let equations of equilibrium be formed for an elementary portion of this layer bounded by a meridian plane and a plane parallel to it at an infinitesimal distance h, and by two radial planes perpendicular to the meridian inclined to each other at an infinitesimal angle ψ; the section of such an element by the meridian plane is represented by $ABB'A'$ (Fig. 1).

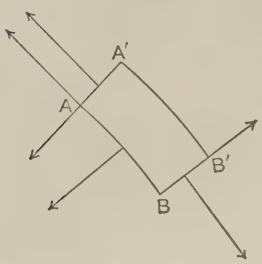

FIG. 1.

We have

$$AB = a\psi, \qquad AA' = -\tfrac{2}{3}ae_1 P_2,$$

$$BB' = AA' + \psi \frac{\partial}{\partial\theta}(-\tfrac{2}{3}ae_1 P_2) = AA' + ae_1 \psi \sin 2\theta.$$

The stresses per unit area at the point A are

$$\widehat{rr} = \lambda\Delta + 2\mu\frac{\partial u_r}{\partial r} = \left[4\lambda\alpha - 2(\lambda + \tfrac{2}{3}\mu)e - \tfrac{2}{3}(\lambda + 2\mu)x\frac{de}{dx}\right]P_2, \tag{27}$$

$$\begin{aligned}\widehat{\theta\theta} &= \lambda\Delta + 2\mu\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\\ &= \left[4\lambda\alpha - 2(\lambda + \tfrac{2}{3}\mu)e - \tfrac{2}{3}\lambda x\frac{de}{dx}\right]P_2 + 4\mu\alpha\cos 2\theta,\end{aligned} \tag{28}$$

$$\widehat{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) = \mu\left(e + x\frac{d\alpha}{dx}\right)\sin 2\theta. \tag{29}$$

From these expressions values may be found for the total normal and tangential forces on the faces AB, AA', BB' of the element, while the total bodily force is the weight of the element, which is equal to ρg into its volume. By writing the two equations of equilibrium for the element, and retaining only terms of lowest order in the small quantities e, α, it is found that

$$-\widehat{rr} + \rho g(\tfrac{2}{3}eaP_2) = 0, \tag{30}$$

$$\widehat{r\theta} = 0. \tag{31}$$

The equations to be satisfied at the surface $r = a$ are therefore

$$(32)\qquad \frac{\widehat{rr}}{P_2} = 4\lambda\alpha - 2(\lambda + \tfrac{2}{3}\mu)e - \tfrac{2}{3}(\lambda + 2\mu)x\frac{de}{dx} = \tfrac{2}{3}\rho gae,$$

$$(33)\qquad \frac{\widehat{r\theta}}{\sin 2\theta} = \mu\left(e + x\frac{d\alpha}{dx}\right) = 0.$$

§ 4. *Preliminary discussion of special cases.*

The problem is now reduced to that of solving the simultaneous ordinary differential equations (25) and (26), subject to the conditions expressed by (32) and (33). Before proceeding to the general solution, it is of interest to note the form assumed by the equations in special cases for which the solutions are known. Such cases are the following:

(A) A non-gravitating body acted upon by bodily disturbing forces having a potential proportional to $P_2 r^2$ (or free from bodily forces), and

(1) With any given surface stresses of type corresponding to the bodily forces (i. e., the radial stress proportional to P_2 and the tangential stress to $\sin 2\theta$), or

(2) With any given surface displacements of the same type (i. e., the radial displacement proportional to P_2 and the tangential to $\sin 2\theta$).

(B) A gravitating incompressible body with bodily and surface forces and surface displacements as in any of the cases under (A).

Of these cases the complete solution will here be given only for that of a non-gravitating body with surface free from stress. The modifications in the analysis required by the other cases under (A) are sufficiently obvious, while the corresponding case of (B) is given later on as a limiting case of the general solution for a gravitating compressible body.

It may be remarked, also, that after the general solution has been given it will be obvious how it must be modified if the surface conditions are changed.

(a) *Solution for the case of a non-gravitating body acted upon by bodily forces but free from surface forces.* In this case all the terms in equations (25) and (26) which represent bodily forces disappear except those representing the disturbing forces, and the equations become

$$(34)\qquad 4(\lambda+\mu)x\frac{d\alpha}{dx} - 8\mu\alpha - \tfrac{2}{3}(\lambda+2\mu)x^2\frac{d^2e}{dx^2} - \tfrac{8}{3}(\lambda+2\mu)x\frac{de}{dx} + 4\mu e - \tfrac{2}{3}\rho gacx^2 = 0,$$

$$(35)\qquad \mu x^2\frac{d^2\alpha}{dx^2} + 4\mu x\frac{d\alpha}{dx} - 2(3\lambda+5\mu)\alpha + (\lambda+\mu)x\frac{de}{dx} + (3\lambda+5\mu)e + \tfrac{1}{2}\rho gacx^2 = 0.$$

The surface conditions (32) and (33) become

$$(36)\qquad 4\lambda\alpha - 2(\lambda + \tfrac{2}{3}\mu)e - \tfrac{2}{3}(\lambda + 2\mu)x\frac{de}{dx} = 0, \qquad e + x\frac{d\alpha}{dx} = 0 \quad (x=1).$$

It is easily shown that these equations may all be satisfied by values of e and α of the forms

$$e = A_0 + A_2 x^2, \tag{37}$$

$$\alpha = B_0 + B_2 x^2, \tag{38}$$

by properly determining the constants A_0, A_2, B_0, B_2.

To satisfy the differential equations we must have

$$B_0 = \tfrac{1}{2} A_0, \tag{39}$$

while A_2 and B_2 must satisfy the two equations

$$\begin{aligned} 8\lambda B_2 - \tfrac{4}{3}(5\lambda + 7\mu) A_2 - \tfrac{2}{3}\rho g a c &= 0, \\ -6\lambda B_2 + (5\lambda + 7\mu) A_2 + \tfrac{1}{2}\rho g a c &= 0, \end{aligned} \tag{40}$$

which are seen to be identical. The surface conditions give two additional relations among the four constants; these are

$$4\lambda(B_0 + B_2) - 2(\lambda + \tfrac{2}{3}\mu)(A_0 + A_2) - \tfrac{4}{3}(\lambda + 2\mu) A_2 = 0, \tag{41}$$

$$A_0 + A_2 + 2B_2 = 0. \tag{42}$$

Solving these equations, we have

$$A_0 = \frac{4\lambda + 3\mu}{19\lambda + 14\mu} \frac{\rho g a}{\mu} c, \qquad A_2 = -\frac{3\lambda + 2\mu}{2(19\lambda + 14\mu)} \frac{\rho g a}{\mu} c,$$

$$B_0 = \frac{4\lambda + 3\mu}{2(19\lambda + 14\mu)} \frac{\rho g a}{\mu} c, \qquad B_2 = -\frac{5\lambda + 4\mu}{4(19\lambda + 14\mu)} \frac{\rho g a}{\mu} c.$$

Hence, finally, the values

$$e = \frac{c}{2(19\lambda + 14\mu)} [2(4\lambda + 3\mu) - (3\lambda + 2\mu) x^2] \frac{\rho g a}{\mu}, \tag{43}$$

$$\alpha = \frac{c}{4(19\lambda + 14\mu)} [2(4\lambda + 3\mu) - (5\lambda + 4\mu) x^2] \frac{\rho g a}{\mu}, \tag{44}$$

which agree with the known solution for this case.

(*b*) *Non-gravitating body with given surface forces or surface displacements.* If the two surface stresses are not zero but are known functions of θ of the type above specified, the last members of (32) and (33) become known constants. If the surface displacements are given and of the type specified, (32) and (33) are replaced by equations obtained in an obvious manner.

(*c*) *Non-gravitating spherical shell.* The solution for a spherical shell may be obtained by assuming

$$e = A_{-5} x^{-5} + A_{-3} x^{-3} + A_0 + A_2 x^2, \tag{45}$$

$$\alpha = B_{-5} x^{-5} + B_{-3} x^{-3} + B_0 + B_2 x^2, \tag{46}$$

the constants being determined to satisfy the conditions at the inner and outer boundaries. It may in fact be shown that these equations satisfy (34) and (35) if A_0, B_0, A_2, B_2 satisfy (39) and (40), and the remaining coefficients the conditions

$$\mu A_{-3} - (3\lambda + 5\mu) B_{-3} = 0, \tag{47}$$

$$A_{-5} + 3B_{-5} = 0. \tag{48}$$

There are thus four constants available for satisfying the boundary conditions.

(*d*) *Incompressible gravitating body.* In this case all the gravitational forces which are due to changes of density vanish, but the attraction of the surface inequality remains. The potential due to this surface layer has already been given, being

$$\tfrac{8}{15}\pi\gamma\rho e_1 r^2 P_2 = -\frac{2g}{5a} e_1 r^2 P_2.$$

This gives in equations (25) and (26) the terms

$$-\tfrac{4}{5}\rho g a e_1 x^2 \qquad \text{and} \qquad \tfrac{3}{5}\rho g a e_1 x^2.$$

Since e_1 is a constant, the solution may proceed as in case (*a*), by assuming

$$e = A_0 + A_2 x^2, \tag{49}$$

$$\alpha = B_0 + B_2 x^2. \tag{50}$$

In satisfying the equations which determine the constants, special reasoning is required by the fact that λ is infinite. This presents no serious difficulty, however, and will not be here given, since the case of incompressibility is hereinafter treated as a special case of the general problem of a gravitating sphere. To the solution of the general problem we now proceed.

§ 5. *Solution of equations in the general case.*

Inspection of the differential equations (25) and (26), and the form of the solution in the special cases above considered, suggest the trial of power series for the values of e and α.

The substitution of

$$e = A_m x^m, \qquad \alpha = B_m x^m \tag{51}$$

results in the following terms in the two equations: In (25),

$$\begin{aligned}\big([4(\lambda+\mu)m - 8\mu]B_m - [\tfrac{2}{3}(\lambda+2\mu)(m^2+3m) - 4\mu]A_m\big)x^m \\ + \rho g a\left[\frac{2(m^2+8m+6)}{3m(m+5)}(6B_m - (m+3)A_m)x^{m+2}\right. \\ \left. - \frac{12}{5m}(2B_m - A_m)x^2\right].\end{aligned} \tag{52}$$

In (26),

$$
\begin{aligned}
&\big([\mu(m^2+3m)-2(3\lambda+5\mu)]B_m+[(\lambda+\mu)m+(3\lambda+5\mu)]A_m\big)x^m \\
(53)\qquad &+\rho ga\Big[-\frac{3}{m(m+5)}(6B_m-(m+3)A_m)x^{m+2} \\
&\qquad\qquad +\frac{9}{5m}(2B_m-A_m)x^2\Big].
\end{aligned}
$$

Notice first that the coefficients of x^2 in these two expressions are in a constant ratio independent of m; hence if m be given a series of values, the coefficients of x^2 in the two equations may be made to vanish by a single relation among the coefficients.

Again, notice that terms in x^{m+2} result only from the following terms in the values of e and α:

$$A_m x^m + A_{m+2}x^{m+2}, \qquad B_m x^m + B_{m+2}x^{m+2};$$

and that the coefficients of x^{m+2} in both equations will vanish if the following two equations are satisfied:

$$
\begin{aligned}
&[4(\lambda+\mu)m+8\lambda]B_{m+2}-[\tfrac{2}{3}(\lambda+2\mu)m(m+7)+\tfrac{4}{3}(5\lambda+7\mu)]A_{m+2} \\
(54)\qquad &\qquad +\rho ga\frac{2(m^2+8m+6)}{3m(m+5)}[6B_m-(m+3)A_m]=0,
\end{aligned}
$$

$$
\begin{aligned}
&[\mu m(m+7)-6\lambda]B_{m+2}+[(\lambda+\mu)m+5\lambda+7\mu]A_{m+2} \\
(55)\qquad &\qquad -\rho ga\frac{3}{m(m+5)}[6B_m-(m+3)A_m]=0.
\end{aligned}
$$

Finally, notice that the terms of lowest degree in the series for e and α will introduce into the two equations terms of like degree which must vanish identically. That is, the terms

$$A_m x^m,\ B_m x^m$$

of lowest degree must be such that the coefficients of x^m in (52) and (53) will vanish. The values of m for which this is possible without the vanishing of A_m and B_m are found by equating to zero the determinant of the coefficients of A_m and B_m in these expressions; this determinant reduces to

$$\tfrac{2}{3}\mu(\lambda+2\mu)(m-2)m(m+3)(m+5).$$

The lowest power of x in the two series must therefore be either

$$x^2,\ x^0,\ x^{-3} \text{ or } x^{-5}.$$

Since negative powers of x are excluded by physical considerations (because the solution must apply to the point $x=0$), we conclude that the terms available

for the solution constitute ascending series of even powers of x beginning with the power zero. It is easily found also that we must have * $B_0 = \frac{1}{2}A_0$.

We now assume

$$e = A_0 + A_2x^2 + A_4x^4 + \cdots, \tag{56}$$

$$\alpha = \tfrac{1}{2}A_0 + B_2x^2 + B_4x^4 + \cdots, \tag{57}$$

and determine the relations which must hold among the coefficients in order that the coefficient of every power of x in each of two equations may vanish identically.

From the above discussion it is evident that if $m > 0$, the equations to be satisfied are (54) and (55), while the vanishing of the coefficient of x^2 requires that

$$8\lambda B_2 - \tfrac{4}{3}(5\lambda + 7\mu)A_2 - g\rho a\left[\tfrac{4}{5}A_0 + \frac{12}{5}\sum_{m=2}^{\infty}\frac{1}{m}(2B_m - A_m)\right] = \tfrac{2}{3}\rho gac \tag{58}$$

(odd values of m being excluded in the summation).

The solution of equations (54) and (55) gives

$$A_{m+2} = \frac{\mu m^2 + 10\mu m - 6(\lambda - 2\mu)}{\mu m(m+2)(m+5)(m+7)}\left[6B_m - (m+3)A_m\right]\frac{ga\rho}{\lambda + 2\mu}, \tag{59}$$

$$B_{m+2} = -\frac{(\lambda+\mu)m + 5\lambda + 4\mu}{\mu m(m+2)(m+5)(m+7)}\left[6B_m - (m+3)A_m\right]\frac{ga\rho}{\lambda + 2\mu}, \tag{60}$$

$$6B_{m+2} - (m+5)A_{m+2} = -\frac{m+6}{m(m+5)}\left[6B_m - (m+3)A_m\right]\frac{ga\rho}{\lambda + 2\mu}. \tag{61}$$

By successive applications of the last equation may be derived the following values of the coefficients of x^4 and higher powers, n having values 0, 1, 2, $\cdots$:

$$A_{4+2n} = -\tfrac{1}{24}(6B_2 - 5A_2)\left(-\frac{ga\rho}{\lambda+2\mu}\right)^{n+1}\left[4(n^2+7n+9) - 6\frac{\lambda}{\mu}\right]\frac{n+3}{7\cdot 9\cdots(2n+9)}, \tag{62}$$

$$B_{4+2n} = \tfrac{1}{24}(6B_2 - 5A_2)\left(-\frac{ga\rho}{\lambda+2\mu}\right)^{n+1}\left[2(n+3) + (2n+7)\frac{\lambda}{\mu}\right]\frac{n+3}{7\cdot 9\cdots(2n+9)}, \tag{63}$$

so that the values of e and α may be written

$$\begin{aligned} e = {} & A_0 + A_2x^2 + \tfrac{1}{24}(6B_2 - 5A_2)\frac{ga\rho}{\lambda + 2\mu}x^4 \\ & \times\sum_0^{\infty}(-1)^n\left[4(n^2+7n+9) - 6\frac{\lambda}{\mu}\right]\frac{n+3}{7\cdot 9\cdots(2n+9)}\left(\frac{ga\rho}{\lambda+2\mu}x^2\right)^n, \end{aligned} \tag{64}$$

* This is evident algebraically. It is also evident otherwise that if the solution has the assumed form, B_0 must equal $\frac{1}{2}A_0$; for A_0 and B_0 are the values of e and α at the center, where these quantities will be related as in the case of a homogeneous strain.

$$\alpha = \tfrac{1}{2}A_0 + B_2x^2 - \tfrac{1}{24}(6B_2 - 5A_2)\frac{ga\rho}{\lambda + 2\mu}x^4$$

(65)
$$\times \sum_0^\infty (-1)^n \left[2(n+3) + (2n+7)\frac{\lambda}{\mu}\right] \frac{n+3}{7\cdot 9 \cdots (2n+9)} \left(\frac{ga\rho}{\lambda + 2\mu}x^2\right)^n,$$

in which A_0, A_2, B_2 must satisfy (58) and the surface conditions (32) and (33).

For substitution in (58) we find (for $n = 0, 1, 2, \cdots$)

(66)
$$\frac{2B_{4+2n} - A_{4+2n}}{4+2n} = \tfrac{1}{12}(6B_2 - 5A_2)\left(-\frac{ga\rho}{\lambda + 2\mu}\right)^{n+1} \times \left(n + 6 + \frac{\lambda}{\mu}\right)\frac{n+3}{7\cdot 9 \cdots (2n+9)}.$$

In order to express as concisely as possible the values of e and α and the three equations to determine the constants, the following notation may be employed:

(67) $$L(z) = \sum_0^\infty (-1)^n \left[4(n^2 + 7n + 9) - 6\frac{\lambda}{\mu}\right] \frac{n+3}{7\cdot 9 \cdots (2n+9)} z^n,$$

(68) $$M(z) = \sum_0^\infty (-1)^n \left[2(n+3) + (2n+7)\frac{\lambda}{\mu}\right] \frac{n+3}{7\cdot 9 \cdots (2n+9)} z^n,$$

(69) $$N(z) = \sum_0^\infty (-1)^n \left[n + 6 + \frac{\lambda}{\mu}\right] \frac{n+3}{7\cdot 9 \cdots (2n+9)} z^n,$$

(70) $$L'(z) = \sum_0^\infty (-1)^n \left[4(n^2 + 7n + 9) - 6\frac{\lambda}{\mu}\right] \frac{(n+2)(n+3)}{7\cdot 9 \cdots (2n+9)} z^n,$$

(71) $$M'(z) = \sum_0^\infty (-1)^n \left[2(n+3) + (2n+7)\frac{\lambda}{\mu}\right] \frac{(n+2)(n+3)}{7\cdot 9 \cdots (2n+9)} z^n,$$

(72) $$b = \frac{ga\rho}{\lambda + 2\mu}.$$

The solution then takes the following form, the last two of the five equations being derived from the surface conditions (32) and (33):

(A) $$e = A_0 + A_2x^2 + \tfrac{1}{24}(6B_2 - 5A_2)bx^4 L(bx^2),$$

(B) $$\alpha = \tfrac{1}{2}A_0 + B_2x^2 - \tfrac{1}{24}(6B_2 - 5A_2)bx^4 M(bx^2),$$

(C)
$$4(A_0 + A_2) + \frac{140}{3b(\lambda/\mu + 2)}A_2 + (6B_2 - 5A_2)\left(2 - \frac{20\lambda/\mu}{3b(\lambda/\mu + 2)} - bN(b)\right) + \tfrac{10}{3}c = 0,$$

$$(D)\quad \begin{aligned}&\left[2+\left(\frac{\lambda}{\mu}+2\right)b\right](A_0+A_2)+4A_2-(6B_2-5A_2)\\&\times\left\{\frac{\lambda}{\mu}-\frac{b}{24}\left[\left(3\frac{\lambda}{\mu}+2+\left(\frac{\lambda}{\mu}+2\right)b\right)L(b)+6\frac{\lambda}{\mu}M(b)+2\left(\frac{\lambda}{\mu}+2\right)L'(b)\right]\right\}=0,\end{aligned}$$

$$(E)\qquad 3(A_0+A_2)+5A_2+(6B_2-5A_2)\left[1-\frac{b}{8}\left[L(b)-2M'(b)\right]\right]=0.$$

The functions represented by the infinite series may be otherwise expressed, depending ultimately upon the exponential integral

$$\int_0^x e^{\frac{1}{2}bx^2}dx.$$

For the purposes of computation, however, it is doubtful whether they can be reduced to expressions more convenient than the series, which converge with satisfactory rapidity unless b is relatively large. In applying the results to a body comparable in size and mass with the earth, for example, it can hardly be supposed that b is as great as 2; and for $b=2$ it is found that eight or ten terms of the series suffice, even in case of the function L', which is the least rapidly convergent of all.

§ 6. *Modified procedure for solving equations.*

It may be well to point out that by changing the procedure the algebraic work involved in the solution of equations (25) and (26) can be somewhat abridged.

The equations of equilibrium of a volume element may be abbreviated by introducing the elementary rotations ϖ_r, ϖ_θ, ϖ_ϕ. The general forms of these equations are given in Part IV of this paper, equations (117), $\cdots$, (123). When the strain is symmetrical about the initial line of the system of coordinates, ϖ_r and ϖ_θ vanish, and the equations reduce to the two following, which are identical with (1) and (2);

$$(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r\sin\theta}\frac{\partial}{\partial\theta}(\varpi_\theta\sin\theta)+\rho F_r=0,$$

$$(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}+\frac{2\mu}{r\sin\theta}\frac{\partial}{\partial r}(r\varpi_\phi\sin\theta)+\rho F_\theta=0.$$

If we now write

$$\Delta=yP_2,\qquad 2\varpi_\phi=z\frac{\partial P_2}{\partial\theta},$$

and introduce everywhere y and z instead of e and α, equations (25) and (26) become

$$(\lambda + 2\mu)x\frac{dy}{dx} + 6\mu z$$

$$+ \rho g a\left[\frac{9}{5x^3}\int_0^x y'x'^4\,dx' - \frac{6x^2}{5}\int_x^1 \frac{y'dx'}{x'} + yx^2 - (\tfrac{4}{5}e_1 + \tfrac{2}{3}c)x^2\right] = 0,$$

$$(\lambda + 2\mu)y + \mu\frac{d(xz)}{dx}$$

$$- \rho g a\left[\frac{3}{5x^3}\int_0^x y'x'^4 dx' + \frac{3x^2}{5}\int_x^1 \frac{y'dx'}{x'} + (\tfrac{2}{5}e_1 + \tfrac{1}{3}c)x^2\right] = 0.$$

It is easily shown that these are satisfied by series of the following forms:

$$y = C_2x^2 + C_4x^4 + \cdots, \qquad z = D_2x^2 + D_4x^4 + \cdots,$$

in which the coefficients satisfy the equations

$$\frac{C_{m+2}}{C_m} = -\frac{m+6}{m(m+5)}\cdot\frac{\rho g a}{\lambda + 2\mu},$$

$$\frac{D_{m+2}}{C_m} = \frac{1}{m(m+5)}\cdot\frac{\rho g a}{\mu},$$

and an additional relation obtained by equating to 0 the coefficients of x^2 after substituting the values of y and z in the differential equations. One coefficient thus remains arbitrary.*

It is now possible to determine e and α from the equations

$$4\alpha - 2e - \tfrac{2}{3}x\frac{de}{dx} = y, \qquad \tfrac{2}{3}\left(e - 2\alpha - x\frac{d\alpha}{dx}\right) = z.$$

Thus if we assume

$$e = \Sigma A_m x^m, \qquad \alpha = \Sigma B_m x^m,$$

we find

$$(m+3)A_m - 6B_m = -\tfrac{3}{2}C_m,$$

$$A_m - (m+2)B_m = \tfrac{3}{2}D_m,$$

from which A_m and B_m are determined in terms of C_m and D_m (and therefore in terms of the one arbitrary constant above mentioned), except in the case † $m = 0$. In this case we may take A_0 arbitrarily and make

$$B_0 = \tfrac{1}{2}A_0.$$

* Another solution consists of ascending series beginning with $C_{-3}x^{-3}$ and $D_{-3}x^{-3}$; but negative powers of x are not permissible in the physical problem.

† The case $m = -5$ is excluded by physical considerations.

That is, the values of e and α contain the same powers of x as the values of y and z, together with the absolute terms A_0 and $\frac{1}{2}A_0$. Two arbitrary constants remain for satisfying the surface conditions.

Although this process results in some saving of algebraic work, it does not appear to suggest any shortening of the labor involved in numerical computations such as those given below.

It is of some interest to note the form of the differential equation resulting from the elimination of z:

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y + \frac{\rho g a}{\lambda + 2\mu}\left(x^3\frac{dy}{dx} + 6x^2y\right) = 0.$$

We obtain also the equation

$$x^2\frac{d^2z}{dx^2} + 2x\frac{dz}{dx} - 6z - \frac{\rho g a}{\mu}x^2y = 0;$$

but the complete elimination of y obviously leads to an equation of the fourth order in z.

§ 7. *External potential and inertia-constant.*

One of the quantities which indicates the yielding of the body is the change in the external potential, or (what amounts to the same thing) in the difference between the principal moments of inertia.

Let I denote the moment of inertia about a diameter before strain, C and A the principal moments after strain (the former being taken about the axis of symmetry), and let

$$f = \frac{C - A}{I}. \tag{73}$$

The value contributed to $C - A$ by an elementary shell of thickness dr and density

$$-\rho\Delta = -\rho\left(4\alpha - 2e - \tfrac{2}{3}r\frac{de}{dr}\right)P_2$$

is

$$\tfrac{4}{5}\pi r^4\rho\left(4\alpha - 2e - \tfrac{2}{3}r\frac{de}{dr}\right)dr,$$

while the part contributed by the surface inequality is

$$\tfrac{8}{15}\pi\rho a^5 e_1;$$

hence

$$C - A = \frac{4\pi\rho}{5}\int_0^a\left(4\alpha - 2e - \tfrac{2}{3}r\frac{de}{dr}\right)r^4\,dr + \tfrac{8}{15}\pi\rho a^5 e_1. \tag{74}$$

Integrating by parts the term containing de/dr, we obtain

$$(75)\qquad C-A=\frac{16\pi\rho}{15}\int_0^a(3\alpha+e)r^4\,dr=\frac{16\pi\rho a^5}{15}\int_0^1(3\alpha+e)x^4\,dx,$$

and

$$(76)\qquad f=\frac{C-A}{I}=2\int_0^1(3\alpha+e)x^4\,dx.$$

This accords with the result already given for the potential due to the change of configuration. Thus the external potential is known to have the value

$$-\frac{\gamma(C-A)}{r^3}P_2,$$

while it is also represented by the first term of the value of V given in (20) if the upper limit of integration be a; i. e., we must have

$$-\frac{\gamma(C-A)}{r^3}P_2=-\frac{4gP_2}{5ar^3}\int_0^a(3\alpha'+e')r'^4\,dr'=-\frac{16\pi\rho\gamma P_2}{15r^3}\int_0^a(3\alpha+e)r^4\,dr,$$

which agrees with the result given above.

Using the values above found for e and α, we may write

$$(77)\quad f=\frac{C-A}{I}=A_0+A_2+(6B_2-5A_2)\left[\frac{1}{7}+\frac{b}{12}\left(2G(b)-3\frac{\lambda}{\mu}F(b)\right)\right],$$

in which

$$(78)\qquad F(b)=\sum_0^\infty(-1)^n\frac{n+3}{7\cdot 9\cdots(2n+9)}b^n,$$

$$(79)\qquad G(b)=\sum_0^\infty(-1)^n\frac{(n+1)(n+3)}{7\cdot 9\cdots(2n+9)}b^n.$$

§8. *Case of incompressibility.*

In the case of incompressibility $\lambda=\infty$, $b=0$. Putting $b'=ga\rho/\mu$, we find

$$(80)\qquad \begin{aligned}
bL(b)&=\frac{ga\rho}{\lambda+2\mu}\left(-6\frac{\lambda}{\mu}\right)\frac{3}{7\cdot 9}=-\tfrac{2}{7}b',\\
bM(b)&=\frac{ga\rho}{\lambda+2\mu}\left(7\frac{\lambda}{\mu}\right)\frac{3}{7\cdot 9}=\tfrac{1}{3}b',\\
bN(b)&=\frac{ga\rho}{\lambda+2\mu}\left(\frac{\lambda}{\mu}\right)\frac{3}{7\cdot 9}=\tfrac{1}{21}b',\\
bL'(b)&=\frac{ga\rho}{\lambda+2\mu}\left(-6\frac{\lambda}{\mu}\right)\frac{6}{7\cdot 9}=-\tfrac{4}{7}b',\\
bM'(b)&=\frac{ga\rho}{\lambda+2\mu}\left(7\frac{\lambda}{\mu}\right)\frac{6}{7\cdot 9}=\tfrac{2}{3}b'.
\end{aligned}$$

Equation (C) now requires that

$$6B_2 - 5A_2 = 0, \tag{81}$$

$$(6B_2 - 5A_2)\frac{\lambda}{\mu} = \frac{3b'}{20}\left[4A_0 + 4\left(1 + \frac{35}{3b'}\right)A_2 + \tfrac{10}{3}c\right]. \tag{82}$$

while (D) and (E) give

$$(6B_2 - 5A_2)\frac{\lambda}{\mu} = (2 + b')A_0 + (6 + b')A_2, \tag{83}$$

$$3A_0 + 8A_2 = 0. \tag{84}$$

Solving, we obtain

$$A_0 = \frac{8b'}{38 + 4b'}c, \qquad A_2 = -\frac{3b'}{38 + 4b'}c, \qquad B_2 = -\frac{5b'}{2(38 + 4b'}c, \tag{85}$$

while all other coefficients in the values of e and α vanish. We thus have

$$\frac{e}{c} = \frac{b'}{38 + 4b'}(8 - 3x^2), \tag{86}$$

$$\frac{\alpha}{c} = \frac{b'}{2(38 + 4b')}(8 - 5x^2), \tag{87}$$

which agree with the known solution for the case of incompressibility.

We find also

$$f = A_0 + A_2 = \frac{5b'c}{38 + 4b'}, \tag{88}$$

which is identical with the surface value of e, as should be the case when $\lambda = \infty$.

§ 9. *Case of zero rigidity.*

This is the only case in which the solution fails. If $\mu = 0$, equations (25) and (26) cannot be satisfied unless $\Delta = 0$; and if $\Delta = 0$, the surface condition (32) cannot be satisfied unless $\lambda = \infty$.

§ 10. *Numerical results.*

In applying the formulas above found, it is convenient to choose a definite value of λ/μ and make computations for a series of values of b or $g\rho a/(\lambda + 2\mu)$. Some of the results thus found are given in the accompanying tables and diagrams,* which for the most part are self-explanatory. The quantities of particular interest are the surface values of e and α and f. As an aid to the

* In all the diagrams it is to be understood that ordinates are multiplied by c.

verification of the computations, the values of the functions L, M, etc., are included in the tables.

TABLE I. *Results for the case* $\lambda/\mu = 1$.

(All numbers below the horizontal line are to be multiplied by c. Numbers in square brackets [ ] obtained by interpolation ; all others computed.)

$b = \frac{ga\rho}{\lambda + 2\mu}$	2	1.8	1.6	1.4	1.2	1	.8	.667
$\mu/ga\rho$	.167	.185	.208	.238	.278	.333	.417	.500
$L(b)$	.8946	.9347	.9773	1.0224	1.0703	1.1212	1.1753	1.2133
$M(b)$	.4618	.4746	.4879	.5018	.5163	.5315	.5474	.5585
$N(b)$	.2579	.2641	.2706	.2774	.2844	.2917	.2994	.3026
$L'(b)$	1.4000	1.4978	1.6042	1.7194	1.8444	1.9801	2.1273	2.2326
$M'(b)$	.7987	.8319	.8668	.9039	.9434	.9853	1.0298	1.0610
$F(b)$	.03798	.03880	.03964	.04052	.04142	.04236	.04333	.04400
$G(b)$	.02999	[.031]	[.033]	[.034]	[.036]	.03758	.03936	.04061
$6B_2 - 5A_2$	3.710	2.857	2.141	1.552	1.067	.7004	.4152	.2701
$A_0 = e_0$	.5146	.5251	.5218	.5058	.4758	.4327	.3760	.3303
A_2	—.5752	—.4954	—.4228	—.3570	—.2964	—.2405	—.1883	—.1551
B_2	.139	.0633	.0045	—.0388	—.0677	—.0837	—.0877	—.0842
e_1	.216	.230	.238	.241	.237	.225	.204	.1843
f	.436	.415	.390	.361	.327	.289	.246	.213
α_1	.253	.224	.196	.169	.142	.117	.0927	.0767

TABLE II. *Results for the case* $\lambda/\mu = 2$.

(Numbers below the horizontal line to be multiplied by e. Numbers in square brackets obtained by interpolation ; all others computed.)

$b = \frac{ga\rho}{\lambda + 2\mu}$	2	1.8	1.6	1.4	1.2	1	.8	.667
$\mu/ga\rho$	.125	.139	.156	.179	.208	.250	.3125	.375
$L(b)$	.6667	.7019	.7394	.7793	.8218	.8671	.9153	.9493
$M(b)$	.7117	.7312	.7516	.7729	.7952	.8185	.8428	.8597
$N(b)$	.2959	.3029	.3103	.3179	.3258	.3341	.3427	.3467
$L'(b)$	.9922	1.0771	1.1697	1.2706	1.3806	1.5005	1.6311	1.7367
$M'(b)$	1.2321	1.2829	1.3365	1.3933	1.4537	1.5179	1.5860	1.6328
$F(b)$	.03798	[.0388]	[.0396]	[.0405]	[.0414]	.04236	.04333	.04400
$G(b)$	.02999	[.031]	[.033]	[.034]	[.036]	.03758	.03936	.04061
$6B_2 - 5A_2$	3.749	2.915	2.209	1.623	1.142	.7577	.4596	.3056
$A_0 = e_0$	.5537	.5852	.5984	.5935	.5698	.528	.4666	.4144
A_2	—.4655	—.4309	—.3938	—.3542	—.3118	—.267	—.2195	—.1862
B_2	.2369	.1267	.0400	—.0247	—.0695	—.0962	—.1063	—.1042
e_1	.2964	.3078	.3135	.3131	.3049	.289	.2611	.236
f	.519	.496	.469	.438	.401	.358	.307	.269
α_1	.2914	.259	.228	.199	.170	.142	.1140	.096

TABLE III. *Results for the case* $\lambda/\mu = 3$.

(Numbers below the horizontal line to be multiplied by c. Numbers in square brackets obtained by interpolation ; all others computed.)

$b = \frac{g\alpha\rho}{\lambda + 2\mu}$	2	1.8	1.6	1.4	1.2	1	.8	.667
$\mu/g\alpha\rho$	.10	.111	.125	.143	.167	.200	.250	.300
$L(b)$	.4388	.4691	.5016	.5362	.5733	.6129	.6553	.6853
$M(b)$	.9616	.9880	1.0154	1.0441	1.0740	1.1054	1.1381	1.1610
$N(b)$	.3329	.3417	.3499	.3584	.3672	.3764	.3861	.3906
$L'(b)$	.5840	.6566	.7354	.8218	.9168	1.0212	1.1349	1.2414
$M'(b)$	1.6654	1.7339	1.8060	1.8826	1.9641	2.0505	2.1423	2.2046
$F(b)$	.03798	.03880	.03964	.04052	.04142	.04236	.04333	.04400
$G(b)$	.02999	[.031]	[.033]	[.034]	[.036]	.03758	.03936	.04061
$6B_2 - 5A_2$	3.910	3.032	2.320	1.721	1.224	.823	.497	.343
$A_0 = e_0$	.545	.598	.643	.654	.642	.605	.532	.489
A_2	−.340	−.348	−.351	−.339	−.317	−.285	−.239	−.214
B_2	.368	.215	.094	.0043	−.0602	−.1003	−.1163	−.1215
e_1	.3487	[.365]	.370	.369	.360	.341	[.314]	.281
f	.580	.556	.533	.501	.463	.417	[.370]	.318
α_1	.327	.290	.258	.226	.195	.164	.131	.112

In Figs. 2 and 3 are shown curves representing e_1, α_1 and f as functions of $\mu/g\rho a$, for certain constant values of λ/μ. It is seen that, as λ/μ increases, the curves for e_1 and f both approach the curve (A) which represents the case $\lambda = \infty$. For a given value of μ both e_1 and f increase with λ; in other words, the effect of compressibility is to decrease both the ellipticity and the external potential corresponding to a certain rigidity. This effect is much more important upon e_1 than upon f. Thus if $\mu = .35\rho g a$, a variation of λ/μ from 1 to ∞ would change f/c only from .279 to .289, while e_1/c would range from .221 to .289. For smaller values of μ the effect of compressibility is greater. An interesting feature is the fact that for a given value of λ/μ there is a maximum value of e_1; thus if $\lambda = \mu$, a decrease of μ below about $.24\rho g a$ causes e_1 to decrease.

Fig. 3 shows also that the value of α_1 is increased to an important extent by compressibility.

II. STRAIN OF AN ELASTIC SPHERE COVERED BY A SHALLOW OCEAN.

§ 1. *Statement of problem.*

In applying the theory of the deformation of a sphere to the case of the earth, an important question is the effect of the ocean. Some light on the order of magnitude of this effect is thrown by the solution of the following problem:

To determine the strain of a gravitating elastic sphere, the surface of which is covered by a shallow ocean in equilibrium.

The solution above given may without difficulty be extended to this case, as follows.

§ 2. *Forces due to ocean inequality.*

Let $c'\rho =$ density of ocean,

$\mathsf{e} =$ ellipticity of ocean surface.

Then there must be introduced into the solution the forces due to an ocean layer of thickness

$$-\tfrac{2}{3}(\mathsf{e}-e_1)aP_2.$$

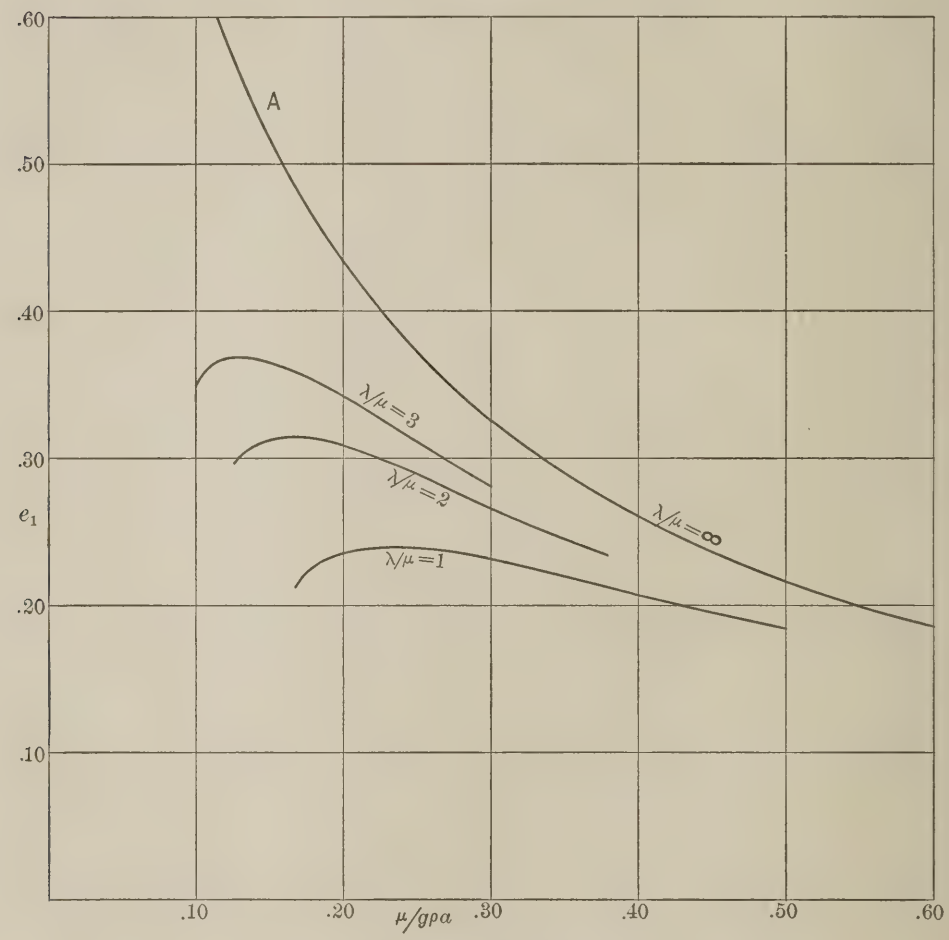

FIG. 2. Curves showing effect of compressibility upon surface ellipticity.

These forces will be (*a*) the bodily force due to the attraction of the layer, and (*b*) the surface pressure due to the weight of the layer.

(*a*) The attraction is given by the potential

$$V' = -\tfrac{8}{15}\pi\gamma c'\rho(\mathsf{e}-e_1)r^2P_2 = -\frac{2g}{5a}c'(\mathsf{e}-e_1)r^2P_2, \tag{89}$$

which adds to (23) and (24) the terms

$$\rho \frac{\partial V'}{\partial r} = -\frac{4g\rho}{5a} c'(\mathrm{e} - e_1) r P_2,$$

$$\frac{\rho}{r}\frac{\partial V'}{\partial \theta} = \frac{3g\rho}{5a} c'(\mathrm{e} - e_1) r \sin 2\theta,$$

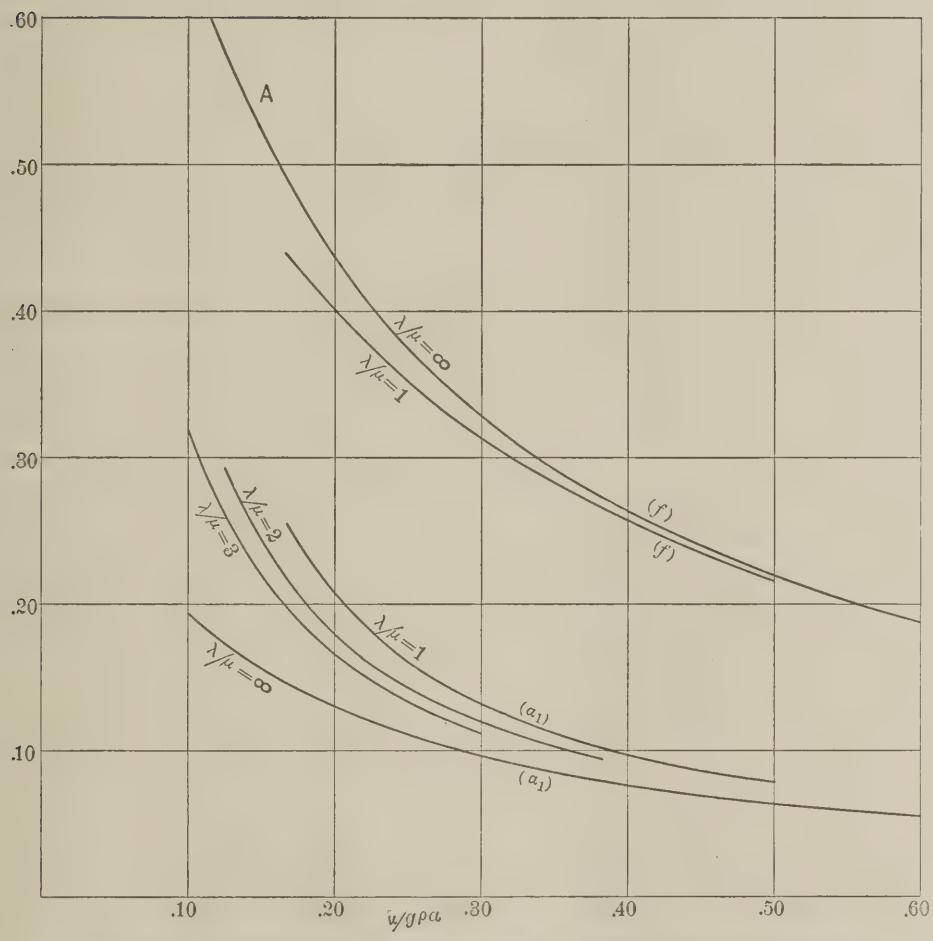

FIG. 3. Curves showing effect of compressibility upon angular displacement and external potential.

and therefore to (25) and (26) the terms

$$-\tfrac{4}{5} g\rho a c'(\mathrm{e} - e_1) x^2, \tag{90}$$

$$\tfrac{3}{5} g\rho a c'(\mathrm{e} - e_1) x^2. \tag{91}$$

(*b*) The weight of the ocean inequality per unit area is

$$-\tfrac{2}{3}c'\rho ga(\mathrm{e}-e_1)P_2,$$

hence there must be added to the second member of (32) the term

$$\tfrac{2}{3}\rho gac'(\mathrm{e}-e_1). \tag{92}$$

§ 3. *Value of ellipticity of ocean surface.*

The value of e (the ellipticity of a surface which is level under the action of gravity and the disturbing forces) may be expressed in terms of f, e_1 and c, as follows:

For the solid sphere the value of $C-A$ is If, while for the ocean inequality it is $Ic'(\mathrm{e}-e_1)$. Hence by a familiar result in the theory of potential,

$$I[f+c'(\mathrm{e}-e_1)] = \tfrac{2}{3}Ma^2(\mathrm{e}-\tfrac{1}{2}c). \tag{93}$$

Or, since for a sphere of uniform density $I=\frac{2}{5}Ma^2$, we have

$$f+c'(\mathrm{e}-e_1)=\tfrac{5}{3}(\mathrm{e}-\tfrac{1}{2}c), \tag{94}$$

whence

$$\mathrm{e}=\frac{6f-6c'e_1+5c}{10-6c'}, \tag{95}$$

and

$$\mathrm{e}-e_1=\frac{6f-10e_1+5c}{10-6c'}. \tag{96}$$

The terms to be added to (25), (26) and (32) thus become

$$\rho ga\left[-\tfrac{4}{5}c'\frac{6f-10e_1+5c}{10-6c'}x^2\right], \tag{97}$$

$$\rho ga\left[\tfrac{3}{5}c'\frac{6f-10e_1+5c}{10-6c'}x^2\right], \tag{98}$$

$$\rho ga\left[\tfrac{2}{3}c'\frac{6f-10e_1+5c}{10-6c'}\right]. \tag{99}$$

§ 4. *Statement of solution.*

The values of e and α for this case are expressed by the same equations (A) and (B) as before, but two of the three equations (C), (D) and (E) which determine the constants must be changed as follows.

In (C) replace $\frac{10}{3}c$ by $[50c+6c'(6f-10e_1)]/[3(5-3c')]$.

Add to the first member of (D):

$$\left(\frac{\lambda}{\mu}+2\right)b\,\frac{c'(5c+6f-10e_1)}{10-6c'}.$$

Equation (E) remains unchanged.

Equations (C) and (D) are thus replaced by the following:

$$(C_1)\qquad \begin{aligned} &4(A_0+A_2)+\frac{140}{3b(\lambda/\mu+2)}A_2 \\ &\quad+(6B_2-5A_2)\left(2-\frac{20\lambda/\mu}{3b(\lambda/\mu+2)}-bN(b)\right) \\ &\qquad+\frac{12c'}{5-3c'}f-\frac{20c'}{5-3c'}e_1+\frac{50}{3(5-3c')}c=0, \end{aligned}$$

$$(D_1)\qquad \begin{aligned} &\left[2+\left(\frac{\lambda}{\mu}+2\right)b\right]A_0+\left[6+\left(\frac{\lambda}{\mu}+2\right)b\right]A_2-(6B_2-5A_2) \\ &\quad\times\left\{\frac{\lambda}{\mu}-\frac{b}{24}\left[\left(3\frac{\lambda}{\mu}+2+\left(\frac{\lambda}{\mu}+2\right)b\right)L(b)+6\frac{\lambda}{\mu}M(b)+2\left(\frac{\lambda}{\mu}+2\right)L'(b)\right]\right\} \\ &\qquad+\left(\frac{\lambda}{\mu}+2\right)b\left(\frac{3c'}{5-3c'}f-\frac{5c'}{5-3c'}e_1+\frac{5c'}{2(5-3c')}c\right)=0. \end{aligned}$$

When f and e_1 are replaced by their values

$$(100)\qquad e_1=A_0+A_2+\tfrac{1}{24}(6B_2-5A_2)bL(b),$$

$$(101)\qquad f=A_0+A_2+(6B_2-5A_2)\left[\frac{1}{7}+\frac{b}{12}\left(2G(b)-3\frac{\lambda}{\mu}F(b)\right)\right],$$

the equations become

$$(C')\qquad \begin{aligned} &\frac{20(1-c')}{5-3c'}(A_0+A_2)+\frac{140}{3b(\lambda/\mu+2)}A_2+(6B_2-5A_2) \\ &\quad\times\left[2-\frac{20\lambda/\mu}{3b(\lambda/\mu+2)}-bN(b)+\frac{4c'}{5-3c'}\left(\tfrac{3}{7}+\tfrac{1}{2}bG(b)-\frac{3\lambda}{4\mu}bF(b)-\tfrac{5}{24}bL(b)\right)\right] \\ &\qquad+\frac{50}{3(5-3c')}c=0, \end{aligned}$$

$$(D')\qquad \begin{aligned} &\left[2+\left(\frac{\lambda}{\mu}+2\right)b\frac{5(1-c')}{5-3c'}\right](A_0+A_2)+4A_2-(6B_2-5A_2) \\ &\quad\times\left\{\frac{\lambda}{\mu}-\frac{1}{24}\left[\left(3\frac{\lambda}{\mu}+2+\left(\frac{\lambda}{\mu}+2\right)b\right)bL(b)+6\frac{\lambda}{\mu}bM(b)+2\left(\frac{\lambda}{\mu}+2\right)bL'(b)\right]\right. \\ &\quad\left.-\left(\frac{\lambda}{\mu}+2\right)b\frac{c'}{5-3c'}\left[\tfrac{3}{7}+\tfrac{1}{2}bG(b)-\frac{3\lambda}{4\mu}bF(b)-\tfrac{5}{24}bL(b)\right]\right\} \\ &\qquad+\left(\frac{\lambda}{\mu}+2\right)b\frac{5c'}{2(5-3c')}c=0. \end{aligned}$$

To these we add equation (E) unchanged:

$$(E')\quad 3(A_0+A_2)+5A_2+(6B_2-5A_2)\left[1+\frac{b}{8}\big(L(b)-2M'(b)\big)\right]=0.$$

These three equations determine the constants A_0, A_2, B_2.

§ 5. *Value of inertia-constant for combined sphere and ocean.*

Let f' represent the value of $(C-A)/I$ for the whole system, then

$$f'=f+c'(e-e_1),$$

which by (96) reduces to

$$(102)\qquad f'=\frac{5}{5-3c'}f-\frac{5c'}{5-3c'}e_1+\frac{5c'}{2(5-3c')}c.$$

§ 6. *Case of incompressibility.*

If $\lambda=\infty$, equations (A) and (B) hold, and equations (C') and (D') require that

$$(103)\qquad 6B_2-5A_2=0.$$

Also, on writing

$$(104)\qquad \frac{\lambda}{\mu}(6B_2-5A_2)=C,$$

(C'), (D') and (E') become

$$(105)\qquad C-7A_2-\frac{3(1-c')}{5-3c'}b'(A_0+A_2)-\frac{5b'}{2(5-3c')}c=0,$$

$$(106)\qquad C-4A_2-\left(2+\frac{5(1-c')}{5-3c'}b'\right)(A_0+A_2)-\frac{5c'b'}{2(5-3c')}c=0,$$

$$(107)\qquad 3(A_0+A_2)+5A_2=0.$$

The solution of these equations gives

$$(108)\qquad \begin{aligned} A_0&=\frac{20(1-c')b'}{19(5-3c')+10(1-c')b'}c,\\ A_2&=-\frac{15(1-c')b'}{38(5-3c')+20(1-c')b'}c,\\ B_2&=-\frac{25(1-c')b'}{76(5-3c')+40(1-c')b'}c.\end{aligned}$$

If $c'=0$, these reduce to (85), as of course they should.

§ 7. *Numerical results.*

For the purpose of showing the order of importance of the ocean effect, computations have been made with a value of c' equal to the ratio of the density of sea-water to the mean density of the earth. This value is taken as $5/27$.

The computations made cover the two cases $\lambda/\mu = 1$, $\lambda/\mu = \infty$, and the results are shown in the accompanying table and diagrams.

For the case of incompressibility

$$(109)\quad A_0 = \frac{22b'}{114 + 11b'}c = \frac{22}{114\dfrac{\mu}{g\rho a} + 11}c, \qquad A_2 = -\tfrac{3}{8}A_0, \qquad B_2 = -\tfrac{5}{16}A_0,$$

$$(110)\qquad \frac{e}{c} = \frac{22}{114\dfrac{\mu}{g\rho a} + 11}(1 - \tfrac{3}{8}x^2),$$

$$(111)\qquad \frac{\alpha}{c} = \frac{11}{114\dfrac{\mu}{g\rho a} + 11}(1 - \tfrac{5}{8}x^2),$$

$$(112)\qquad \frac{e_1}{c} = \frac{f}{c} = \frac{1}{c}(A_0 + A_2) = \frac{5}{\dfrac{456}{11}\dfrac{\mu}{g\rho a} + 4}.$$

The last expression is written in a form for comparison with the value without ocean effect:

$$(113)\qquad \frac{e_1}{c} = \frac{f}{c} = \frac{5}{38\dfrac{\mu}{g\rho a} + 4}.$$

From (102),

$$(114)\qquad f' = \tfrac{11}{12}e_1 + \tfrac{5}{48}c.$$

Values of e_1, f and f' for this case, also corresponding values without ocean effect, are given in Table IV, while the same results are shown graphically in Fig. 4.

For $\lambda = \mu$, computations have been made by means of equations (A'), (B'), (C'), (D'), (E'), with the results given in Table V and shown graphically in Fig. 5.

Inspection of the curves shows that the effect of the ocean on the strain of the sphere is quite appreciable even in the case of incompressibility, and is very considerable in the case $\lambda = \mu$. A fact of equal importance in the interpretation of observational results as to the yielding of the earth is shown by the curves representing f'. These are nearly identical for the two cases $\lambda = \infty$, $\lambda = \mu$ (Figs. 4 and 5), and in both cases are very considerably above the curves representing f. It thus appears that an important part of the change of external potential is due to the ocean — a result which will be referred to again in the following discussion of the yielding of the earth.

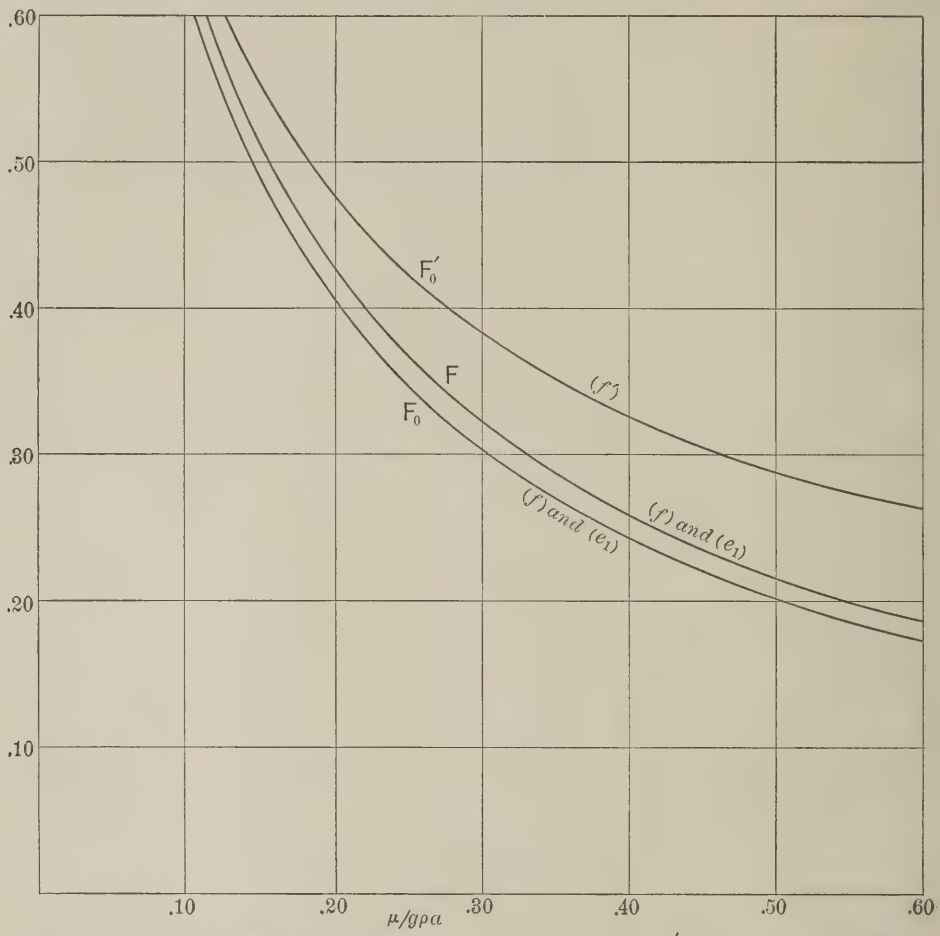

FIG. 4. Curves showing effect of ocean in case $\lambda = \infty$. (F_0 and F_0' with ocean, F without.)

TABLE IV. *Effect of ocean in case* $\lambda = \infty$.

$\frac{\mu}{g\rho a}$	Without ocean.		With ocean.		
	$\frac{a_1}{c}$	$\frac{e_1}{c} = \frac{f}{c}$	$\frac{a_1}{c}$	$\frac{e_1}{c} = \frac{f}{c}$	$\frac{f'}{c}$
0	.375	1.250	.375	1.250	1.250
.05	.254	.848	.247	.823	.858
.10	.193	.642	.184	.613	.666
.15	.155	.516	.146	.488	.551
.20	.129	.431	.122	.407	.477
.25	.111	.370	.104	.348	.423
.30	.097	.325	.091	.304	.383
.35	.087	.289	.081	.270	.352
.40	.078	.260	.073	.243	.327
.45	.071	.237	.066	.221	.307
.50	.065	.217	.061	.202	.289
.55	.060	.201	.056	.187	.275
.60	.056	.187	.052	.173	.263

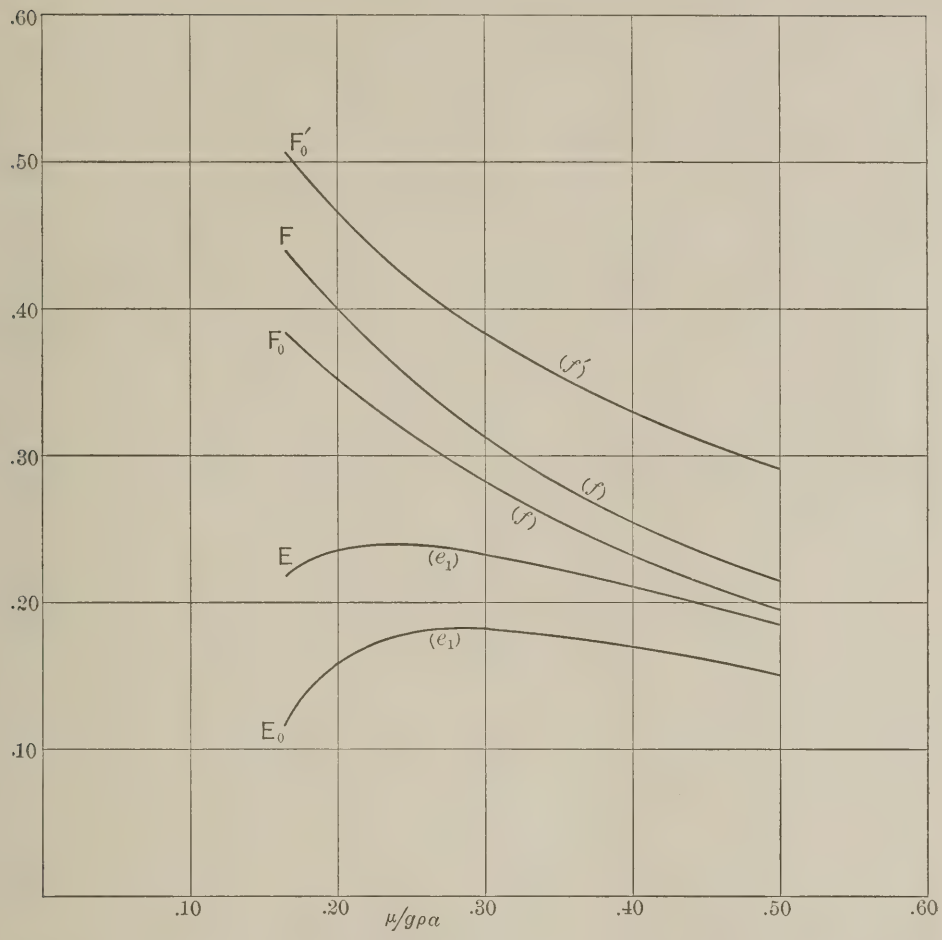

FIG. 5. Curves showing effect of ocean in case $\lambda = \mu$. (Curves E, F without ocean; E_0, F_0, F_0' with ocean.)

TABLE V. *Effect of ocean in case* $\lambda = \mu$.

b	$\frac{\mu}{g\rho a}$	Without ocean.			With ocean.			
		$\frac{a_1}{c}$	$\frac{e_1}{c}$	$\frac{f}{c}$	$\frac{a_1}{c}$	$\frac{e_1}{c}$	$\frac{f}{c}$	$\frac{f'}{c}$
2	.167	.253	.216	.436	.257	.119	.379	.505
1.8	.185	.224	.230	.415	.229	.145	.365	.485
1.6	.208	.196	.238	.390	.198	.163	.345	.458
1.4	.238	.169	.241	.361	.170	.176	.322	.429
1.2	.278	.142	.237	.327	.143	.181	.294	.397
1.0	.333	.117	.225	.289	.118	.179	.263	.363
.8	.417	.093	.204	.246	.094	.167	.225	.322
.667	.500	.077	.184	.213	.077	.151	.194	.291

III. Effect of compressibility upon estimates of the rigidity of the earth.

The estimates of the rigidity of the earth, made originally by Lord Kelvin and since by others, are based upon a comparison of what is known regarding the actual yielding of the earth under known disturbing forces with the computed yielding of an ideal body under like forces. This ideal body is a homogeneous, elastic, incompressible sphere of mass and size equal to those of the earth. It is, of course, not to be expected that such a comparison can lead to anything more than a rough average value of the modulus of rigidity of the material of the earth, since the actual earth is certainly far from uniform in density, and probably far from uniform in elastic properties. It is, however, a matter of some interest to notice in what way these estimates are modified when compressibility is taken into account. In answering this question it is needful to refer briefly to the present status of the evidence regarding the actual yielding of the earth.*

§ 1. *Nature of evidence of yielding of the earth.*

Evidence regarding the actual yielding of the earth relates to the two quantities above represented by e_1 and f. The former is the surface ellipticity due to strain, the latter measures the change in the gravitational potential at the earth's surface due to the strain. The number f is, in fact, the value of $(C - A)/I$ due to the strain. It may be remarked that for a homogeneous *incompressible* body e_1 and f are equal, but for a compressible body f depends upon the values of both e and α throughout the body.

There are three lines of evidence of yielding of the earth to tidal or centrifugal forces: (*a*) Tidal observations, (*b*) experiments with the horizontal pendulum, (*c*) the movement of the axis of rotation within the earth as determined by latitude observations.

(*a*) *Nature of evidence from tidal observations.* The height of an equilibrium-tide at any locality can be computed on the assumption that the solid earth is unyielding, also on the assumption that f and e_1 have any definite values. A comparison of these results with the observed tidal oscillation gives evidence as to the actual values of f and e_1. It may, in fact, be shown without difficulty that if the attraction of the ocean itself be neglected, the ratio of the actual fluctuation of the water relative to the solid ground to the fluctuation on the assumption of no yielding of the solid earth is

* A clear and non-technical summary of this evidence is given by G. H. Darwin in a paper entitled "*The Rigidity of the Earth,*" in Rivista di Scienza, vol. 5 (1909). See also a paper by A. E. H. Love entitled "*The Yielding of the Earth to Disturbing Forces,*" in the Proceedings of the Royal Society, Series A, Vol. 84.

$$\frac{c+f-2e_1}{c}=1+\frac{f}{c}-2\frac{e_1}{c}.$$

It is here assumed that the tide has its equilibrium value, an assumption which can reasonably be made only in the case of tides of long period. It is therefore only the fortnightly and monthly tides that have been available for this comparison.

It should be said also that for full generality the term f/c must be multiplied by $3I/Ma^2$. For a homogeneous sphere the value of this is $6/5$, but for the earth it is very nearly 1. It is the actual value for the earth that must be used in the application made below in § 2.

(*b*) *Nature of the evidence from experiments with the horizontal pendulum.* If the change in the direction of apparent gravity relative to the earth at any locality due to tidal forces can be measured, a comparison of this effect with the effect computed on the assumption of absolute rigidity of the earth will give evidence as to the actual yielding of the earth. The actual effect is so minute that it must be greatly magnified in order to become measurable. This magnification has been accomplished by means of the horizontal pendulum. It is not difficult to show that the actual deflection of such a pendulum due to tidal forces bears the same ratio to the deflection that would occur if the earth did not yield as the amplitude of an equilibrium tide bears to the amplitude it would have if the earth were unyielding. The quantity determined from horizontal pendulum observations is therefore identical with that inferred from tidal observations.

(*c*) *Nature of the evidence from the periodical variation of latitudes.* The periodic motion of the axis of rotation within the earth furnishes evidence as to the yielding of the earth to centrifugal forces. This evidence is of a different character from that obtained from the tides or the horizontal pendulum, in that it leads to an estimate of the value of f/c instead of to a relation between f/c and e_1/c. The change in the value of $(C-A)/I$ due to the yielding of the earth to given centrifugal forces may be inferred directly from a comparison of the actual period of the free nutation of the earth's axis with the period computed for an unyielding body having the earth's actual figure. It may, in fact, be shown that the actual nutation period is to a close approximation the same as that of an unyielding body having the figure which the earth would assume if centrifugal forces were annulled.*

*This was shown by S. S. HOUGH (Philosophical Transactions of the Royal Society, ser. *A*, 1896) for the case of a homogeneous, incompressible elastic spheroid, and has been proved by J. LARMOR and others to hold without other restriction as to the character of the body than that its density and elastic properties have approximate spherical symmetry. A proof of this was given by the present writer in a paper presented to the American Mathematical Society, San Francisco Section, Dec. 20, 1902, of which only an abstract was published, Bulletin of this Society, Vol. 9 (1903), p. 299. The reasoning employed was identical with that published by LARMOR, Proceedings of the Royal Society, ser. *A*, vol. 82 (1909).

§ 2. *Numerical results expressing actual yielding of the earth.*

The actual value of $(C-A)/I$ for the earth is $1/306$, so that the period of free nutation if it were absolutely rigid would be 306 sidereal days. The period of the movement of the axis about its mean position, as inferred from the observations of the International Latitude Service carried on systematically since 1899, has varied from 427 days to 442 days. The former gives

$$f = \tfrac{1}{306} - \tfrac{1}{427} = .00093\,;$$

the latter,

$$f = \tfrac{1}{306} - \tfrac{1}{442} = .00100.$$

Since the ratio of equatorial centrifugal force to gravity is $c = 1/289$, the corresponding values of f/c are .269 and .289. The former of these values has hitherto usually been accepted. It is nearly equal to $4/15$.

The first analysis of tidal observations, with reference to the question of the yielding of the earth, was made by G. H. Darwin, the data and results being given in *Thomson and Tait's Natural Philosophy.* From all the data then available which were regarded as trustworthy, the value found for the ratio of the actual tidal amplitude to that computed on the assumption of an absolutely rigid earth was $.676 \pm .076$; but a separate analysis of the more consistent observations gave $.931 \pm .056$. The result was not regarded by the author as conclusive.* Analyses since made by others have been confirmatory of the former of these values, and about the same result is inferred from observations with the horizontal pendulum.†

If this result be accepted, we have (very nearly)

$$1 + \frac{f}{c} - 2\frac{e_1}{c} = \frac{2}{3}. \tag{115}$$

And if in this we substitute

$$\frac{f}{c} = \frac{4}{15},$$

there results

$$\frac{e_1}{c} = \frac{3}{10}.$$

§ 3. *Comparison of actual yielding with that of an ideal elastic solid.*

It is pointed out by Love ‡ that the above values of f/c and e_1/c cannot coexist in the case of a homogeneous, incompressible elastic solid having the earth's size and mass, and it is easily seen from the results found in the first

* "We thus see that the more consistent observations seem to bring out the tides more nearly to their theoretical equilibrium values with no elastic yielding of the solid." G. H. Darwin, in *Thomson and Tait's Natural Philosophy,* Part II, p. 460 (edition of 1890).

† W. Schweydar, Beiträge zur Geophysik, vol. 9 (1908); O. Hecker, *Veröffentlichung des Kgl. Preussischen Geodätischen Institutes,* No. 32, Berlin (1907).

‡ Loc. cit.

part of this paper that the discrepancy is increased rather than diminished if the effect of compressibility is considered. This is shown clearly by Fig. 6, in which the curves represent the relation between e_1/c and f/c for several values of λ/μ, while the straight line AB represents equation (115), or

$$2\frac{e_1}{c} - \frac{f}{c} = \frac{1}{3}. \tag{116}$$

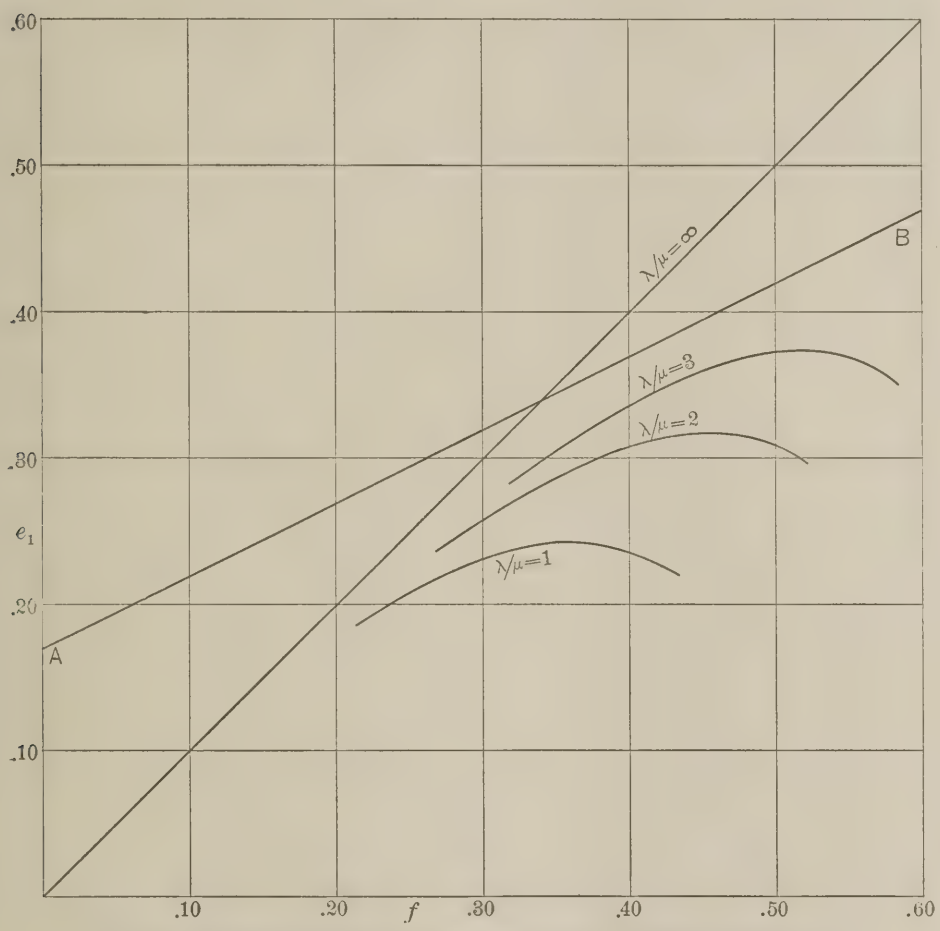

FIG. 6. Curves showing relation between e_1 and f for several values of λ/μ.

As λ/μ increases, the curves approach the limiting line $e_1 = f$; and until λ/μ reaches a value considerably greater than 3, the curves will fail to intersect AB. It is obvious that if λ/μ be taken great enough so that (116) can be satisfied, the value of f/c will be greater than $\frac{4}{15}$; being in fact $\frac{1}{3}$ for $\lambda = \infty$, and greater than $\frac{1}{3}$ for any finite value of λ/μ.

It may be remarked that the discrepancy between

$$\frac{f}{c}=\frac{1}{3}, \qquad \frac{e_1}{c}=\frac{1}{3},$$

and

$$\frac{f}{c}=\frac{4}{15}, \qquad \frac{e_1}{c}=\frac{3}{10},$$

is after all not remarkably great, considering the great delicacy of the observations upon which equation (116) is based. It may perhaps be still open to question whether the accepted values are so thoroughly reliable as to exclude the possibility of errors of the magnitude of these discrepancies.*

It may be noted, also, that a value of $1+f/c-2e_1/c$ much less than the greater of Darwin's results would remove the discrepancy. Thus the following values would be consistent for the ideal homogeneous sphere:

$\frac{\lambda}{\mu}$	$\frac{f}{c}$	$\frac{e_1}{c}$	$1+\frac{f}{c}-2\frac{e_1}{c}$
∞	.270	.270	.730
3	.270	.245	.780
2	.270	.235	.800
1	.270	.220	.830

Although the value of the surface ellipticity accepted as most reliable is more nearly consistent with the computed value for an incompressible than for a compressible sphere, it can hardly be supposed that this furnishes any important evidence as to the actual compressibility of the materials composing the earth. As stated above, only a roughly approximate agreement is to be expected between the behavior of the actual earth and that of an ideal body assumed to be uniform in density and elastic properties. Moreover, there are certain questions not generally considered which appear to require attention in any complete study of the theoretical basis of the comparison. One of these questions relates to the effect of the ocean.

* The most elaborate and apparently the most trustworthy series of observations heretofore made with the horizontal pendulum are those of HECKER (loc. cit.). A remarkable feature of the results obtained for the lunar effect upon the yielding of the earth is the variation of this effect with the azimuth of the moon. The value found for the ratio of the actual deflection of gravity to the deflection computed for a rigid earth is a minimum when the moon is on the meridian, and increases continually with the azimuth. The minimum value of the ratio is about 0.36, for azimuth 45° it is about 0.50, and for azimuth 67° about 0.90; and although as the azimuth approaches 90° the ratio becomes indeterminate, it appears to approach unity. No satisfactory explanation of this variation seems to have been suggested. If it is accepted as proving that the yielding of the earth varies greatly with the azimuth of the disturbing body, it appears to render of little value conclusions based upon the ordinary theory which assumes that the yielding is symmetrical about the axis of symmetry of the disturbing forces. At all events these results can hardly be regarded as establishing any definite value of the ratio of yielding. [The above numerical values of the ratio for different azimuths were obtained by measurement from the diagram given by HECKER showing the observed lunar effect and the effect computed for a rigid earth; assuming that the deflection of apparent gravity always agrees in azimuth with the moon.] HECKER's results are discussed by G. H. DARWIN in the paper already cited. This paper was received too late to permit here any quotation of suggestive comments.

§ 4. *Effect of the ocean on estimates of the yielding of the earth.*

The question as to the effect of the ocean involves two parts, — the first, what effect has the ocean on the actual yielding of the solid earth, and the second, what direct effect has the ocean on the observational results?

While no exact answer can be given to either of these questions, something can be inferred as to the degree of importance of the ocean effect from the results reached in Part II of this paper, and shown in Tables IV and V, and in Figs. 4 and 5. These results apply to the two cases $\lambda = \infty$ and $\lambda/\mu = 1$. It appears that the effect of the ocean upon the strain of the sphere is quite appreciable in both cases, and in the case $\lambda = \mu$ becomes of considerable importance. Although the angular displacement α_1 is affected but slightly, both e_1 and f are very considerably changed by the influence of the ocean.

Moreover, while these quantities are both materially decreased, the total external potential due to the solid sphere and ocean is considerably greater than the potential when the ocean is absent. This is true both for $\lambda = \mu$ and for $\lambda = \infty$, as is seen from the values of f'/c. (It is of interest to note that the curves showing the relation between f'/c and $\mu/g\rho a$ are nearly identical for the two cases.)

In estimating the value of μ from the value of $(C - A)/I$ inferred from the prolongation of the period of free nutation, it is therefore important to know whether the quantity determined applies to the solid earth alone or to the combined solid earth and ocean. Thus if it applies to the solid earth alone, the value $f/c = 0.27$ would correspond to

$$\frac{\mu}{g\rho a} = \text{about } 0.37,$$

while if 0.27 is the value of f'/c, we find

$$\frac{\mu}{g\rho a} = \text{about } 0.57.$$

The former would give $\mu = 1.30 \times 10^{12}$, while the latter would give $\mu = 2 \times 10^{12}$. These values are in C. G. S. units (dynes per square centimeter), the value of $g\rho a$ in these units being taken as 3.5×10^{12}.

Whether the change in the value of $(C - A)/I$ due to centrifugal forces, as inferred from the nutation period, applies to the solid earth alone, or whether an important part of it is due to the change in configuration of the ocean, is a question which must be answered in the light of the theory of the effect of yielding upon the nutation period. It seems a reasonable assumption that, for a disturbance for so long a period as 427 days, the ocean always has its equilibrium figure. If this be assumed in applying the equation of angular momen-

tum,* it is difficult to avoid the conclusion that the change in the value of $(C-A)/I$ inferred from the prolongation of the nutation period includes the yielding of the ocean; in other words, that the number $4/15$ is the value of f'/c rather than of f/c.

What effect the ocean may have upon observations with the horizontal pendulum is a more difficult question, because the equilibrium theory must be discarded in dealing with tides of short period.

IV. Strain of a compressible, gravitating elastic sphere due to disturbing forces of which the potential is a spherical harmonic of any degree not less than the second.

The following solution of this problem follows the general lines of the solution already given for the case in which the degree of the harmonic is 2. It in fact includes that as a special case. In comparing the algebraic results it must be noted that there is a slight change in the notation; the quantities e and α as used below must be multiplied by the factor $-\frac{3}{2}$ in order to agree with e and α in the solution given above.

§ 1. *Physical analysis of the problem.*

The physical analysis is identical with that already given in the special case, and need not be repeated.

§ 2. *Formation of the equations of equilibrium of an element of volume.*

(*a*) *General form of equations in polar coördinates.* The polar coördinates being r, θ, ϕ, let the corresponding displacements be

$$u_r,\ u_\theta,\ u_\phi,$$

the cubical expansion Δ, and the three components of rotation ϖ_r, ϖ_θ, ϖ_ϕ. Also let the components of total bodily force (per unit volume) be ρF_r, ρF_θ, ρF_ϕ. Then the equations of equilibrium for a volume element are the following:†

$$(117)\qquad (\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\varpi_\phi\sin\theta)-\frac{\partial\varpi_\theta}{\partial\phi}\right]=-\rho F_r,$$

$$(118)\qquad (\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}-\frac{2\mu}{r\sin\theta}\left[\frac{\partial\varpi_r}{\partial\phi}-\frac{\partial}{\partial r}(r\varpi_\phi\sin\theta)\right]=-\rho F_\theta,$$

$$(119)\qquad (\lambda+2\mu)\frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}-\frac{2\mu}{r}\left[\frac{\partial}{\partial r}(r\varpi_\theta)-\frac{\partial\varpi_r}{\partial\theta}\right]=-\rho F_\phi;$$

* In the manner employed by Larmor in the paper cited above.

† See Love, *Treatise on the Mathematical Theory of Elasticity*, 2d edition (1906), p. 138, for the first members of these equations; p. 56, for the values of Δ, ϖ_r, ϖ_θ, ϖ_ϕ.

while the cubical expansion and the components of rotation are given by the following equations:

$$\Delta = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 u_r \sin\theta) + \frac{\partial}{\partial\theta}(r u_\theta \sin\theta) + \frac{\partial}{\partial\phi}(r u_\phi)\right], \tag{120}$$

$$2\varpi_r = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(r u_\phi \sin\theta) - \frac{\partial}{\partial\phi}(r u_\theta)\right], \tag{121}$$

$$2\varpi_\theta = \frac{1}{r \sin\theta}\left[\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(r u_\phi \sin\theta)\right], \tag{122}$$

$$2\varpi_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right]. \tag{123}$$

(*b*) *Assumptions.* The potential of the disturbing forces, being a spherical solid harmonic of degree i, is proportional to

$$S_i r^i,$$

in which S_i is a spherical surface harmonic, and therefore satisfies the partial differential equation

$$\frac{1}{\sin^2\theta}\frac{\partial^2 S_i}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_i}{\partial\theta}\right) + i(i+1)S_i = 0. \tag{124}$$

It will be assumed that

$$u_r = u S_i, \tag{125}$$

$$u_\theta = v\frac{\partial S_i}{\partial\theta}, \tag{126}$$

$$u_\phi = v\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}, \tag{127}$$

in which u and v are independent of θ and ϕ. These assumptions give by virtue of (124)

$$\Delta = \left[\frac{1}{r^2}\frac{d(r^2 u)}{dr} - i(i+1)\frac{v}{r}\right]S_i, \tag{128}$$

$$2\varpi_r = 0, \tag{129}$$

$$2\varpi_\theta = \frac{1}{r}\left[u - \frac{d(rv)}{dr}\right]\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}, \tag{130}$$

$$2\varpi_\phi = -\frac{1}{r}\left[u - \frac{d(rv)}{dr}\right]\frac{\partial S_i}{\partial\theta}, \tag{131}$$

and reduce (117), (118) and (119) to the following:

$$(132)\quad \left[(\lambda+2\mu)\frac{d}{dr}\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right)+i(i+1)\frac{\mu}{r^2}\left(\frac{d(rv)}{dr}-u\right)\right]S_i=-\rho F_r,$$

$$(133)\quad \left[(\lambda+2\mu)\frac{1}{r}\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right)+\frac{\mu}{r}\frac{d}{dr}\left(\frac{d(rv)}{dr}-u\right)\right]\frac{\partial S_i}{\partial\theta}=-\rho F_\theta,$$

$$(134)\quad \left[(\lambda+2\mu)\frac{1}{r}\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right)+\frac{\mu}{r}\frac{d}{dr}\left(\frac{d(rv)}{dr}-u\right)\right]\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}=-\rho F_\phi.$$

(c) *Bodily forces.* We now proceed to express the values of (1) the disturbing forces, (2) the gravitational force due to the change of density of the attracted element, (3) the attraction due to the change of configuration of the attracting mass.

(1) Disturbing forces. Let the potential of the disturbing forces be

$$(135)\qquad W_i=-\frac{2}{3i}cga\frac{r^i}{a^i}S_i;$$

then the terms in ρF_r, ρF_θ, ρF_ϕ representing these forces are

$$(136)\qquad \rho\frac{\partial W_i}{\partial r}=-\tfrac{2}{3}c\rho ga\frac{r^{i-1}}{a^i}S_i,$$

$$(137)\qquad \frac{\rho}{r}\frac{\partial W_i}{\partial\theta}=-\frac{2}{3i}c\rho ga\frac{r^{i-1}}{a^i}\frac{\partial S_i}{\partial\theta},$$

$$(138)\qquad \frac{\rho}{r\sin\theta}\frac{\partial W_i}{\partial\phi}=-\frac{2}{3i}c\rho ga\frac{r^{i-1}}{a^i}\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}.$$

(2) Attraction of original mass-configuration upon increment of mass of element. The attraction of the original mass-configuration per unit mass upon an element distant r from the center is

$$-g\frac{r}{a}.$$

The increment of density is

$$(139)\qquad -\rho\Delta=-\rho\left[\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right]S_i.$$

Hence the attraction per unit volume, due to the increment of density of the attracted element, is

$$(140)\qquad \begin{aligned}\left(-g\frac{r}{a}\right)(-\rho\Delta)&=g\rho\frac{r}{a}\left[\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right]S_i\\&=g\rho a\frac{r}{a^2}\left[\frac{du}{dr}+2\frac{u}{r}-i(i+1)\frac{v}{r}\right]S_i.\end{aligned}$$

(3) Attraction due to increment of density of attracting mass. The potential at a point (r, θ, ϕ) due to a shell of density $D'S_i$, radius r' and thickness dr' is

$$\frac{4\pi\gamma}{2i+1}\frac{r'^{i+2}}{r^{i+1}}D'S_i dr' \qquad (r > r'), \tag{141}$$

$$\frac{4\pi\gamma}{2i+1}\frac{r^i}{r'^{i-1}}D'S_i dr' \qquad (r < r'). \tag{142}$$

On replacing $D'S_i$ by

$$-\rho\Delta' = \rho\left[i(i+1)\frac{v'}{r'} - \frac{1}{r'^2}\frac{d(r'^2u')}{dr'}\right]S_i,$$

the potential due to the change of density of the sphere is given by the expression

$$\begin{aligned}\frac{3gS_i}{(2i+1)a}\Bigg[\frac{1}{r^{i+1}}\int_0^r\left(i(i+1)\frac{v'}{r'} - \frac{1}{r'^2}\frac{d(r'^2u')}{dr'}\right)r'^{i+2}dr' \\ + r^i\int_r^a\left(i(i+1)\frac{v'}{r'} - \frac{1}{r'^2}\frac{d(r'^2u')}{dr'}\right)\frac{dr'}{r'^{i-1}}\Bigg].\end{aligned} \tag{143}$$

To this must be added the potential due to the surface inequality of density ρ and thickness $u_1 S_i$:

$$\frac{3g}{2i+1}u_1\frac{r^i}{a^i}S_i. \tag{144}$$

Adding (143) and (144), and reducing by integrating by parts the last term in each definite integral, we see that the total potential may be expressed in the form

$$\begin{aligned}V_i = \frac{3gS_i}{(2i+1)a}\Bigg[\frac{i}{r^{i+1}}\int_0^r\left((i+1)\frac{v'}{r'} + \frac{u'}{r'}\right)r'^{i+2}dr' \\ + (i+1)r^i\int_r^a\left(i\frac{v'}{r'} - \frac{u'}{r'}\right)\frac{dr'}{r'^{i-1}}\Bigg].\end{aligned} \tag{145}$$

The components of bodily force due to this potential are (per unit volume)

$$\begin{aligned}\rho\frac{\partial V_i}{\partial r} = \frac{3g\rho}{(2i+1)a}\Bigg[-\frac{i(i+1)}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'} + \frac{u'}{r'}\right)r'^{i+2}dr' \\ + i(i+1)r^{i-1}\int_r^a\left(i\frac{v'}{r'} - \frac{u'}{r'}\right)\frac{dr'}{r'^{i-1}} + (2i+1)u\Bigg]S_i,\end{aligned} \tag{146}$$

$$\begin{aligned}\frac{\rho}{r}\frac{\partial V_i}{\partial\theta} = \frac{3g\rho}{(2i+1)a}\Bigg[\frac{i}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'} + \frac{u'}{r'}\right)r'^{i+2}dr' \\ + (i+1)r^{i-1}\int_r^a\left(i\frac{v'}{r'} - \frac{u'}{r'}\right)dr'\Bigg]\frac{\partial S_i}{\partial\theta},\end{aligned} \tag{147}$$

$$\frac{\rho}{r\sin\theta}\frac{\partial V_i}{\partial\phi}=\frac{3g\rho}{(2i+1)a}\left[\frac{i}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'}+\frac{u'}{r'}\right)r'^{i+2}dr'\right.$$
$$\left.+(i+1)r^{i-1}\int_r^a\left(i\frac{v'}{r'}-\frac{u'}{r'}\right)dr'\right]\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}. \tag{148}$$

(4) Components of total bodily force. Combining all the bodily forces, we find

$$\rho F_r=\rho ga\left\{-\tfrac{2}{3}c\frac{r^{i-1}}{a^i}+\frac{r}{a^2}\left(\frac{du}{dr}+2\frac{u}{r}-i(i+1)\frac{v}{r}\right)+3\frac{u}{a^2}\right.$$
$$+\frac{3i(i+1)}{(2i+1)a^2}\left[-\frac{1}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'}+\frac{u'}{r'}\right)r'^{i+2}dr'\right.$$
$$\left.\left.+r^{i-1}\int_r^a\left(i\frac{v'}{r'}-\frac{u'}{r'}\right)\frac{dr'}{r'^{i-1}}\right]\right\}S_i, \tag{149}$$

$$\rho F_\theta=\rho ga\left\{-\frac{2}{3i}c\frac{r^{i-1}}{a^i}+\frac{3}{(2i+1)a^2}\left[\frac{i}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'}+\frac{u'}{r'}\right)r'^{i+2}dr'\right.\right.$$
$$\left.\left.+(i+1)r^{i-1}\int_r^a\left(i\frac{v'}{r'}-\frac{u'}{r'}\right)dr'\right]\right\}\frac{\partial S_i}{\partial\theta}, \tag{150}$$

while ρF_ϕ differs from ρF_θ only in the substitution of $1/\sin\theta\cdot\partial S_i/\partial\phi$ for $\partial S_i/\partial\theta$. These values substituted in (132), (133) and (134) give the three equations of equilibrium for a volume element. It is seen that the factors involving θ and ϕ cancel, also that the second and third equations are identical. Hence the problem is reduced to the solution of two simultaneous ordinary differential equations for determining u and v in terms of r.

(*d*) *Final form of equations of interior equilibrium.* We now get the equations in final form by substituting in (132) and (133) the above values of ρF_r and ρF_θ. In writing the equations we shall replace u and v by new functions

$$e=\frac{u}{r},\qquad \alpha=\frac{v}{r}, \tag{151}$$

and introduce as independent variable $x=r/a$. The equations then take the following forms:

$$(\lambda+2\mu)x^2\frac{d^2e}{dx^2}+4(\lambda+2\mu)x\frac{de}{dx}-i(i+1)\mu e-i(i+1)(\lambda+\mu)x\frac{d\alpha}{dx}$$
$$+2i(i+1)\mu\alpha+\rho ga\left[-\tfrac{2}{3}cx^i+\left(x\frac{de}{dx}+6e-i(i+1)\alpha\right)x^2\right.$$
$$\left.+\frac{3i(i+1)}{2i+1}\left(-\frac{1}{x^{i+1}}\int_0^x[(i+1)\alpha'+e']x'^{i+2}dx'+x^i\int_x^1(i\alpha'-e')\frac{dx'}{x'^{i-1}}\right)\right]=0, \tag{152}$$

$$(153)\qquad \begin{aligned}(\lambda+\mu)x\frac{de}{dx}+(3\lambda+5\mu)e+\mu x^2\frac{d^2\alpha}{dx^2}+4\mu x\frac{d\alpha}{dx}-[i(i+1)\lambda+2(i^2+i-1)\mu]\alpha\\ +\rho g a\left[-\frac{2c}{3i}x^i+\frac{3}{2i+1}\left(\frac{i}{x^{i+1}}\int_0^x[(i+1)\alpha'+e']x'^{i+2}dx'\right.\right.\\ \left.\left.+(i+1)x^i\int_x^1(i\alpha'-e')\frac{dx'}{x'^{i-1}}\right)\right]=0.\end{aligned}$$

§ 3. *Surface conditions.*

The stress upon the surface of the actual strained body is assumed to be 0. This is equivalent to assuming the body to extend to or beyond the original boundary $r=a$ in every direction, and assuming the stress on this spherical surface to be a normal stress equal to the weight of the surface inequality of thickness $u_1 S_i$ or $e_1 a S_i$, e_1 being the surface value of e.

That this is correct to the first order of small quantities may be shown by the reasoning outlined above in the solution of the case $i=2$, which need not be here repeated. Assuming its correctness, the boundary conditions are the following:

$$(154)\qquad \left.\begin{aligned}\widehat{rr}&=-\rho g a e_1 S_i\\ \widehat{r\theta}&=0\\ \widehat{r\phi}&=0\end{aligned}\right\}\text{ when } x=1.$$

In these we must put

$$(155)\qquad \begin{aligned}\widehat{rr}&=\lambda\Delta+2\mu\frac{\partial u_r}{\partial r},\\ \widehat{r\theta}&=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right),\\ \widehat{r\phi}&=\mu\left(\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}+\frac{1}{r\sin\phi}\frac{\partial u_r}{\partial\phi}\right).\end{aligned}$$

By means of (125), (126), (127) and (151) these take the forms,

$$(156)\qquad \begin{aligned}\widehat{rr}&=\left[(\lambda+2\mu)x\frac{de}{dx}+(3\lambda+2\mu)e-i(i+1)\lambda\alpha\right]S_i,\\ \widehat{r\theta}&=\mu\left(x\frac{d\alpha}{dx}+e\right)\frac{\partial S_i}{\partial\theta},\\ \widehat{r\phi}&=\mu\left(x\frac{d\alpha}{dx}+e\right)\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi},\end{aligned}$$

so that the three equations (154) reduce to the two equations,

$$(157)\qquad (\lambda+2\mu)x\frac{de}{dx}+(3\lambda+2\mu)e-i(i+1)\lambda\alpha=-\rho gae,$$

$$(158)\qquad x\frac{d\alpha}{dx}+e=0,$$

which must be satisfied for $x=1$.

[It is easy to see what equations must replace these if different surface conditions are given; for example, if there are known surface stresses proportional to S_i, $\partial S_i/\partial\theta$, $1/\sin\theta\cdot\partial S_i/\partial\phi$, or given surface displacements of a similar type.]

§ 4. *Solution of equations.*

A solution of (152) and (153) satisfying (157) and (158) is obtained by assuming

$$(159)\qquad e=\sum A_m x^m,\qquad \alpha=\sum B_m x^m.$$

The substitution of $A_m x^m$, $B_m x^m$ for e and α yields in (152) the terms

$$(160)\qquad \begin{aligned}&\left\{[(\lambda+2\mu)(m^2+3m)-\mu i(i+1)]A_m-i(i+1)[(\lambda+\mu)m-2\mu]B_m\right\}x^m\\&+\rho ga\left\{[(m+6)A_m-i(i+1)B_m]x^{m+2}\right.\\&\left.+\frac{3i(i+1)}{2i+1}\left[-\frac{A_m+(i+1)B_m}{m+i+3}x^{m+2}+\frac{A_m-iB_m}{m-i+2}x^{m+2}-\frac{A_m-iB_m}{m-i+2}x^i\right]\right\},\end{aligned}$$

and in (153) the terms

$$(161)\qquad \begin{aligned}&\left\{[(\lambda+\mu)m+(3\lambda+5\mu)]A_m+[\mu(m^2+3m)-(i(i+1)\lambda+2(i^2+i-1)\mu)]B_m\right\}x^m\\&+\rho ga\frac{3}{2i+1}\left[\frac{i[A_m+(i+1)B_m]}{m+i+3}x^{m+2}\right.\\&\left.+\frac{(i+1)(A_m-iB_m)}{m-i+2}x^{m+2}-\frac{(i+1)(A_m-iB_m)}{m-i+2}x^i\right].\end{aligned}$$

Notice first that the terms in x^i in these two expressions are in the same ratio as the original terms of degree i in (152) and (153); therefore, the terms of degree i in both equations can be made to vanish by a single relation among the coefficients in the assumed series (159).

Again, for the terms of lowest degree, the coefficients of x^m in (160) and (161) must vanish, and this is possible without the vanishing of A_m and B_m only if the determinant of the coefficients of A_m and B_m vanishes. The value of this determinant is found to be

$$(162)\qquad (\lambda+2\mu)\mu(m-i)(m-i+2)(m+i+1)(m+i+3).$$

The four values of m for which this vanishes, with the corresponding values of B_m/A_m, are as follows:

m	$\dfrac{B_m}{A_m}=\dfrac{(\lambda+2\mu)(m^2+3m)-\mu i(i+1)}{i(i+1)[(\lambda+\mu)m-2\mu]}$
i	$\dfrac{(i+3)\lambda+(i+5)\mu}{(i+1)[i\lambda+(i-2)\mu]}$
$i-2$	$\dfrac{1}{i}$
$-i-1$	$-\dfrac{(i-2)\lambda+(i-4)\mu}{i(i+1)\lambda+i(i+3)\mu}$
$-i-3$	$-\dfrac{1}{i+1}$

Negative powers of x being excluded by physical considerations, the series may start with terms of degree $i-2$, provided i is not less than 2. If i is positive and less than 2, the lowest degree available would be i.

We here assume i equal to or greater than 2, and let the series (159) take the forms

$$(163)\qquad \begin{aligned} e &= A_{i-2}x^{i-2}+A_ix^i+A_{i+2}x^{i+2}+\cdots,\\ \alpha &= \frac{1}{i}A_{i-2}x^{i-2}+B_ix^i+B_{i+2}x^{i+2}+\cdots. \end{aligned}$$

From (160) and (161) it is seen that (excluding the terms of degree i) the coefficients of x^{m+2} resulting from the substitution of (163) in (152) and (153) will vanish if the following equations are satisfied:

$$(164)\qquad \begin{aligned} &[(\lambda+2\mu)(m+2)(m+5)-\mu i(i+1)]A_{m+2}-i(i+1)[(\lambda+\mu)(m+2)-2\mu]B_{m+2}\\ &\quad+\rho ga\frac{m^2+8m-(i-3)(i+4)}{(m+i+3)(m-i+2)}[(m+3)A_m-i(i+1)B_m]=0, \end{aligned}$$

$$(165)\qquad \begin{aligned} &[(\lambda+\mu)(m+2)+3\lambda+5\mu]A_{m+2}\\ &\quad+\{\mu(m+2)(m+5)-[i(i+1)\lambda+2(i^2+i-1)\mu]\}B_{m+2}\\ &\quad+\rho ga\frac{3}{(m+i+3)(m-i+2)}[(m+3)A_m-i(i+1)B_m]=0. \end{aligned}$$

By means of these equations the coefficients $A_{i+2}, A_{i+4}, \cdots, B_{i+2}, B_{i+4}, \cdots$ may be expressed in terms of A_i, B_i, leaving thus far arbitrary the three constants A_{i-2}, A_i, B_i.

It remains only to equate to zero the coefficients of x^i, which as already

remarked requires but a single equation. The two differential equations will thus be satisfied by values of e and α of the assumed form, containing two arbitrary constants, provided the series thus determined prove to be convergent. The two arbitrary constants suffice for the satisfaction of the surface conditions.

The solution of (164) and (165) gives

$$(166)\quad A_{m+2} = -\frac{\rho g a}{\lambda + 2\mu} \times \frac{[(m+4)(m+6)\mu - i(i+1)(\lambda+2\mu)][(m+3)A_m - i(i+1)B_m]}{(m-i+2)(m+i+5)(m-i+4)(m+i+3)\mu},$$

$$(167)\quad B_{m+2} = \frac{\rho g a[m(\lambda+\mu)+5\lambda+4\mu][(m+3)A_m - i(i+1)B_m]}{(\lambda+2\mu)(m-i+2)(m+i+5)(m-i+4)(m+i+3)\mu},$$

$$(168)\quad \frac{(m+5)A_{m+2} - i(i+1)B_{m+2}}{(m+3)A_m - i(i+1)B_m} = -\frac{m+6}{(m-i+2)(m+i+3)} \cdot \frac{\rho g a}{\lambda+2\mu}.$$

By means of these equations the values of all coefficients after the first two in each of the assumed series (163) may be expressed in terms of A_i and B_i. These values may be expressed as follows, n taking values 0, 1, 2, $\cdots$:

$$(169)\quad A_{i+2n+2} = (-b)^{n+1}\left[i(i+1)\frac{\lambda}{\mu} + 2i(i+1) - (i+2n+4)(i+2n+6)\right] \times \frac{(i+4)(i+6)\cdots(i+2n+4)[i(i+1)B_i - (i+3)A_i]}{(i+4)[2\cdot 4\cdots(2n+4)][(2i+3)(2i+5)\cdots(2i+2n+5)]},$$

$$(170)\quad B_{i+2n+2} = (-b)^{n+1}\left[(i+2n+5)\frac{\lambda}{\mu} + i + 2n + 4\right] \times \frac{(i+4)(i+6)\cdots(i+2n+4)[i(i+1)B_i - (i+3)A_i]}{(i+4)[2\cdot 4\cdots(2n+4)][(2i+3)(2i+5)\cdots(2i+2n+5)]}.$$

The coefficient of x^i may be found by substituting in (160) or (161) successively $m = i-2, i, i+2, \cdots$, remembering that $iB_{i-2} = A_{i-2}$, and that, as already pointed out, a single equation causes the vanishing of the coefficients of x^i in both (152) and (153). This equation may be written in the following form, after dividing through by $\lambda + 2\mu$ and writing b for $g\rho a/(\lambda+2\mu)$:

$$(171)\quad i(i+1)\left[\left(\frac{\lambda+\mu}{\lambda+2\mu} + \frac{2}{i+1} - \frac{3b}{2(2i+1)}\right)A_i - \left(\frac{i\lambda + (i-2)\mu}{\lambda+2\mu} - \frac{3ib}{2(2i+1)}\right)B_i\right] + \frac{3ib}{2i+1}A_{i-2} - \frac{3i(i+1)b}{2i+1}\sum_0^\infty \frac{A_{i+2+2n} - iB_{i+2+2n}}{2n+4} - \tfrac{2}{3}bc = 0.$$

Also, from (169) and (170),

$$
(172)\quad \begin{aligned}
\frac{A_{i+2+2n} - iB_{i+2+2n}}{2n+4} &= (-b)^{n+1}\left[i\frac{\lambda}{\mu} + 3i + 2(n+3)\right] \\
&\times \frac{[(i+4)(i+6)\cdots(i+2n+4)]\,[(i+3)A_i - i(i+1)B_i]}{(i+4)[2\cdot 4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]},
\end{aligned}
$$

by means of which (171) is reduced to a relation among the three constants A_{i-2}, A_i, B_i.

The differential equations (152) and (153) are thus satisfied by (163), provided the coefficients satisfy (169), (170), (171) and (172); and two of the coefficients still remain arbitrary. By properly assigning these, the surface conditions (157) and (158) may be satisfied. The solution may be expressed in fairly compact form by means of the following functions:

$$
(173)\quad \begin{aligned}
L(z) &= \sum_0^\infty (-1)^n\left[i(i+1)\frac{\lambda}{\mu} + 2i(i+1) - (i+2n+4)(i+2n+6)\right] \\
&\quad \times \frac{(i+4)(i+6)\cdots(i+2n+4)}{(i+4)[2\cdot 4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]} z^n, \\
M(z) &= \sum_0^\infty (-1)^n\left[(i+2n+5)\frac{\lambda}{\mu} + i + 2n + 4\right] \\
&\quad \times \frac{(i+4)(i+6)\cdots(i+2n+4)}{(i+4)[2\cdot 4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]} z^n, \\
N(z) &= \sum_0^\infty (-1)^n\left[i\frac{\lambda}{\mu} + 3i + 2(n+3)\right] \\
&\quad \times \frac{(i+4)(i+6)\cdots(i+2n+4)}{(i+4)[2\cdot 4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]} z^n, \\
L'(z) &= \sum_0^\infty (-1)^n\left[i(i+1)\frac{\lambda}{\mu} + 2i(i+1) - (i+2n+4)(i+2n+6)\right] \\
&\quad \times \frac{(i+2n+2)[(i+4)(i+6)\cdots(i+2n+4)]}{(i+4)[2\cdot 4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]} z^n, \\
M'(z) &= \sum_0^\infty (-1)^n\left[(i+2n+5)\frac{\lambda}{\mu} + i + 2n + 4\right] \\
&\quad \times \frac{(i+2n+2)[(i+4)(i+6)\cdots(i+2n+4)]}{(i+4)[2\cdot 4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]} z^n.
\end{aligned}
$$

The values of e and α, and the three equations for determining A_{i-2}, A_i and B_i, may then be written as follows, the third equation being equivalent to (171), the fourth to (157), and the fifth to (158):

$$(A)\qquad e = A_{i-2}x^{i-2} + A_i x^i + [(i+3)A_i - i(i+1)B_i]\, bx^{i+2} L(bx^2),$$

$$(B)\qquad \alpha = \frac{1}{i} A_{1-2}x^{i-2} + B_i x^i + [(i+3)A_i - i(i+1)B_i]\, bx^{i+2} M(bx^2),$$

$$(C)\qquad \begin{aligned} &\frac{3ib}{2i+1}(A_{i-2} + A_i) + \frac{(4i+6)\mu}{\lambda+2\mu} A_i \\ &+ [(i+3)A_i - i(i+1)B_i]\left(\frac{i\lambda+(i-2)\mu}{\lambda+2\mu} - \frac{3ib}{2(2i+1)} + \frac{3i(i+1)}{2i+1} b^2 N(b)\right) - \tfrac{2}{3}bc = 0, \end{aligned}$$

$$(D)\qquad \begin{aligned} &\left[b\left(\frac{\lambda}{\mu}+2\right) + 2i - 2\right](A_{i-2} + A_i) + 4A_i + [(i+3)A_i - i(i+1)B_i] \\ &\times \left[\frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}+2\right) bL'(b) + \left(3\frac{\lambda}{\mu}+2\right) bL(b) \right. \\ &\qquad \left. + \left(\frac{\lambda}{\mu}+2\right) b^2 L(b) - i(i+1)\frac{\lambda}{\mu} bM(b)\right] = 0, \end{aligned}$$

$$(E)\qquad \begin{aligned} &(i^2-1)(A_{i-2} + A_i) + (2i+1)A_i + \frac{i}{2}[(i+3)A_i - i(i+1)B_i] \\ &\times [-1 + (i+1)bL(b) + (i+1)bM'(b)] = 0. \end{aligned}$$

These correspond to equations (A), (B), (C), (D), (E) obtained in Part I of this paper, and reduce to them when $i = 2$. It must be noted, however, that the notation is slightly different, e, α and the functions (173) differing by constant factors from the corresponding functions in Part I.

DIE NATÜRLICHEN GLEICHUNGEN DER ANALYTISCHEN CURVEN IM EUKLIDISCHEN RAUME*

VON

E. STUDY

Die Grundformeln der gewöhnlichen Differentialgeometrie (und nicht minder zum Beispiel auch die der Differentialgeometrie im Nicht-Euklidischen Raume) haben sämmtlich ihre Quelle in einigen wenigen algebraischen Identitäten, Determinantenrelationen, *die sich im Voraus erschöpfend angeben lassen.* Indem man diesen Gedanken, der in älteren Darstellungen wohl noch nie zu seinem Rechte gekommen ist, sich deutlich vor Augen hält, kommt man zu einer neuen Behandlungsweise des Stoffes, die der Verfasser, zunächst allerdings nur an dem Beispiele der Curven, vor kurzem auseinandergesetzt hat. †

Die gegenwärtige Arbeit enthält eine erste Fortsetzung der genannten Untersuchung. Für den besonders wichtigen Fall, dass der Bogen (oder, bei den Minimalcurven der natürliche Parameter, d. h. einer dieser Parameter) die unabhängige Veränderliche ist, werden vollständige Systeme von Differentialinvarianten aufgestellt (§ 2 und § 5). Als Anwendung dieser kleinen ganz elementaren Theorie ergiebt sich die Thatsache, dass das Integrationsproblem der natürlichen Gleichungen einer regulären Curve äquivalent ist mit dem Problem der Integration gewisser linearer Differentialgleichungen, deren Theorie demgemäss vollständig und mit einfachen Hülfsmitteln entwickelt wird (§6–§8). Im Besonderen folgt hieraus, dass man die natürlichen Gleichungen aller der regulären Curven kennzeichnen und durch Quadraturen auflösen kann, deren Tangenten parallel sind zu den Erzeugenden irgend eines Kegels zweiter Ordnung. Die bekannte Theorie der durch die Gleichung $R : T =$ const. charakterisierten Schraubenlinien (die Rotationskegeln entsprechen) ist hiervon ein besonderer Fall.

Ausserdem enthält die Arbeit eine Integrationsmethode für die natürlichen Gleichungen selbst (in § 1 und § 3, die für sich ein abgeschlossenes Ganzes bilden.) Diese unsre Methode ist von der sonst üblichen völlig verschieden, wenn auch, wie natürlich, der Charakter der integrierenden Hülfsgleichungen derselbe bleibt. Sie besteht darin, dass der allgemeine Fall zurückgeführt

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† *Zur Differentialgeometrie der analytischen Curven* — weiterhin mit A. C. citiert — Transactions of the American Mathematical Society, vol. 10 (1909).

wird auf das entsprechende Problem für Curven auf gegebenem Minimalkegel. Diese Aufgabe wiederum verlangt die Integration einer *Schwarz*'schen Differentialgleichung (oder einer *Riccati*'schen Gleichung). Gelegentlich ergiebt sich hierbei eine geometrische Deutung des *Schwarz*'schen Differentialausdrucks (§ 1.)

Wegen der ebenen singulären Curven, die hier ausgeschlossen bleiben, verweisen wir auf die citierte Abhandlung (A. C., § 5).

§ 1.

Die Curven auf gegebenem Minimalkegel und die Minimalcurven.

Wir werden zunächst krumme Linien betrachten, die auf dem Minimalkegel

$$y_1^2 + y_2^2 + y_3^2 = 0$$

liegen. Um zu einer zweckdienlichen Parameterdarstellung dieses Kegels zu gelangen, fassen wir ihn als Grenzfall einer Kugel

$$y_1^2 + y_2^2 + y_3^2 = \mathfrak{R}^2$$

von nicht verschwindendem Radius auf. Die letzte Gleichung nun, die wir kürzer auch so schreiben können: $(y \mid y) = \mathfrak{R}^2$, kann nach bekanntem (unsres Wissens in allem Wesentlichen von Darboux herrührendem) Verfahren, identisch befriedigt werden durch die Substitutionen

$$y_1 \equiv \mathfrak{R}\,\frac{1 - st}{s - t}, \qquad y_2 \equiv \mathfrak{R} \cdot i\,\frac{1 + st}{s - t}, \qquad y_3 \equiv -\mathfrak{R} \cdot \frac{s + t}{s - t},$$

wobei die uneigentlichen Punkte der Kugel und ausserdem die eigentlichen Punkte von zwei Erzeugenden $\{ y_1 - iy_2 = 0,\ y_3^2 - 1 = 0 \}$ zunächst ausgeschlossen bleiben. In den Parametern s, t werden dann die automorphen Bewegungen der Kugel, nämlich die Drehungen um deren Mittelpunkt, dargestellt durch irgend zwei mit einander identische lineare Transformationen. Diese lassen sich in der Form

$$s^* = \frac{(\alpha_0 - i\alpha_3)s - (\alpha_2 - i\alpha_1)}{(\alpha_2 + i\alpha_1)s + (\alpha_0 + i\alpha_3)}, \qquad t^* = \frac{(\alpha_0 - i\alpha_3)t - (\alpha_2 - i\alpha_1)}{(\alpha_2 + i\alpha_1)t + (\alpha_0 + i\alpha_3)}$$

schreiben. Die Verhältnissgrössen $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3$, deren Quadratsumme nicht verschwinden darf, sind die Euler'schen Parameter der ausgeführten Drehung. Ist diese insbesondere reell, so dürfen auch die Parameter α als reell angenommen werden, und umgekehrt entsprechen reelle Parameter einer reellen Drehung.

Wir führen jetzt einen Grenzübergang aus, indem wir den Quotienten $(s - t) : \mathfrak{R}$ durch einen neuen Parameter $2t_1$ ersetzen, und dann zur Grenze

$t = s$, $\Re = 0$ übergehen. So entstehen, wenn für t_1 wieder t geschrieben wird, die Formeln

$$(1) \qquad y_1 \equiv \frac{1 - s^2}{2t}, \qquad y_2 \equiv i\frac{1 + s^2}{2t}, \qquad y_3 \equiv -\frac{2s}{2t}.$$

Diese Gleichungen dienen also zur Parameterdarstellung einer Kugel vom Radius Null, nämlich des Minimalkegels

$$(2) \qquad y_1^2 + y_2^2 + y_3^2 = 0,$$

wobei jedoch der Scheitel dieses Kegels ($t = \infty$) und überhaupt alle Punkte einer Erzeugenden $\{ y_1 - iy_2 = 0,\ y_3 = 0 \}$ in die Parameterdarstellung nicht (nicht unmittelbar) einbezogen sind.

Aus (1) folgt noch

$$(3) \qquad s \equiv \frac{y_1 + iy_2}{y_3} \equiv -\frac{y_3}{y_1 - iy_2}, \qquad t \equiv \frac{1}{y_1 - iy_2} \equiv -\frac{y_1 + iy_2}{y_3^2}.$$

Für die Parameter s und t erhalten wir jetzt — durch den gleichen Grenzübergang — die Transformationsgleichungen

$$(4) \quad s^* = \frac{(\alpha_0 - i\alpha_3)s - (\alpha_2 - i\alpha_1)}{(\alpha_2 + i\alpha_1)s + (\alpha_0 + i\alpha_3)}, \qquad t^* = \frac{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}{\{(\alpha_2 + i\alpha_1)s + (\alpha_0 + i\alpha_3)\}^2} \cdot t,$$

die zur Darstellung der Drehungen um den Scheitel des Kegels dienen, soweit geeignete Punkte des Kegels selbst in Frage kommen. Die Formeln (4) aber setzen folgenden Lehrsatz in Evidenz:

Bei geeigneter Coordinatenwahl vertauscht die Gruppe der automorphen Bewegungen eines Minimalkegels dessen Punkte durch die erste Lie'sche Erweiterung der Gruppe aller linearen Transformationen eines binären Gebietes.

Das heisst, denkt man sich s als Function eines Parameters p, so zeigt die Transformationsgleichung für t, wie der Differentialquotient $ds : dp$ transformiert wird. Der Satz ist hier deshalb von Bedeutung, weil er die Motivierung unseres Grenzüberganges vervollständigt. Er zeigt, warum es unter Umständen unzweckmässig sein kann, an Stelle des Zeichens t ein Zeichen für $1 : t$ zu gebrauchen, was bei anderer Art der Exposition näher liegen würde. Insbesondere führt unsere Bezeichnung noch zu einer bemerkenswerthen Zusammenfassung der beiden Gleichungen (4) in eine einzige ("duale") Gleichung:

$$(5) \qquad s^* + \epsilon t^* = \frac{(\alpha_0 - i\alpha_3)(s + \epsilon t) - (\alpha_2 - i\alpha_1)}{(\alpha_2 + i\alpha_1)(s + \epsilon t) + (\alpha_0 + i\alpha_3)} \qquad (\varepsilon^2 = 0).*$$

Nach diesen Vorbereitungen können wir nun jede *krumme* analytische Linie

* Vgl. des Verfassers *Geometrie der Dynamen* (Leipzig, 1903), II, § 23, und A. C., S. 40, 41, wo umfassendere Formeln angegeben werden.

auf dem Minimalkegel, oder doch ein Stück einer solchen Linie, dadurch ausdrücken, dass wir s und t als analytische Functionen eines Parameters p darstellen, als Functionen, die natürlich einen gemeinsamen Existenzbereich haben müssen und von denen übrigens s nicht constant sein darf. Unter allen Parameterdarstellungen einer solchen Curve giebt es aber eine, und von einer unwesentlichen Willkür abgesehen, auch nur eine ausgezeichnete, die durch den *natürlichen Parameter* der Curve. *Um auszudrücken, dass p ein "natürlicher" Parameter sei, genügt nicht eine einzige Differentialgleichung, sondern es sind* — wie A. C., S. 34 gezeigt worden ist — *deren zwei erforderlich,*

$$(y'|y')_p \equiv -1, \qquad (yy'y'')_p \equiv -1,$$

die natürlich mit einander verträglich sein müssen und es thatsächlich auch sind. Man hat nun in unserem Falle, wo y durch Vermittelung von s und t von p abhängt,

$$(y'|y')_p \equiv -\left(i\frac{s'}{t}\right)^2, \qquad (yy'y'')_p \equiv -\left(i\frac{s'}{t}\right)^3,$$

also

$$t \equiv i\frac{ds}{dp}. \tag{6}$$

Die auf den nunmehr natürlichen Parameter p bezogene Curve wird mithin durch folgende Formeln dargestellt:

$$y_1 \equiv -i\frac{1-s^2(p)}{2s'(p)}, \qquad y_2 \equiv \frac{1+s^2(p)}{2s'(p)}, \qquad y_3 \equiv i\frac{2s(p)}{2s'(p)}. \tag{7}$$

Die *natürliche Gleichung* der Curve auf dem Minimalkegel (2) aber giebt das Quadrat der Krümmung als Function von p (A. C., Nr. 39, S. 35):

$$\phi(p) \equiv \frac{1}{R^2} \equiv (y''|y'')_p \equiv -\frac{2s_p's_p''' - 3s_p''s_p''}{s_p's_p'}. \tag{8}$$

Man hat also zur Bestimmung der Curve aus ihrer natürlichen Gleichung die Schwarz'sche Differentialgleichung (8) *oder die mit dieser äquivalente, durch die Substitution*

$$\frac{s''}{2s'} \equiv \omega$$

entstehende Riccati'sche Gleichung

$$\frac{d\omega}{dp} \equiv -\frac{\phi}{4} + \omega^2 \tag{9}$$

aufzulösen.

Wir erinnern daran, dass durch drei verschiedene Lösungen der Riccati'schen Gleichung eine Lösung der Schwarz'schen Gleichung auf die einfachste Weise

ausgedrückt werden kann:

$$s \equiv \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3}.*$$

Zugleich ist bewiesen:

Zu jeder Schwarz'schen Differentialgleichung

$$\{s, p\} \equiv \frac{2s's''' - 3s''s''}{2s's'} \equiv -\tfrac{1}{2}\phi(p) \qquad \text{(Nr. 8)}$$

gehört eine Classe unter einander congruenter krummer Linien, die auf dem Minimalkegel $y_1^2 + y_2^2 + y_3^2 = 0$ *liegen, und zu jeder einzelnen Lösung der Gleichung eine bestimmte unter diesen Curven.* p *(allgemein* $p + const$*) ist ein zugehöriger natürlicher Parameter,* ϕ *die in der gemeinsamen natürlichen Gleichung jener Curven auftretende Function, also* $\phi(p)$ *das Quadrat ihrer Krümmung an den zu dem Werthe* p *gehörigen Stellen.*

Es ist natürlich nicht ausgeschlossen, dass eine und dieselbe Curve mehreren, auch unendlich vielen Lösungen der Schwarz'schen Gleichung entspricht. Es existiert dann eine Gruppe automorpher Bewegungen der Curve. Continuierlich ist diese Gruppe nur im Falle der Kreise, nämlich der parabolischen (singulären) $\{\phi \equiv 0\}$ und regulären Kreise $\{\phi \equiv \text{const.}, \neq 0\}$.

Zu beachten ist, dass in obigem Satze der natürliche Parameter *nicht* durch den (zu *zweiwerthigem* Differential gehörigen) Curvenbogen ersetzt werden kann. Giebt man den Werth ϕ als Function des Bogens, so ist damit die zugehörige Curve nur bis auf Bewegungen und Umlegungen bestimmt.

Die Gleichung

$$\frac{1}{R^2} \equiv \phi(p)$$

fungiert auch als natürliche Gleichung für die Minimalcurven, die die Curve auf dem Minimalkegel zum sphärischen Bilde, oder zum Bestandtheil ihres sphärischen Bildes haben. (Siehe darüber A. C., § 6, insbes. S. 37.)

Um also eine (*krumme*) Minimallinie aus ihrer natürlichen Gleichung zu bestimmen, hat man nach Integration der Gleichung (8), die mit der Gleichung {A. C., (51)} identisch ist, noch die in der Formel $\int x\,dp$ zusammengefassten drei Quadraturen auszuführen.

§ 2.

Erläuterungen. Differentialinvarianten der Minimalcurven.

Durch das Vorgetragene wird die Frage nach der Bestimmung einer krummen Minimallinie aus ihrer natürlichen Gleichung auf die wohl einfachste Weise

* Der Zusammenhang beider Gleichungen wird eingehend untersucht in der Abhandlung *Über Riccati'sche und Schwarz'sche Differentialgleichungen*, Archiv der Mathematik und Physik, 1910, die weiterhin mit R. Gl. citirt wird.

beantwortet. Man kann sich indessen die Frage vorlegen, wie diese Methode mit mehreren anderen zusammenhängt, die sich ebenfalls darbieten.

Wir beginnen mit der Aufstellung eines in gewissem Sinne vollständigen Systems von Differentialinvarianten einer krummen Minimallinie. Ein solches wird gebildet von der Function ϕ mit ihren Differentialquotienten nach p:

Benutzt man zur Darstellung einer krummen Minimallinie $x \equiv x(p)$ einen natürlichen Parameter, so werden alle auf diesen bezogenen Semiinvarianten $(i|k)$ und (ikl) ganze rationale Functionen der Function ϕ und ihrer Differentialquotienten, und zwar kommen in diesen Ausdrücken die Differentialquotienten von ϕ höchstens in der $(n-3)$ten Ordnung vor, wenn $i, k, l \leqq n$ sind.

Mit Coefficienten von gleicher Beschaffenheit lassen sich die Semicovarianten $(x^{(k)}|\omega)$ und $(x^{(i)}x^{(k)}\omega)$ linear ausdrücken durch $(x'|\omega)$, $(x''|\omega)$, $(x'''|\omega)$.

Wir gehen zunächst so weit, wie es die Anwendungen in der Regel erfordern werden, nämlich bis zu dem Werthe $n = 4$:

$$
(1)\qquad
\begin{array}{llll}
(1|1) \equiv 0, & * \;, & * \;, & * \;, \\
(1|2) \equiv 0, & (2|2) \equiv -1, & * \;, & * \;, \\
(1|3) \equiv 1, & (2|3) \equiv 0 \;, & (3|3) \equiv \phi \;, & * \;, \\
(1|4) \equiv 0, & (2|4) \equiv -\phi, & (3|4) \equiv \tfrac{1}{2}\phi', & (4|4) \equiv -\phi^2; \\
\cdot & \cdot & \cdot & \cdot \\
 & (123) \equiv -1, & (124) \equiv 0 \;, & \\
 & (134) \equiv \phi \;, & (234) \equiv -\tfrac{1}{2}\phi',^{*} &
\end{array}
$$

Man sieht, dass, wenn der Satz für den Werth $n-1\,(\geqq 4)$ als richtig angenommen wird, er auch für den Werth n gelten muss. In der That erhält man aus den Semiinvarianten $(i|k)$ für $i, k \leqq n-1$ die Semiinvarianten $(k|n)$ für $k < n$ durch je einmalige Differentiation. Durch diese Grössen lassen sich sodann die Semiinvarianten $(ikn)\{i < k < n\}$ ganz und rational ausdrücken, wie man durch Bildung des Productes $-(123)(ikn)$ erkennt. Durch Entwickelung des Productes $-(123)(n|n)$ erhält man sodann den Werth der noch fehlenden Semiinvariante $(n|n)$. Entsprechend beweist man den Satz über die Semicovarianten, von denen wir nur einige wenige anführen wollen:

$$
(2)\quad (12\omega) \equiv -(1|\omega), \qquad (13\omega) \equiv -(2|\omega), \qquad (23\omega) \equiv \phi\cdot(1|\omega) - (3|\omega);
$$

$$
(3)\qquad (4|\omega) \equiv \tfrac{1}{2}\phi'\cdot(1|\omega) + \phi\cdot(2|\omega).
$$

Die wichtigsten unter diesen Gleichungen sind die erste unter (2) und die Gleichung (3). Diese letzte ist offenbar mit der natürlichen Gleichung der

* Diese Tafel ergiebt sich aus den Formeln A. C., Nr. 36, 37, wenn man linker Hand jeden Differentiationsindex durch den nächsthöheren ersetzt.

Minimalcurve äquivalent, da man ihr die Function $\phi(p)$ entnehmen kann. Sie zeigt, dass die Coordinaten $y_k \equiv dx_k/dp$ des Differentialvectors $y \equiv x'_p$ Lösungen einer linearen Differentialgleichung 3. Ordnung sind, die, wegen der ersten Gleichung unter (2), mit ihrer Adjungierten zusammenfällt. *Diese Differentialgleichung*

$$y''' - \phi(p)\cdot y' - \tfrac{1}{2}\phi'(p)\cdot y = 0 \tag{4}$$

ist aber (bekanntlich) die allgemeinste vom zweiten Gliede befreite lineare Gleichung 3. Ordnung, bei der zwischen je drei linear-unabhängigen Lösungen eine homogene quadratische Gleichung besteht. Man kann daher die Theorie der krummen Linien auf Minimalkegeln und weiterhin die Theorie der krummen Minimallinien auch als eine Anwendung der Theorie dieser speciellen linearen Differentialgleichungen 3. Ordnung behandeln. Wie dieser Gedanke weiter auszuführen ist, liegt auf der Hand. Wir verweisen auf die citierte Arbeit (R. Gl., § 12), wo insbesondere auch der Zusammenhang der angedeuteten Methode mit der zuvor angewendeten dargelegt ist.

Als eine weitere Folgerung unserer Theorie heben wir hervor:

Wie zu den regulären Curven, so gehört auch zu den Minimalcurven im Allgemeinen ein System von orthogonalen Triedern, ein die Curve begleitendes Axenkreuz, und entsprechend ein System von Formeln, die den Serret-Frenet'schen Gleichungen analog sind.

Der hier zu Grunde liegende Gedanke ist ganz selbstverständlich, es scheint uns aber nöthig, die correkte Form eines derartigen Satzes festzustellen.

Das angedeutete System von drei zu einander senkrechten Axen (und Einheitsvectoren) existiert an allen den Stellen regulären Verhaltens einer Minimalcurve, wo das Krümmungsmaass ihres sphärischen Bildes nicht verschwindet ($\phi \neq 0$). *Es existiert also überhaupt nicht im Falle der Minimalcurven 3. Ordnung* (*wo* $\phi \equiv 0$ *ist*).

Nennen wir $1/R$ und $1/T$ die Krümmung und die Torsion des sphärischen Bildes unserer Curve, so haben wir, wie bereits abgeleitet (A. C., S. 34),

$$\frac{1}{R} \equiv \sqrt{\phi}, \qquad \frac{1}{T} \equiv \tfrac{1}{2}\frac{\phi'}{\phi} \equiv -\frac{1}{R}\frac{dR}{dp}. \tag{5}$$

Dem Bogen des sphärischen Bildes kann irgend einer der in der Formel

$$s \equiv \sqrt{-1}\, p \ \{+\text{const.}\} \tag{6}$$

zusammengefassten Werthe beigelegt werden. Für die Einheitsvectoren α, β, γ, die zu Tangente, Hauptnormale und Binormale des sphärischen Bildes gehören, — derart, dass

$$(\alpha\beta\gamma) \equiv 1, \ (\beta\gamma\omega) \equiv (\alpha|\omega), \ (\gamma\alpha\omega) \equiv (\beta|\omega), \ (\alpha\beta\omega) \equiv (\gamma|\omega)$$

wird — ergeben sich sodann die Ausdrücke

$$(7)\qquad \begin{gathered}(\alpha|\omega)\equiv-\sqrt{-1}(x''|\omega),\qquad (\beta|\omega)\equiv-R.(x'''|\omega),\\ (\gamma|\omega)\equiv\sqrt{-1}\cdot R\cdot(x''x'''\omega),\end{gathered}$$

wobei die Differentiationen sich, wie zuvor, auf den natürlichen Parameter p beziehen. An den Stellen, wo die Vectoren α, β, γ existieren, kann man dann alle Differentialvectoren x', x'', x''', $\cdots$, linear durch sie ausdrücken, zum Beispiel:

$$(8)\qquad \begin{gathered}x'\equiv-R\{\beta+\sqrt{-1}\gamma\},\qquad x''\equiv\sqrt{-1}\cdot\alpha,\qquad x'''\equiv-\frac{1}{R}\cdot\beta,\\ x''''\equiv\frac{1}{R^2}\cdot\left\{\sqrt{-1}\cdot\alpha+\frac{dR}{dp}(\beta+\sqrt{-1}\gamma)\right\}.\end{gathered}$$

Die *Frenet*'schen Gleichungen für das sphärische Bild nehmen jetzt die Form an:

$$(9)\qquad \frac{d\alpha}{dp}\equiv\frac{\sqrt{-1}}{R}\cdot\beta,\qquad \frac{d\beta}{dp}\equiv-\frac{\sqrt{-1}}{R}\left\{\alpha+\frac{dR}{dp}\gamma\right\},\qquad \frac{d\gamma}{dp}\equiv\frac{\sqrt{-1}}{R}\cdot\frac{dR}{dp}\cdot\beta.$$

Man sieht, dass alle diese Formeln dort illusorisch werden, wo die Function ϕ verschwindet, nicht aber an anderen Stellen regulären Verhaltens von ϕ, oder — was dasselbe ist — nicht an anderen Stellen regulären Verhaltens der Curve. Man mag diese letzten Stellen ($\phi \neq 0$) etwa solche *gewöhnlichen Verhaltens* nennen. Freilich bestehen dann die Minimalcurven 3. Ordnung aus lauter Stellen nicht "gewöhnlichen" Verhaltens.*

Die Gleichungen (9) können schliesslich nach der bei regulären Curven üblichen Methode behandelt werden. Damit hätten wir einen dritten Ansatz zur Bestimmung der durch die Ungleichung $\phi(p)\not\equiv 0$ charakterisierten Minimalcurven aus einer natürlichen Gleichung. Die angedeutete Methode besteht darin, dass man für irgend drei gleichnamige Coordinaten α_κ, β_κ, γ_κ der Einheitsvectoren α, β, γ der Reihe nach Ausdrücke der Form

$$\frac{1-\sigma_\kappa\tau_\kappa}{\sigma_\kappa-\tau_\kappa},\quad i\frac{1+\sigma_\kappa\tau_\kappa}{\sigma_\kappa-\tau_\kappa},\quad -\frac{\sigma_\kappa+\tau_\kappa}{\sigma_\kappa-\tau_\kappa}$$

setzt. Die Grössen σ_κ, τ_κ genügen dann für $\kappa=1, 2, 3$ sämmtlich der Riccati'schen Gleichung

$$(10)\qquad \frac{d\Omega}{dp}\equiv\frac{1}{2R}\left\{\frac{dR}{dp}+2\Omega-\frac{dR}{dp}\Omega^2\right\},$$

deren vollständige Lösung, wie bekannt, die der Gleichungen (9) nach sich zieht, und damit, nach (8), auch $y\equiv x'_p$ und dann durch drei Quadraturen x liefert.

* Man kann übrigens bemerken, dass an Stelle der Gleichungen (7) noch etwas einfachere treten, wenn man anstatt p als Parameter das Integral über den Contingenzwinkel $\sqrt{-1}\,dp:R$ des sphärischen Bildes einführt.

Aber die skizzierte Methode steht hinter unserer ersten zurück, und auch bei Erhaltung ihres Grundgedankens lässt sie sich durch eine einfachere ersetzen.

Der Vortheil, den die Einführung des beweglichen orthogonalen Trieders bietet, beruht zum Theil darauf, dass alle solchen Trieder zu einander congruent sind. Der gleichen Eigenschaft erfreuen sich aber auch Trieder, und zugehörige Tripel gewisser Vectoren a, b, c, die den folgenden Gleichungen genügen:

$$(11)\qquad \begin{aligned} &(a|a) = 0, \quad (b|b) = -1, \quad (c|c) = 0,\\ &(b|c) = 0, \quad (c|a) = -1, \quad (a|b) = 0,\\ &(abc) = 1, \end{aligned}$$

und für die dann weiter auch die Gleichungen gelten:

$$(12)\qquad (bc\omega) = -(c|\omega), \qquad (ca\omega) = -(b|\omega), \qquad (ab\omega) = -(a|\omega).$$

Ein derartiges Tripel von Vectoren, deren zwei die Länge Null haben, setzen wir jetzt an Stelle der drei Einheitsvectoren α, β, γ, nämlich dieses:

$$(13)\qquad \begin{aligned} &a \equiv y \equiv x_p', \qquad b \equiv y' \equiv x_p'',\\ &c \equiv \tfrac{1}{2}\phi\cdot y - y'' \equiv \tfrac{1}{2}\phi\cdot x' - x'''. \end{aligned}$$

Die Formeln zeigen, dass dieses neue System linear-unabhängiger Vectoren, abweichend von dem vorigen, an allen regulären Stellen der betrachteten Minimalcurve existiert, und dass also auch die Minimalcurven 3. Ordnung und ihre sphärischen Bilder hier nicht mehr ausgeschlossen zu werden brauchen. An Stelle der Frenet'schen Gleichungen (9) treten jetzt andere, die insofern einen noch etwas einfacheren Bau haben, als in ihnen die zur Bestimmung von R dienende Quadratwurzel und der Differentialquotient der Function ϕ nicht vorkommen:

$$(14)\qquad \frac{da}{dp} \equiv b, \qquad \frac{db}{dp} \equiv \tfrac{1}{2}\phi\cdot a - c, \qquad \frac{dc}{dp} \equiv -\tfrac{1}{2}\phi\cdot b.$$

Bilden wir nun das Product $(abc)\cdot(\omega|\omega)$, so erhalten wir

$$(\omega|\omega) \equiv -2(a|\omega)(c|\omega) - (b|\omega)^2;$$

es genügen also je drei gleichnamige Coordinaten a_κ, b_κ, c_κ der Vectoren a, b, c der Gleichung

$$2a_\kappa c_\kappa + b_\kappa^2 = -1 \qquad (\kappa = 1, 2, 3).$$

Diese Gleichung wird identisch befriedigt, wenn wir für die drei genannten Coordinaten der Reihe nach Ausdrücke der Form

$$-i\,\frac{1}{\mu_\kappa - \nu_\kappa}, \qquad i\,\frac{\mu_\kappa + \nu_\kappa}{\mu_\kappa - \nu_\kappa}, \qquad i\,\frac{2\mu_\kappa \nu_\kappa}{\mu_\kappa - \nu_\kappa}$$

einsetzen. Die Gleichungen (14) gehen dann über in die Gleichungen

$$(15) \qquad \frac{d\mu}{dp} \equiv -\frac{\phi}{4} + \mu^2, \qquad \frac{d\nu}{dp} \equiv -\frac{\phi}{4} + \nu^2;$$

μ und ν sind also in jedem der drei Fälle ($\kappa = 1, 2, 3$) Lösungen eben der Riccati'schen Gleichung (§ 1, Nr. 9), *die wir zuvor auf anderem Wege gewonnen hatten.* Man kann daher, wie zuvor, nach vollständiger Lösung der Gleichung (9) in § 1 die Coordinaten des gesuchten Vectors $y \equiv x'$ unmittelbar hinschreiben. —

Aus unserer Darlegung geht hervor, dass in dem Bereiche ($\phi \neq 0$), in dem die Gleichungen (§ 1, Nr. 9) *und* (§ 2, Nr. 10) *zugleich Geltung haben, sie mit einander äquivalent sein müssen.* In der That lassen sie sich auch in einander transformieren.

Zunächst wird der Zusammenhang zwischen den Vectorentripeln a, b, c und α, β, γ gefunden:

$$(16) \quad \alpha \equiv -\sqrt{-1}\, b, \qquad \beta \equiv -R\left\{\frac{a}{2R^2} - c\right\}, \qquad \gamma \equiv \sqrt{-1}\, R\left\{\frac{a}{2R^2} + c\right\}.$$

Setzen wir zum Beispiel $\sqrt{-1} = i$, so ergiebt sich der Zusammenhang zwischen den mit σ, τ und μ, ν bezeichneten Grössen in jedem Falle ($\kappa = 1, 2, 3$) in der Form

$$(17) \qquad \sigma_\kappa \equiv \frac{2R\mu_\kappa - 1}{2R\mu_\kappa + 1}, \qquad \tau_\kappa \equiv \frac{2R\nu_\kappa - 1}{2R\nu_\kappa + 1}.$$

Daher wird jeder Lösung ω der Gleichung

$$\frac{d\omega}{dp} \equiv -\frac{\phi}{4} + \omega^2 \qquad \left\{\phi \equiv \frac{1}{R^2}\right\}$$

eine Lösung Ω der Gleichung

$$\frac{d\Omega}{dp} \equiv \frac{1}{2R}\left\{\frac{dR}{dp} + 2\Omega - \frac{dR}{dp}\Omega^2\right\}$$

zugeordnet, und umgekehrt, durch die Substitutionen

$$(18) \qquad \Omega \equiv \frac{2R\omega - 1}{2R\omega + 1}, \qquad \omega \equiv \frac{1}{2R}\,\frac{1 + \Omega}{1 - \Omega}.$$

Aber dieser leicht zu verificierende Satz hat zur selbstverständlichen Voraussetzung, dass ϕ nicht verschwindet. Die durch ω und Ω vermittelten Lösungen des Problems der natürlichen Gleichungen dürfen daher, wenn man genau sein will, *nicht* schlechthin "äquivalent" genannt werden.

§ 3. *Die regulären Curven.*

Das Problem der Bestimmung einer regulären Curve aus ihren natürlichen Gleichungen

$$(1) \qquad \Phi \equiv \Phi(s), \qquad \Psi \equiv \Psi(s) \qquad (\Phi \not\equiv 0)$$

lässt sich auf die in § 1 behandelte Aufgabe zurückführen.

Wie wir wissen, gehören zu jedem Punkte x allgemeiner Lage (A. C., S. 20) der Curve (x) zwei Punkte, von denen aus der zugehörige Krümmungskreis durch Minimalkegel projiciert wird, die beiden *Scheitel* des Punktes x (A. C., S. 47, 48). Nennen wir diese Punkte $x + \xi$ und $x + \eta$, und denken wir uns die Vectoren ξ, η mit ihren Anfangspunkten am Anfangspunkte der Coordinaten angeheftet, so erhalten wir zwei neue auf dem zugehörigen Minimalkegel gelegene Curven $\xi \equiv \xi(s)$, $\eta \equiv \eta(s)$. Diese (die in ihrer ganzen Erstreckung betrachtet, nicht nothwendig von einander verschieden sind) wollen wir die beiden *Begleiter* der regulären Curve nennen. Es besteht nun der Satz:

Sind die natürlichen Gleichungen einer regulären Curve bekannt, so lässt sich die natürliche Gleichung eines jeden ihrer Begleiter explicite angeben. Nach Integration einer dieser beiden Gleichungen findet man die Curve selbst durch Quadratur über ein Vectordifferential.

Wir construieren zunächst, unter Benutzung des Bogens als Parameter, das die reguläre Curve begleitende System von drei Einheitsvectoren (A. C., S. 21, Nr. 6):

$$(2) \qquad \begin{gathered} (\alpha|\omega) \equiv (x'|\omega) \equiv (y|\omega), \qquad (\beta|\omega) \equiv R(x''|\omega) \equiv R(y'|\omega), \\ (\gamma|\omega) \equiv R(x'x''\omega) \equiv R(yy'\omega). \end{gathered}$$

Dann gelten die *Frenet-Serret'schen Gleichungen*,

$$(3) \qquad \frac{d\alpha}{ds} \equiv \frac{\beta}{R}, \qquad \frac{d\beta}{ds} \equiv -\frac{\alpha}{R} + \frac{\gamma}{T}, \qquad \frac{d\gamma}{ds} \equiv -\frac{\beta}{T}.$$

Die beiden Begleiter werden jetzt gegeben durch die Formeln

$$(4) \qquad \xi \equiv R(\beta + i\gamma), \qquad \eta \equiv R(\beta - i\gamma).$$

Hieraus folgt, wenn

$$(5) \qquad \begin{gathered} M \equiv \frac{d\,lg R}{ds} - \frac{i}{T} \equiv -\frac{1}{\Phi}\left\{\frac{1}{2}\frac{d\Phi}{ds} + i\Psi\right\}, \\ N \equiv \frac{d\,lg R}{ds} + \frac{i}{T} \equiv -\frac{1}{\Phi}\left\{\frac{1}{2}\frac{d\Phi}{ds} - i\Psi\right\}, \end{gathered}$$

gesetzt wird,

$$(6) \qquad \frac{d\xi}{ds} \equiv -\alpha + M\xi, \qquad \frac{d\eta}{ds} \equiv -\alpha + N\eta,$$

und

$$\frac{d^2\xi}{ds^2} \equiv -M\cdot\alpha - \frac{1}{R}\cdot\beta + \left\{M^2 + \frac{dM}{ds}\right\}\cdot\xi,$$

$$\frac{d^2\eta}{ds^2} \equiv -N\cdot\alpha - \frac{1}{R}\cdot\beta + \left\{N^2 + \frac{dN}{ds}\right\}\cdot\eta;$$

(7) $$(\xi\xi'\omega) \equiv -i(\xi|\omega), \qquad (\eta\eta'\omega) \equiv i(\eta|\omega);$$

(8) $$(\xi'|\xi') \equiv 1, \qquad (\xi\xi'\xi'') \equiv i,$$
$$(\eta'|\eta') \equiv 1, \qquad (\eta\eta'\eta'') \equiv -i;$$

also, wenn die natürlichen Parameter der beiden Begleiter p und q gennant werden,

$$\frac{dp}{ds} \equiv \frac{(\xi\xi'\xi'')}{(\xi'|\xi')} \equiv i, \qquad \frac{dq}{ds} \equiv \frac{(\eta\eta'\eta'')}{(\eta'|\eta')} \equiv -i \text{ (A. C., Nr. 35)},$$

(9) $$p \equiv is\,(+\text{ const.}), \qquad q \equiv -is\,(+\text{ const.}).$$

Nennen wir ferner $\phi(s)$ und $\psi(s)$ die quadrierten Krümmungsmaasse der beiden begleitenden Curven, so folgt

(10) $$\phi(s) \equiv (\xi''|\xi'')_s \equiv \Phi - 2\frac{dM}{ds} - M^2,$$
$$\psi(s) \equiv (\eta''|\eta'')_s \equiv \Phi - 2\frac{dN}{ds} - N^2.$$

Mithin erhalten wir nach Lösung der beiden Schwarz'schen Gleichungen

(11) $$\{\sigma, s\} \equiv \tfrac{1}{2}\phi(s), \qquad \{\tau, s\} \equiv \tfrac{1}{2}\psi(s)$$

(Vgl. Nr. 9 und § 1, Nr. 8) die beiden Begleiter:

(12) $$\xi_1 \equiv \frac{1-\sigma^2}{2\sigma'}, \qquad \eta_1 \equiv -\frac{1-\tau^2}{2\tau'},$$
$$\xi_2 \equiv i\frac{1+\sigma^2}{2\sigma'}, \qquad \eta_2 \equiv -i\frac{1+\tau^2}{2\tau'},$$
$$\xi_3 \equiv -\frac{2\sigma}{2\sigma'}, \qquad \eta_3 \equiv \frac{2\tau}{2\tau'}.$$

(Die Differentiationen beziehen sich auf s, nicht auf die natürlichen Parameter p, q). Die Formeln Nr. (6) zeigen nunmehr, dass man von den beiden Schwarz'schen Gleichungen (11) nur eine zu lösen nöthig hat, um den Vector α zu finden. Schliesslich erhält man durch Quadratur über α den Punkt x:

(13) $$-\xi + \int M\xi\,ds \equiv x \equiv -\eta + \int N\eta\,ds.$$

Hiermit ist die Aufgabe vollständig gelöst. Die beiden Vectorintegrale liefern offenbar die zur gesuchten Curve gehörigen *Minimalevoluten*, oder, falls diese nicht existieren, die sie vertretenden Punkte (A. C., S. 48).

Angemerkt zu werden verdient vielleicht noch der Ausdruck

$$\mathfrak{R} = R \cdot T \cdot \sqrt{MN} \tag{14}$$

für den Radius der Schmiegungskugel.

§ 4. *Andere Ableitung des in § 3 gefundenen Resultates. Erläuterungen.*

Da die Minimalvectoren ξ, η zu dem Einheitsvector α senkrecht sind, so können die Parameter σ, τ nicht wesentlich verschieden sein von den *symmetrischen Coordinaten des Punktes α der Einheitskugel.* Überdies zeigt der in § 1 ausgeführte Grenzübergang, dass beide Arten von Grössen vollkommen identisch sind. Die Umkehrung dieses Gedankenganges muss offenbar zu einer zweiten Herleitung der in § 3 erhaltenen Formeln führen, und auch diese scheint uns Interesse zu bieten.

Wir befriedigen also jetzt die Gleichung

$$y_1^2 + y_2^2 + y_3^2 = 1, \tag{1}$$

in der wir die Coordinaten y_κ mit denen des Vectors α identifizieren ($y \equiv \alpha$) identisch durch die Substitutionen

$$\alpha_1 \equiv \frac{1 - \sigma\tau}{\sigma - \tau}, \qquad \alpha_2 \equiv i\frac{1 + \sigma\tau}{\sigma - \tau}, \qquad \alpha_3 \equiv -\frac{\sigma + \tau}{\sigma - \tau}, \tag{2}$$

wobei nur die Punkte von zwei geradlinigen Erzeugenden der Einheitskugel ausgeschlossen bleiben (§ 1).

Bei Berechnung der Coordinaten des zweiten Einheitsvectors $\beta \equiv R\, d\alpha/ds$ ist zu berücksichtigen, dass sich

$$\left(\frac{ds}{R}\right)^2 \equiv \frac{4 d\sigma\, d\tau}{(\sigma - \tau)^2}$$

ergiebt; es folgt daher

$$\beta_1 \equiv \frac{1}{2R}\left\{\begin{matrix} \dfrac{1-\sigma^2}{2\sigma'} - \\ -\dfrac{1-\tau^2}{2\tau'} \end{matrix}\right\}, \quad \beta_2 \equiv \frac{1}{2R}\left\{\begin{matrix} i\dfrac{1+\sigma^2}{2\sigma'} - \\ -i\dfrac{1+\tau^2}{2\tau'} \end{matrix}\right\}, \quad \beta_3 \equiv \frac{1}{2R}\left\{\begin{matrix} -\dfrac{2\sigma}{2\sigma'} + \\ +\dfrac{2\tau}{2\tau'} \end{matrix}\right\}, \tag{3}$$

$$\gamma_1 \equiv \frac{1}{2R}\left\{\begin{matrix} -i\dfrac{1-\sigma^2}{2\sigma'} - \\ -i\dfrac{1-\tau^2}{2\tau'} \end{matrix}\right\}, \quad \gamma_2 \equiv \frac{1}{2R}\left\{\begin{matrix} \dfrac{1+\sigma^2}{2\sigma'} + \\ +\dfrac{1+\tau^2}{2\tau'} \end{matrix}\right\}, \quad \gamma_3 \equiv \frac{1}{2R}\left\{\begin{matrix} i\dfrac{2\sigma}{2\sigma'} + \\ +i\dfrac{2\tau}{2\tau'} \end{matrix}\right\}. \tag{4}$$

Die Formeln (3) und (4) zeigen, dass die Verbindungen

$$(5) \qquad \xi \equiv R(\beta + i\gamma), \qquad \eta \equiv R(\beta - i\gamma)$$

nur von je einer der Grössen σ, τ abhängen: man erhält eben die Formeln (12) des vorigen Paragraphen. Man wird also die Functionen σ und τ einzeln zu bestimmen suchen. Damit kommt man, nach § 1, wieder zu den Formeln (10) und (11) des § 3. Da auch die Gleichungen (6) in § 3 zur Verfügung stehen, so hat man auch auf diese Art die vollständige Lösung des Problems.

Man kann auch so zu Werke gehen, dass man nach Bestimmung etwa des Vectors ξ zunächst den Vector η sucht. Der Vector α, auf den es ankommt, ergiebt sich dann aus der evidenten Gleichung

$$(6) \qquad (\xi\eta\omega) \equiv -2iR^2 \cdot (\alpha|\omega),$$

die, im vorigen § angewendet, unmittelbar die Gleichungen (1) geliefert haben würde. Der Vector η aber ergiebt sich, wenn ξ bekannt ist, ohne Weiteres, da offenbar

$$(7) \qquad \frac{\xi + \eta}{2} \equiv R^2 \frac{d\alpha}{ds} \equiv \frac{1}{\Phi}\frac{d}{ds}\left\{ M\xi - \frac{d\xi}{ds} \right\} \equiv \frac{1}{\Phi}\frac{d}{ds}\left\{ N\eta - \frac{d\eta}{ds} \right\}$$

ist.

Ferner muss, wenn eine Lösung $\sigma(s)$ oder $\tau(s)$ einer der beiden Schwarz'schen Gleichungen bekannt ist, die entsprechende Lösung $\tau(s)$ oder $\sigma(s)$ der anderen eindeutig sich bestimmen lassen. Man erhält, wenn man in irgend eine der Gleichungen (2) die Werthe der Coordinaten α_κ aus der Gleichung (6) in § 3, und dann die Werthe der Coordinaten ξ_κ oder η_κ aus (12) einträgt, unmittelbar die gesuchten Formeln:

$$(8) \qquad \tau \equiv \sigma - \frac{2\sigma'\sigma'}{M\sigma' + \sigma''}, \qquad \sigma \equiv \tau - \frac{2\tau'\tau'}{N\tau' + \tau''}.$$

Die Lösungen

$$(9) \qquad \mu \equiv \frac{\sigma''}{2\sigma'}, \qquad \nu \equiv \frac{\tau''}{2\tau'}$$

der beiden Riccati'schen Gleichungen

$$(10) \qquad \frac{d\mu}{ds} \equiv \frac{\phi}{4} + \mu^2, \qquad \frac{d\nu}{ds} \equiv \frac{\psi}{4} + \nu^2,$$

die den betrachteten Schwarz'schen Gleichungen zugeordnet sind, hängen durch eine lineare Substitution zusammen, die ebenfalls leicht zu berechnen ist. Man hat nämlich, nach (8):

$$(11) \qquad M \equiv \frac{2\sigma'}{\sigma - \tau} - \frac{\sigma''}{\sigma'}, \qquad N \equiv \frac{2\tau'}{\tau - \sigma} - \frac{\tau''}{\tau'}.$$

Da ausserdem

$$4\sigma'\tau' \equiv \Phi \cdot (\sigma - \tau)^2 \qquad \text{(Nr. 2)},$$

so folgt

$$(12) \qquad \left(M + \frac{\sigma''}{\sigma'}\right)\left(N + \frac{\tau''}{\tau'}\right) + \Phi \equiv 0,$$

oder

$$(13) \qquad 4\mu\nu + 2N\mu + 2M\nu + \frac{1}{\Phi^2}\left\{\Phi^3 + \Psi^2 + \frac{1}{4}\left(\frac{d\Phi}{ds}\right)^2\right\} \equiv 0.$$

Die Discriminante dieser Gleichung ist, wie es sein muss, nicht identisch gleich Null.

Durch das Gesagte ist auch schon die Aufgabe gelöst, die Functionen Φ, Ψ zu finden, wenn die Functionen ϕ, ψ gegeben sind. Nach vollständiger Lösung der beiden Gleichungen (10) oder der entsprechenden Schwarz'schen Gleichungen kennt man σ, τ, M, N, und man hat

$$(14) \qquad \Phi \equiv e^{-\int (M+N)ds} \equiv \frac{4\sigma'\tau'}{(\sigma-\tau)^2}, \qquad \Psi \equiv \frac{i}{2}\Phi\cdot(M-N).$$

Ungleich der Function Φ können dabei die Functionen ϕ, ψ ganz beliebig (mit gemeinsamem Existenzbereich) angenommen werden. Nur können selbstverständlich, im Falle sie identisch sind, nicht auch identische Lösungen σ, τ der dann einzigen Schwarz'schen Gleichung benutzt werden.

Die zu gegebenen Functionen ϕ, ψ gehörigen Curven sind natürlich nicht alle unter einander congruent. Vielmehr stehen je zwei unter ihnen in der Beziehung, dass ihre Tangentenindicatrices zu einander eigentlich-kreisverwandt sind. Congruent sind alle die Curven, die erhalten werden, wenn man die Grössen σ, τ derselben linearen Transformation unterwirft (§ 1).

Die Gleichung $\phi \equiv \psi$ kennzeichnet die Familie der *regulären Schraubenlinien*, der Curven nämlich, für die $R:T$ einen constanten Werth hat, also die krummen geodätischen Linien auf unebenen Cylinderflächen, oder die Curven, deren Tangentenindicatrices (reguläre oder parabolische) Kreise sind. Dass diese Curven durch Quadraturen bestimmt werden können, ergiebt sich hier aus der Thatsache, dass aus jeder Lösung der zugehörigen Riccati'schen Gleichung durch die Transformation (13) eine zweite abgeleitet werden kann. (R. Gl., § 6). Doch wird man hier, wie auch in anderen Fällen, näher liegende und elementarere Methoden vorziehen (vgl. § 8). Die Gleichung $M - N \equiv$ const. kennzeichnet die *Curven constanter Torsion*, insbesondere die Gleichung $M - N \equiv 0$ *die regulären ebenen Curven* — die zugleich Schraubenlinien sind —; die Gleichung $M + N \equiv$ const. ist charakteristisch für *die Curven constanter Krümmung*.

Ist die betrachtete reguläre Curve *relle*, so sind ihre Begleiter conjugiert-complex, und die Gleichungen $\phi \equiv \phi(s)$, $\bar{\psi} \equiv \bar{\psi}(\bar{s})$ drücken dieselbe analytische Abhängigkeit aus: Dieses ist die Bedingung dafür, dass unter den zu ϕ, ψ gehörigen regulären Curven sich reelle finden. Ist überdies ein *reeller*

Zug vorhanden, und ist $\{ s \equiv \bar{s} \}$ ein entsprechendes Werthgebiet des reellen Bogens, so müssen die Functionen ϕ und ψ, nach Potenzen von $s - s_0 \equiv \bar{s} - \bar{s}_0$ entwickelt, conjugiert-complexe Coefficienten haben.

§ 5. *Differentialinvarianten der regulären Curven.*

Wie im Falle der Minimalcurven, so hängt auch im Falle der regulären Curven das Problem der natürlichen Gleichungen mit der Theorie gewisser linearer Differentialgleichungen zusammen. Eine solche von der dritten Ordnung wird nämlich geliefert durch die lineare Relation, die immer zwischen den Vectoren x', x'', $x_{,}''$, x'''' stattfinden muss. Um ihre Coefficienten zu bestimmen, hat man die auf den Bogen bezogenen Semiinvarianten (234), (341), (412), (123), — die eben dadurch in Invarianten übergegangen sind — auszudrücken durch die Functionen Φ, Ψ und ihre Differentialquotienten. So entsteht, ähnlich wie in § 2, eine allgemeinere Frage, die durch den folgenden Lehrsatz beantwortet wird:

Benutzt man bei einer regulären Curve den Bogen als Parameter, so gehen dadurch die Semiinvarianten $(i|\kappa)$ *und* $(i\kappa l)$ *für* i, κ, $l \leqq n$ *in rationale Functionen von*

$$\Phi,\ \Phi',\ \Phi'',\ \ldots,\ \Phi^{(n-2)};\ \Psi,\ \Psi',\ \Psi'',\ \ldots,\ \Psi^{(n-3)}$$

über und zwar treten in den Nennern dieser Ausdrücke lediglich Potenzen der absoluten Invariante Φ *auf.*

Mit ebenso beschaffenen Coefficienten lassen sich alle Semicovarianten $(\kappa|\omega)$ *und* $(i\kappa\omega)$ — *für* $i < \kappa \leqq n$ — *linear ausdrücken durch*

$$(x'|\omega), \qquad (x''|\omega), \qquad (x'x''\omega).$$

Wir gehen wieder zunächst bis zum Werthe $n = 4$ und erhalten dann, wenn wir neben den Zeichen

$$\Phi \equiv (2|2)_s, \qquad \Psi \equiv (123)_s$$

noch die Abkürzung

$$\text{(1)} \qquad X \equiv \frac{1}{\Phi}\left\{ \Phi^3 + \Psi^2 + \frac{1}{4}\left(\frac{d\Phi}{ds}\right)^2 \right\}$$

gebrauchen, die folgende Tabelle, die die für die Anwendungen wichtigsten Formeln umfasst:

$$\text{(2)} \qquad \begin{array}{lll} (1|1) \equiv 1, & *, & *, \\ (1|2) \equiv 0, & (2|2) \equiv \Phi, & *, \\ (1|3) \equiv -\Phi, & (2|3) \equiv \frac{1}{2}\Phi', & (3|3) \equiv X, \\ (1|4) \equiv -\frac{3}{2}\Phi', & (2|4) \equiv \frac{1}{2}\Phi'' - X, & (3|4) \equiv \frac{1}{2}X', \end{array}$$

$$(4|4) \equiv \frac{1}{\Phi}\left[\tfrac{9}{4}\Phi\Phi'\Phi' + \Psi'\Psi' + (\tfrac{1}{2}\Phi'' - X)^2\right],$$

. . . . . .

$$(2)\qquad \begin{aligned} &(123) \equiv \Psi, \qquad (134) \equiv \frac{1}{\Phi}\left\{\frac{\Phi'\Psi' - \Psi\Phi''}{2} + \Psi X\right\}, \\ &(124) \equiv \Psi', \qquad (234) \equiv \tfrac{1}{2}\{2\Phi\Psi' - 3\Psi\Phi'\}. \end{aligned}$$

Wir ersetzen jetzt $n(\geqq 4)$ durch $n-1$, und nehmen für diesen Werth den Satz als richtig an. Wir erhalten dann durch Differentiation alle Semiinvarianten der Form $(i|n)$ für $i < n$, also alle bilinearen Semiinvarianten mit Ausnahme von $(n|n)$. Ebenso entstehen durch Differentiation alle Semiinvarianten $(i\kappa n)$, wo $i < \kappa < n$, mit Ausnahme derer, für die $\kappa = n-1$ ist. Durch Bildung der Producte $(1|1)(i, n-1, n)$ für $i = 2, 3, \cdots, n-2$ erkennt man sodann, dass alle diese Grössen sich als ganze Functionen der schon berechneten Ausdrücke und der einen Semiinvariante $(1, n-1, n)$ darstellen lassen. Die gleiche Eigenschaft aber hat das Product dieses letzten Differentialausdrucks mit $\Phi \equiv (2|2)$. Die noch fehlende Semiinvariante $(n|n)$ erhält man schliesslich, gleichfalls in der behaupteten Form, durch Bildung des Quadrates $(12n)^2$.

Der zweite Theil des aufgestellten Satzes ergiebt sich am einfachsten durch Bildung des Productes $(12n)(12\omega)$:

$$(3)\qquad \Phi\cdot(n|\omega) \equiv \Phi\cdot(1|n)\cdot(1|\omega) + (2|n)\cdot(2|\omega) + (12n)\cdot(12\omega).$$

$$(4)\qquad \Phi(mn\omega) \equiv \Phi\cdot(1mn)\cdot(1|\omega) + (2mn)\cdot(2|\omega) + (12|mn)\cdot(12\omega). \quad (m < n).$$

Wie die Herleitung zeigt, bilden die gefundenen Grössen ein in gewissem Sinne *vollständiges System absoluter* — in ihrer Gesammtheit nicht rationaler — *Differentialinvarianten und Differentialcovarianten* der betrachteten regulären Curve.

Zum Beispiel wird die Bedingung dafür, dass eine reguläre Curve im weiteren Sinne des Wortes sphärisch ist (uneben-sphärisch oder eben), allgemein ausgedrückt durch die Gleichung

$$2\Phi\Phi_s'\cdot\Psi_s' + (\Phi_s'\Phi_s' - 2\Phi\Phi_s'' + 4\Psi^2)\Psi \equiv 0.$$

(Vgl. A. C., Nr. 60).

Auf dem hier vorgezeichneten Wege wird man, unter Anderem, auch die Bedingung dafür ermitteln können, dass die Tangentenindicatrix einer regulären Curve ein sphärischer Kegelschnitt oder wenigstens ein Stück einer solchen Curve ist. Die Differentialgleichung der Kegelschnitte ist in der hierfür nöthigen allgemeinen Form vom Verfasser aufgestellt worden.* Man hat darnach die in der Gleichung

$$\begin{aligned} 9\cdot(123)^2\cdot\{(126) + 10\cdot(234) + 5\cdot(135)\} - \\ - 45\cdot(123)\cdot(124)\cdot\{2\cdot(134) + (125)\} + 40\cdot(124)^3 \equiv 0 \end{aligned}$$

vorkommenden Differentialinvarianten durch Φ, Ψ und die Differentialquotienten

* Leipziger Berichte, Bd. 53, (1901) S. 349.

dieser Functionen (bis zu denen dritter Ordnung) auszudrücken.

Die auszuführende Rechnung ist natürlich umständlich. Im nächsten § werden wir deshalb die gesuchte Bedingung auf einem anderen Wege, und zwar mit ganz kurzer Rechnung ableiten. $(\Theta \equiv 0)$.

§ 6. *Über gewisse lineare Differentialgleichungen 3. Ordnung.*

Die Formel (3) in § 5 zeigt, dass im Falle einer *ebenen* regulären Curve der Differentialvector $y \equiv x'$ der linearen Differentialgleichung 2. Ordnung

$$y'' - \frac{1}{2}\frac{\Phi'}{\Phi}y' + \Phi y \equiv 0$$

genügt, deren allgemeine Lösung in der Form

$$c_1 \cos \int \sqrt{\Phi}\, ds + c_2 \sin \int \sqrt{\Phi}\, ds$$

enthalten ist. Wir schliessen diesen Fall von der ferneren Betrachtung aus, nehmen also an, dass die niederste lineare Differentialgleichung, der der Vector y genügt, von der dritten Ordnung ist,

$$(1) \qquad y''' + 3Ay'' + 3By' + Cy \equiv 0 \qquad (\Psi \not\equiv 0).$$

Die Werthe der Coefficienten A, B, C können dann den Formeln des vorigen § entnommen werden:

$$(2) \qquad \begin{aligned} A &\equiv -\frac{1}{3}\frac{(124)}{(123)} \equiv -\frac{1}{3}\frac{\Psi'}{\Psi}, \\ B &\equiv \frac{1}{3}\frac{(134)}{(123)} \equiv \frac{1}{3\Phi}\left\{ X + \frac{\Phi'}{2}\frac{\Psi'}{\Psi} - \frac{\Phi''}{2} \right\}, \\ C &\equiv -\frac{(234)}{(123)} \equiv -\frac{1}{2}\left\{ 2\Phi . \frac{\Psi'}{\Psi} - 3\Phi' \right\}, \\ X &\equiv \frac{1}{\Phi}\left\{ \Phi^3 + \Psi^2 + \frac{1}{4}\left(\frac{d\Phi}{ds}\right)^2 \right\}. \end{aligned}$$

Wie die Ableitung zeigt, ist in dieser Form jede beliebige lineare homogene Differentialgleichung 3. Ordnung enthalten, bei der (mindestens) eine aus drei linear-unabhängigen Lösungen gebildete quadratische Form von nicht verschwindender Discriminante einen von Null verschiedenen constanten Werth hat.

Man wird dann nämlich drei solche Lösungen y_1, y_2, y_3 derart auswählen können, dass

$$(3) \qquad y_1^2 + y_2^2 + y_3^2 \equiv 1$$

wird: Diese Functionen y_1, y_2, y_3 der unabhängigen Veränderlichen s könne

sonach als die Coordinaten der Tangentenindicatrix einer regulären *unebenen* Curve aufgefasst werden.

Es entsteht nunmehr die Frage, durch was für Besonderheiten die Coefficienten A, B, C der hier auftretenden linearen Differentialgleichungen 3. Ordnung gekennzeichnet sind; wie man diese Coefficienten in die Form (2) setzen, und wie man schliesslich die Auflösung einer gegebenen Gleichung der Art bewirken kann.

Wir nehmen also an, dass die Gleichungen (2) bestehen. Aus den Gleichungen für A und C folgt dann

$$\Psi' \equiv -3A\Psi, \qquad \Phi' \equiv \tfrac{2}{3}C - 2A\Phi,$$

$$\Phi'' \equiv \tfrac{2}{3}C' - \tfrac{4}{3}AC + 2\{2A^2 - A'\}\Phi;$$

durch Substitution dieser Werthe in die Gleichung für B ergiebt sich

$$C^2 - 3\{C' + 3AC\}\Phi + 9\{A' + 2A^2 - 3B\}\Phi^2 + 9\{\Phi^3 + \Psi^2\} \equiv 0,$$

und hieraus, durch nochmalige Differentiation und Elimination von Φ', Ψ':

$$2AC^2 + \{C'' + AC' - A'C + 12BC - 14A^2C\}\Phi$$
$$- 3\{A'' - 3B' + 2C + 12AB - 8A^3\}\Phi^2 + 18A\{\Phi^3 + \Psi^2\} \equiv 0.$$

Aus den beiden letzten Gleichungen kann nunmehr die Summe

$$C^2 + 9\{\Phi^3 + \Psi^2\}$$

eliminiert werden.

Wir setzen zur Abkürzung

$$F \equiv A' + 2A^2 - 3B, \qquad G \equiv C' + 3AC, \tag{4}$$

so dass obige Gleichung die Form

$$C^2 - 3G\cdot\Phi + 9F\Phi^2 + 9\{\Phi^3 + \Psi^2\} \equiv 0 \tag{5}$$

annimmt, ferner

$$\Theta \equiv A'' + 6AA' + 4A^3 - 3B' - 6AB + 2C,$$

$$\mathrm{H} \equiv C'' + 7A\cdot C' + \{-A' + 4A^2 + 12B\}\cdot C,$$

oder

$$\Theta \equiv \frac{dF}{ds} + 2AF + 2C, \tag{6}$$

$$\mathrm{H} \equiv \frac{dG}{ds} + 4AG - 4CF, \tag{7}$$

und erhalten, als Eliminationsresultat, die Gleichung:

$$3\Theta\Phi - \mathrm{H} \equiv 0. \tag{8}$$

Aus der abgeleiteten linearen Differentialgleichung für Φ ergiebt sich schliess-

lich eine Gleichung fur Θ und H:

$$\Theta \mathrm{H}' - \mathrm{H}\Theta' \equiv 2\Theta \{ C\Theta - A\mathrm{H} \}. \tag{9}$$

Dieser, die in den Differentialquotienten von A, B, C bis zur dritten Ordnung ansteigt, müssen also die Functionen A, B, C jedenfalls genügen. Das ist indessen nicht hinreichend. Damit der eingeschlagene Weg auch in umgekehrter Richtung gangbar sei, muss sich vielmehr ein nicht identisch verschwindender Werth von Φ, und hierauf — durch Lösung einer reinen quadratischen Gleichung — ein ebenfalls nicht identisch verschwindender Werth von Ψ berechnen lassen. Es muss also, wie die Gleichung (8) zeigt,

$$\textit{entweder} \quad \Theta \not\equiv 0, \mathrm{H} \not\equiv 0 \quad \textit{oder} \quad \Theta \equiv 0, \mathrm{H} \equiv 0$$

sein.

Im ersten Fall kann man Φ bestimmen: Es kommt also nur noch die weitere Einschränkung hinzu, dass der dann zu berechnende Werth von Ψ nicht identisch gleich Null wird.

Machen wir die zweite Annahme, so folgt aus $\Theta \equiv 0$, dass eine quadratische Form von y_1, y_2, y_3 identisch verschwindet,* und dass folglich *alle* quadratischen Formen von y_1, y_2, y_3, die einem gewissen Büschel angehören, constante Werthe haben. Umgekehrt liegt auch, wenn das eintritt, immer der zweite Fall vor. Es kann dann das genannte Büschel das Quadrat einer linearen Form enthalten, deren Werth natürlich ebenfalls constant sein muss. Die Bedingung hierfür ist $C \equiv 0$.

Wir werden hiernach, in der Theorie der hier in Betracht kommenden speciellen linearen Differentialgleichungen drei Hauptfälle unterscheiden, deren einige dann noch weiter zu gliedern sein werden. Auf diese Unterscheidung bezieht sich der folgende Lehrsatz, der durch das Vorgetragene begründet ist:

Wenn zwischen drei passend gewählten linear-unabhängigen Lösungen einer linearen homogenen Differentialgleichung 3. Ordnung eine Gleichung der Form (3) *besteht, so liegen folgende Möglichkeiten vor:*

I. *Es verschwindet keine quadratische Form von y_1, y_2, y_3 identisch. Der Ort des Punktes y ist dann eine sphärische Curve oder Stück einer solchen,*† *die uneben und auch verschieden ist von einem sphärischen Kegelschnitt.*

II. *Es verschwindet eine derartige Form, die vorgelegte Gleichung hat aber keine constante Lösung. Der Ort des Punktes y ist dann ein unebener sphärischer Kegelschnitt, oder Stück eines solchen.*

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*Siehe z. B. PICARD, *Traité d'analyse*, t. III, p. 556, oder R. Gl., Nr. 42.

†Auch in der Theorie der linearen Differentialgleichungen ist es nicht üblich, die Begriffe Curve und Curvenstück zu unterscheiden. Man kann aber zuweilen Aussagen über den ganzen Verlauf, oder nahezu über den ganzen Verlauf einer Curve machen, und wir wünschen, wo dies der Fall ist, es hervortreten zu lassen. So entziehen sich der Parameterdarstellung $x \equiv x(s)$ höchstens isolierte Stellen, während auf der Curve (y) der Parameter s sehr wohl eine natürliche Grenze haben kann. Deshalb sagen wir im Texte "Curve oder Curvenstück," nicht einfach "Curve." Freilich ist auch diese Terminologie noch nicht ganz zufriedenstellend.

III. *Die vorgelegte Gleichung hat eine constante Lösung. Der Ort des Punktes y ist dann ein (regulärer oder parabolischer) Kreis, oder Stück eines solchen Kreises (der jedoch immer von einem Hauptkreis verschieden ist).*

Durch das Vorhergehende schon erledigt ist die erste dieser Annahmen:

Damit der Fall I *eintrete, ist nothwendig und hinreichend, dass die Functionen* Θ, H *der Gleichung* (9) *genügen, aber nicht identisch verschwinden, und dass ausserdem der Ausdruck*

$$C^2\Theta^3 - G\Theta^2\mathrm{H} + F\Theta\mathrm{H}^2 + \tfrac{1}{3}\mathrm{H}^3 \tag{10}$$

nicht identisch gleich Null ist.

Durch Berechnung der Functionen Φ, Ψ (Nr. 8, 5) *lässt sich dann die Integration der vorgelegten Gleichung zurückführen auf die Integration einer Schwarz'schen Differentialgleichung* (§ 3).

Soviel wird sich erreichen lassen, so lange nicht weitere Besonderheiten eintreten.*

§ 7. *Fortsetzung: Lineare Differentialgleichungen 3. Ordnung, die mit der Theorie der elliptischen Functionen zusammenhängen.*

Wir wenden uns jetzt zur Betrachtung des zweiten der in § 6 unterschiedenen Fälle, setzen also nunmehr voraus, dass $\Theta \equiv 0$, $\mathrm{H} \equiv 0$, aber $C \not\equiv 0$ ist. Dann werden sich in dem Büschel quadratischer Formen von y_1, y_2, y_3, die constante Werthe haben, entweder drei Formen von verschwindender Discriminante finden, oder deren zwei, oder eine, und jede von diesen muss einen nicht verschwindenden Werth haben. Die Form, die den Werth Null annimmt, hat eine nicht verschwindende Discriminante. Eine zweite Form von nicht verschwindender Discriminante kann willkürlich ausgewählt werden, und sie liefert dann die Gleichung der Einheitskugel, bei passender Wahl der Coordinaten in der üblichen Form

$$y_1^2 + y_2^2 + y_3^2 = 1. \tag{1}$$

Überdies wird man es erreichen können, dass die hinzutretende zweite Gleichung entweder die Form

$$\kappa_1 y_1^2 + \kappa_2 y_2^2 + \kappa_3 y_3^2 = 0 \qquad (\kappa_1 \neq \kappa_2 \neq \kappa_3 \neq 0) \tag{2a}$$

annimmt, oder zweitens die Form

$$y_2(y_2 + iy_3) = \text{const., besser}$$

$$y_1^2 + y_2^2 + y_3^2 + 2\kappa y_2(y_2 + iy_3) = 0 \qquad (\kappa \neq 0, -1), \tag{2b}$$

oder drittens die Form $y_1(y_2 + iy_3) = \text{const.}$, insbesondere z. B. $= -1$, wofür wir besser schreiben können

$$y_1^2 + y_2^2 + y_3^2 + 2y_1(y_2 + iy_3) = 0. \tag{2c}$$

* Diese Fälle lassen sich bis jetzt nicht erschöpfend angeben. R. Gl., Einleitung.

Das vorgelegte Problem nimmt also nunmehr eine präcisere Form an: Erstens werden Kriterien für diese drei Möglichkeiten zu finden sein, und zweitens Lösungstripel, die je eines der aufgestellten Gleichungspaare befriedigen. Zugleich hat sich ergeben:

Im Falle II *ist der Ort der Punkte* y *entweder eine — vollständige oder unvollständige — sogenannte elliptische Curve 4. Ordnung* (a), *oder eine rationale Curve 4. Ordnung mit Doppelpunkt auf dem absoluten Kegelschnitt* (b), *oder endlich eine Curve 4. Ordnung mit Spitze* (c).

Wir nehmen jetzt an, dass der Fall (a) vorliegt. Dann können wir die Gleichungen (1) und (2a) identisch befriedigen durch Einführung elliptischer Coordinaten:

$$(3)\quad y_1 \equiv \sqrt{\kappa_2 - \kappa_3}\sqrt{z - e_1},\quad y_2 \equiv \sqrt{\kappa_3 - \kappa_1}\sqrt{z - e_2},\quad y_3 \equiv \sqrt{\kappa_1 - \kappa_2}\sqrt{z - e_3},$$

indem wir nämlich drei Constante e_1, e_2, e_3 gemäss den Bedingungen

$$e_1 + e_2 + e_3 = 0,$$

$$(\kappa_2 - \kappa_3)e_1 + (\kappa_3 - \kappa_1)e_2 + (\kappa_1 - \kappa_2)e_3 = -1,$$

$$\kappa_1(\kappa_2 - \kappa_3)e_1 + \kappa_2(\kappa_3 - \kappa_1)e_2 + \kappa_3(\kappa_1 - \kappa_2)e_3 = 0$$

bestimmen:

$$e_1 = \frac{1}{3}\,\frac{\kappa_2(\kappa_3 - \kappa_1) - \kappa_3(\kappa_1 - \kappa_2)}{(\kappa_2 - \kappa_3)(\kappa_3 - \kappa_1)(\kappa_1 - \kappa_2)} \qquad \text{(etc.)}.$$

Setzen wir dann noch

$$(\kappa_2 - \kappa_3)e_2e_3 + (\kappa_3 - \kappa_1)e_3e_1 + (\kappa_1 - \kappa_2)e_1e_2 = e_0,$$

so bestehen die Ungleichungen

$$(4)\qquad e_0 \neq e_1 \neq e_2 \neq e_3,$$

und es lassen sich die Verhältnissgrössen $\kappa_1 : \kappa_2 : \kappa_3$ wieder durch die Verhältnissgrössen $e_0 : e_1 : e_2 : e_3$ ausdrücken:

$$\kappa_1 = -\frac{(e_2 - e_0)(e_3 - e_0)}{(e_2 - e_3)(e_3 - e_1)(e_1 - e_2)} \qquad \text{(etc.)}.$$

Führen wir ferner die Abkürzung

$$\Pi \equiv \sqrt{-1}\cdot\sqrt{(e_0 - e_1)(e_0 - e_2)(e_0 - e_3)}$$

ein, so kann eine Abhängigkeit zwischen dieser Quadratwurzel und den Wurzelgrössen $\sqrt{\kappa_2 - \kappa_3}$, $\sqrt{\kappa_3 - \kappa_1}$, $\sqrt{\kappa_1 - \kappa_2}$ wie folgt erklärt werden:

$$\Pi = (e_2 - e_3)(e_3 - e_1)(e_1 - e_2)\cdot\sqrt{\kappa_2 - \kappa_3}\sqrt{\kappa_3 - \kappa_1}\sqrt{\kappa_1 - \kappa_2}.$$

Nunmehr kann man, von den Gleichungen (3) ausgehend, die Functionen Φ

und Ψ darstellen

$$(5)\qquad \Phi \equiv (y'|y') \equiv -\left\{\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}\right\}^2 \cdot (z-e_0),$$

$$(6)\qquad \Psi \equiv (yy'y'') \equiv -\left\{\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}\right\}^3 \cdot \Pi,$$

ferner aber auch die Coefficienten der linearen Differentialgleichung 3. Ordnung, der die Functionen y_1, y_2, y_3 genügen:

$$(7)\qquad \begin{aligned} A &\equiv -\frac{1}{3}\frac{(yy'y''')}{(yy'y'')} \equiv -\frac{d}{ds}\log\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}, \\ B &\equiv \frac{1}{3}\frac{(yy''y''')}{(yy'y'')} \equiv -\frac{1}{3}\frac{z'''}{z'} + \\ &\quad + \frac{d}{ds}\log\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}\cdot\frac{z''}{z'} + 3\left\{\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}\right\}^3\cdot z, \\ C &\equiv -\frac{(y'y''y''')}{(yy'y'')} \equiv -\frac{3}{2}\left\{\frac{z'}{2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}}\right\}^2\cdot z'. \end{aligned}$$

Diese Gleichungen (7) liefern nun das vollständige System der Bedingungen, denen die Functionen A, B, C zu genügen haben, und sie lassen sich dann auch unschwer nach e_1, e_2, e_3, z auflösen (die durch die weitere Gleichung $e_1 + e_2 + e_3 = 0$ verbunden sind). Am einfachsten geht man wohl so zu Werke, dass man zwischen s und z noch eine zur Lösung des Integrationsproblems übrigens nicht nöthige Hülfsveränderliche u einschiebt — das zu der Curve (y) gehörige Integral 1. Gattung* — vermöge der Substitutionen:

$$(8)\qquad z \equiv \wp(u), \qquad \frac{dz}{du} \equiv \wp'(u) \equiv -2\sqrt{z-e_1}\sqrt{z-e_2}\sqrt{z-e_3}.$$

Die Gleichungen (7), deren System wir sogleich noch etwas ergänzen wollen, nehmen dann, bei Gebrauch der schon eingeführten Abkürzungen

$$(9)\qquad F \equiv A' + 2A^2 - 3B, \qquad G \equiv C' + 3AC$$

eine übersichtlichere Form an:

$$(10)\qquad \begin{aligned} A &\equiv -\frac{u''}{u'}, \qquad F \equiv 3\wp(u)\cdot(u')^2, \\ B &\equiv -\frac{1}{3}\frac{u'''}{u'} + \left(\frac{u''}{u'}\right)^2 - \wp(u)\cdot(u')^2, \\ C &\equiv -\tfrac{3}{2}\wp'(u)\cdot(u')^3, \qquad G \equiv -\tfrac{3}{2}\wp''(u)\cdot(u')^4, \end{aligned}$$

* Wir entnehmen hier der Theorie der elliptischen Functionen lediglich einige übliche *Zeichen.*

oder also

$$(11)\qquad \begin{aligned} \frac{ds}{du} &\equiv -e^{\int A ds}, & \wp(u) &\equiv \tfrac{1}{3}F\cdot e^{2\int A ds}, \\ \wp'(u) &\equiv \tfrac{2}{3}C\cdot e^{3\int A ds}, & \wp''(u) &\equiv -\tfrac{2}{3}G\cdot e^{4\int A ds}. \end{aligned}$$

Hieraus, und aus den in (8) enthaltenen Gleichungen

$$\{\wp'(u)\}^2 \equiv 4\{\wp(u)-e_1\}\{\wp(u)-e_2\}\{\wp(u)-e_3\}$$

oder

$$\{\wp'(u)\}^2 \equiv 4\wp^3(u) - g_2\wp(u) - g_3,$$

$$\wp''(u) \equiv 4\wp^2(u) - \tfrac{1}{2}g_2, \qquad \wp'''(u) \equiv 12\wp(u)\wp'(u)$$

folgen nun wieder die auf anderem Wege schon abgeleiteten Bedingungsgleichungen

$$(12)\qquad \Theta \equiv F' + 2AF + 2C \equiv 0, \qquad \mathrm{H} \equiv G' + 4AG - 4CF \equiv 0.$$

Nehmen wir umgekehrt an, dass neben $C \not\equiv 0$ diese Identitäten bestehen, so erhalten wir aus (10) oder (11) zunächst drei Gleichungen zur Bestimmung von e_1, e_2, e_3, nämlich:

$$(13)\qquad \begin{aligned} e_1 + e_2 + e_3 &= 0 \\ g_2 &\equiv -4\{e_2e_3 + e_3e_1 + e_1e_2\} \equiv \tfrac{4}{3}\{F^2 + G\}e^{4\int A ds} \\ g_3 &\equiv 4e_1e_2e_3 \equiv -\tfrac{4}{27}\{2F^3 + 3FG + 3C^2\}e^{6\int A ds}. \end{aligned}$$

Zufolge (12) werden dann diese Grössen constant. Es liegt aber kein Grund dafür vor, dass e_1, e_2, e_3 von einander verschieden ausfallen sollten. Es ergiebt sich also noch eine weitere, eben für den behandelten Fall (a) charakteristische Bedingung:

$$(14)\qquad \tfrac{1}{16}\{g_2^3 - 27g_3^2\}e^{-12\int A ds} \equiv \tfrac{1}{27}\{4(F^2+G)^3 - (2F^3+3FG+3C^2)^2\} \not\equiv 0.$$

Ist auch diese Voraussetzung erfüllt, so erhält man schliesslich durch Einsetzen der Function

$$(15)\qquad z \equiv \tfrac{1}{3}Fe^{2\int A ds}$$

in die Wurzelgrössen $\sqrt{z - e_\kappa}$ drei linear-unabhängige Lösungen der gegebenen Gleichung. Es ist nämlich, wie aus $\Theta \equiv 0$, d. h. $F' + 2AF \equiv -2C$ und $C \not\equiv 0$ folgt, die aus (15) berechnete Function z nicht constant. Wie dann, nach Annahme einer passend gewählten Constanten e_0, die verlangten Particularlösungen y_1, y_2, y_3 gefunden werden können, wurde schon oben gezeigt.

Aber die in der Formel (15) noch vorkommende Quadratur ist überflüssig. Wir denken uns in dem Integral $\int A ds$ der unteren Grenze einen bestimmten

Werth s_0 beigelegt, und schreiben gleichzeitig die Constante e_0 etwas anders:

$$\int A ds = \int_{s_0}^{s} A ds, \qquad e_0 = e_0^* e^{\int_{s_0}^{s_1} A ds}.$$

Dann leuchtet ein, dass bei Verschiebung von s_0 die Functionen

$$e_0,\ e_1,\ e_2,\ e_3 \text{ und } z$$

alle denselben constanten Factor annehmen, so dass y_1^2, y_2^2, y_3^2 ganz ungeändert bleiben und y_1, y_2, y_3 ebenfalls ungeändert bleiben können.

Die gefundenen Formeln enthalten daher, entsprechend dem Umstande, dass nach Festlegung der Einheitskugel in dem mehrfach genannten Büschel das System der Functionen y_1^2, y_2^2, y_3^2 eindeutig bestimmt sein muss, nur einen wesentlichen Parameter e_0^*. Insbesondere ergiebt sich, dass man, je nach Umständen, eine der drei Constanten $g_3 : g_2$, g_2, g_3 willkürlich vorschreiben und so die in (15) vorkommende Exponentialgrösse bestimmen kann. Man hat dazu entweder die Gleichung

$$(16, \alpha) \qquad e^{2\int A ds} \equiv -\frac{9(F^2 + G)}{2F^3 + 3FG + 3C^2} \cdot \frac{g_3}{g_2},$$

oder, wenn nämlich

$$2F^3 + 3FG + 3C^2 \equiv 0, \qquad g_3 = 0$$

ist, die Gleichung

$$(16, \beta) \qquad e^{2\int A ds} \equiv \tfrac{1}{2} \sqrt[2]{\frac{3g_2}{F^2 + G}}$$

oder endlich, wenn $F^2 + G \equiv 0$, $g_2 = 0$ ist, die Gleichung

$$(16, \gamma) \qquad e^{2\int A ds} \equiv -\tfrac{3}{2} \sqrt[3]{\frac{2g_3}{2F^3 + 3F^2G + 3C^2}}$$

zu benutzen.

Der Fall II*a*) *liegt also dann vor, wenn zu den Bedingungen* $C \not\equiv 0$, $\Theta \equiv 0$, $H \equiv 0$ *die Ungleichung* (14) *tritt. Die Auflösung der gegebenen Gleichung erfordert dann lediglich algebraische Operationen.*

Dass von den eingeführten Wurzelgrössen keine durch einen rationalen Ausdruck ersetzt werden kann, so lange nicht weitere (in unseren Voraussetzungen nicht enthaltene) Besonderheiten vorliegen, lässt sich unschwer einsehen.

Zur Bestimmung des auf der Curve (y) überall endlichen Integrals

$$(17) \qquad u \equiv \int ds \cdot e^{-\int A ds}$$

braucht man nach dem Vorgetragenen nur die Differentialgleichung für y zu kennen, und man erhält dann diese Function durch Quadratur über gewisse Wurzelgrössen.

In der vorausgehenden Darlegung haben wir darauf Rücksicht genommen, die den Formeln innewohnenden Symmetrie-Eigenschaften hervortreten zu lassen. Will man vielmehr auf den Fall reeller Curven und namentlich derer mit reellen Zügen besondere Rücksicht nehmen, so sind einige der getroffenen Festsetzungen entsprechend zu ändern.

Etliche der abgeleiteten Formeln lassen sich noch etwas weiter entwickeln, wenn man $e_0 = \wp(v)$ setzt ($v \not\equiv 0, \omega_1, \omega_2, \omega_3$ modd. $2\omega_1, 2\omega_2$) und dann zu $\mathfrak{S}$- oder Θ-Functionen übergeht.

§ 8. *Fortsetzung: Grenzfälle.*

Die Betrachtung der Fälle II*b*) und II*c*) wird wesentlich erleichtert durch die Bemerkung, dass im vorigen § die mit den Formeln (5) einsetzende Entwickelung grossentheils unabhängig davon ist, ob die Grössen e_1, e_2, e_3 von einander verschieden sind, oder nicht. Nur bleiben die auf diese Art ermittelten Lösungen in den Grenzfällen nicht linear-unabhängig. Die fehlenden Lösungen aber kann man unschwer durch Grenzübergänge erhalten.

Im Falle II*b*), *der durch die Bedingungen* $C \not\equiv 0$, $\Theta \equiv 0$, $\mathrm{H} \equiv 0$ *und*

$$(1) \qquad 4(F^2 + G)^3 - (2F^3 + 3FG + 3C^2)^2 \equiv 0, \qquad F^2 + G \not\equiv 0$$

gekennzeichnet wird, setzen wir $e_2 = e_3$, und dem entsprechend

$$e_1 = 2\eta, \quad e_2 = -\eta, \quad e_3 = -\eta, \quad \zeta \equiv \frac{z}{\eta}.$$

Wir erhalten dann, wenn ζ aus der Gleichung

$$(2) \qquad \zeta \equiv \frac{z}{\eta} \equiv \frac{1}{3\eta} F e^{2\int A ds} \equiv -\frac{2F(F^2 + G)}{2F^3 + 3FG + 3C^2}$$

bestimmt wird, für ζ einen nicht constanten Werth (da $F \equiv 0$ und auch schon $F' + 2AF \equiv 0$, ebenso wie $G \equiv 0$, das Verschwinden von C zur Folge haben würden). Die Ausdrücke

$$(3) \qquad (\zeta - 2)^{\frac{1}{2}}, \quad (\zeta + 1)^{\frac{1}{2}}, \quad (\zeta + 1)^{-\frac{1}{2}}$$

stellen dann ein System von drei linear-unabhängigen Lösungen dar. Dieses wird leicht durch ein System y_1, y_2, y_3 von der verlangten Form (§ 7, Nr. 2*b*) ersetzt:

$$y_1 \equiv \lambda\sqrt{\zeta - 2}, \quad y_2 \equiv \frac{1}{2\kappa\lambda}\frac{1}{\sqrt{\zeta + 1}}, \quad y_3 \equiv i\left\{\lambda\sqrt{\zeta + 1} + \frac{1}{2\kappa\lambda}\frac{1}{\sqrt{\zeta + 1}}\right\},$$

$$\lambda = \sqrt{-\frac{\kappa + 1}{3\kappa}}.$$

Im Falle II*c*) treten zu den Bedingungen $C \not\equiv 0$, $\Theta \equiv 0$, $\mathrm{H} \equiv 0$ die

Gleichungen

$$F^2 + G \equiv 0, \qquad 2F^3 + 3FG + 3C^2 \equiv 0.$$

Diese ziehen aber die Gleichungen $\Theta \equiv 0$, $\mathrm{H} \equiv 0$ nach sich. In der That ist jetzt nothwendig $F \not\equiv 0$, $G \not\equiv 0$; man kann daher

$$\Xi \equiv -\frac{C}{F} \equiv \frac{1}{3}\frac{G}{C}$$

setzen, woraus

$$A \equiv \Xi - \frac{\Xi'}{\Xi}, \qquad F \equiv 3\Xi^2,$$

(5) $$B \equiv \frac{1}{3}\left\{\frac{d}{ds}\left(\Xi - \frac{\Xi'}{\Xi}\right) + 2\left(\Xi - \frac{\Xi'}{\Xi}\right)^2 - 3\Xi^2\right\},$$

$$C \equiv -3\Xi^3, \qquad G \equiv -9\Xi^4$$

und $\Theta \equiv 0$, $\mathrm{H} \equiv 0$ folgt. *Die Bedingungen dieses Falles sind also*

(6) $$C \not\equiv 0, \quad F^2 + G \equiv 0, \quad F^3 - 3C^2 \equiv 0;$$

man kann alle hierher gehörigen linearen Differentialgleichungen 3. Ordnung mit Hülfe einer nicht constanten, sonst aber völlig willkürlichen Function $\Xi(s)$ bilden.

Wird, entsprechend den Entwickelungen des vorigen Paragraphen,

$$z \equiv \tfrac{1}{3} F e^{2\int A ds}, \quad \text{und} \quad Z = \sqrt{z}$$

gesetzt, so folgt

(7) $$Z \equiv e^{\int \Xi ds},$$

und man erhält in den Ausdrücken

(8) $$Z, \quad Z^{-1}, \quad Z^{-3}$$

ein System von drei linear-unabhängigen Lösungen der vorgelegten Gleichung. Die in (7) vorkommende Quadratur ist nicht zu entbehren, so lange die willkürliche Function $\Xi(s)$ nicht specialisiert wird. Ein System von drei Lösungen y_1, y_2, y_3, die der in § 7 aufgestellten Forderung (2c) oder also den Gleichungen

(9) $$y_1^2 + (y_2 + iy_3)(y_2 - iy_3) \equiv 1, \qquad 2y_1(y_2 + iy_3) \equiv -1,$$

genügen, ist

(10) $$y_1 \equiv \frac{i}{2}\cdot\frac{1}{Z}, \qquad y_2 \equiv \frac{i}{2}\left\{Z - \frac{1}{Z} - \frac{1}{4Z^3}\right\}, \qquad y_3 \equiv \frac{1}{2}\left\{Z + \frac{1}{Z} + \frac{1}{4Z^3}\right\}.$$

Man kann aber jetzt die Forderung (9) offenbar auf unendlich viele Weisen erfüllen.

Es hat sich also ergeben:

Im Falle II*b*) *erfordert die Lösung der vorgelegten Gleichung nur algebraische Operationen, im Falle* II*c*) (*höchstens und in der Regel*) *eine Quadratur.*

Wichtiger als diese stets imaginären Grenzfälle sind die der Annahme $C \equiv 0$ entsprechenden.

Setzen wir also jetzt voraus, dass $C \equiv 0$, $\Theta \equiv 0$, sei — woraus $\mathrm{H} \equiv 0$ folgt — so ergiebt die Gleichung $C \equiv 0$, dass

$$\Phi \equiv \Xi^2, \qquad \Psi \equiv \kappa \Xi^3 \qquad (\kappa = \text{const.}, \neq 0) \tag{11}$$

gesetzt werden kann. Es folgt dann

$$\begin{gathered} A \equiv -\frac{\Xi'}{\Xi}, \qquad F \equiv -(1+\kappa^2)\Xi^2, \qquad G \equiv 0, \\ B \equiv \frac{1}{3}\left\{-\frac{\Xi''}{\Xi} + 3\left(\frac{\Xi'}{\Xi}\right)^2 + (1+\kappa^2)\Xi^2\right\}; \end{gathered} \tag{12}$$

man kann wieder alle hierher gehörigen Gleichungen mit Hülfe einer nicht identisch verschwindenden, aber sonst willkürlichen Function $\Xi(s)$ bilden.

Sind die Coefficienten A, B gegeben, so sind zwei Fälle zu unterscheiden:

$$C \equiv 0, \qquad \Theta \equiv 0, \qquad F \not\equiv 0, \tag{13a}$$

$$C \equiv 0, \qquad F \equiv 0. \tag{13b}$$

Im ersten Falle kann man der Constanten κ einen geeigneten Werth nach Belieben beilegen, und dann Ξ bestimmen:

$$\Xi \equiv \frac{1}{\sqrt{1+\kappa^2}}\sqrt{-F} \qquad (\kappa \neq 0, i, -i). \tag{14a}$$

Ein System von drei linear-unabhängigen Lösungen, die den hier etwa zu stellenden Forderungen

$$y_1^2 + y_2^2 + y_3^2 \equiv 1, \qquad y_1 \equiv \text{const.} \qquad (\text{const.} \neq 0, 1, -1) \tag{15a}$$

genügen, ist gegeben durch die Formeln

$$\begin{gathered} y_1 = \frac{\kappa}{\sqrt{1+\kappa^2}}, \qquad y_2 \equiv \frac{1}{\sqrt{1+\kappa^2}} \cos \int \sqrt{-F}\, ds, \\ y_3 \equiv \frac{1}{\sqrt{1+\kappa^2}} \sin \int \sqrt{-F}\, ds. \end{gathered} \tag{16a}$$

Die geometrische Bedeutung dieses Formelsystems und die Unentbehrlichkeit der ausgeführten Quadratur sind evident. Für die zugehörigen Curven $x \equiv \int y\, ds$ hat man

$$\frac{R}{T} \equiv \pm \kappa \qquad (\kappa \neq 0, i, -i); \tag{17a}$$

diese Curven sind Schraubenlinien, die auf gewöhnlichen unebenen Cylindern geodätische Linien sind.

Ist dagegen $F \equiv 0$, so folgt $\kappa = \pm i$, und Ξ wird durch Quadratur ermittelt:

$$(14b) \qquad \Xi \equiv e^{-\int A ds}$$

Man kann dann auf unendlich viele Arten — deren einzelne noch von zwei willkürlichen Parametern abhängt — ein System von Lösungen y_1, y_2, y_3 herstellen, die den Forderungen

$$(15b) \qquad y_1^2 + y_2^2 + y_3^2 = 1, \qquad y_2 + i y_3 = 1$$

genügen. Man kommt zu derartigen Formeln am schnellsten, wenn man unter den auf der Einheitskugel verlaufenden parabolischen Kreisen — die alle zu einander schon vermöge automorpher Bewegungen der Einheitskugel congruent sind — einen geeigneten auswählt,

$$y_1 \equiv -ip, \qquad y_2 \equiv 1 + \tfrac{1}{2}p^2, \qquad y_3 \equiv \frac{i}{2}p^2,$$

und dann die Veränderliche p, die für den parabolischen Kreis ein natürlicher Parameter ist (A. C., § 5), als eine zu suchende Function von s behandelt. Man erhält

$$p \equiv i \int \Xi ds \equiv i \int ds\, e^{-\int A ds},$$

$$(16b) \quad y_1 \equiv \int \Xi ds, \qquad y_2 \equiv 1 - \frac{1}{2}\left\{\int \Xi ds\right\}^2, \qquad y_3 \equiv -\frac{i}{2}\left\{\int \Xi ds\right\}^2.$$

Die zugehörigen Schraubenlinien $x \equiv \int y ds$ entsprechen der Gleichung

$$(17b) \qquad \frac{R}{T} \equiv \pm i.$$

Sie sind also geodätische Linien auf (unebenen) Cylindern mit Minimalgeraden als Erzeugenden.

Die Annahme $A \equiv 0$ kennzeichnet in beiden Fallen (a), (b) die (unebenen) *gemeinen* Schraubenlinien, die entweder (a) transcendent oder (b) rational, sogenannte Lion'sche Curven (dritter Ordnung) sind.

Es hat sich also ergeben:

In den durch die Bedingungen (13) *gekennzeichneten Fällen* IIIa) *und* IIIb) *verlangt die Integration der vorgelegten Gleichung höchstens eine* (a) *oder zwei* (b) *Quadraturen.*

Dass diese auch im Falle (b) nicht entbehrlich sind, so lange nicht noch weitere Annahmen hinzukommen, ist evident.

Das in § 6 *gestellte Problem ist hiermit vollständig gelöst. Damit zugleich sind die natürlichen Gleichungen der regulären Curven hergestellt worden,*

deren Tangentenindicatrix ein sphärischer Kegelschnitt ist; es sind die Bedingungen ermittelt, unter denen eine durch ihre natürlichen Gleichungen gegebene Curve dieser Familie angehört, und es ist ferner auch die Integration dieser Gleichungen geleistet.

Leicht lassen sich nunmehr auch, mit gleichartigem Ergebniss, die natürlichen Gleichungen der regulären Curven behandeln, deren Haupt- oder Binormalenindicatrix ein spärischer Kegelschnitt ist. Wir gehen hierauf nicht mehr ein.

Schlussbemerkungen.

Die in § 6–§ 8 behandelten linearen Differentialgleichungen gehören zu denen, auf die die von Herrn E. *Vessiot* begründete Integrationstheorie Anwendung findet.* In der That stimmen die von uns ausgeführten Operationen *der Art nach* genau überein mit denen, die sich bei Anwendung der Grundsätze des genannten Autors ergeben würden. Wir haben indessen von dieser schönen Theorie mit Absicht keinen Gebrauch gemacht. Einmal wollten wir die an den Leser zu stellenden Anforderungen nicht höher spannen, als eben nöthig war. Dann aber wird eine möglichst zweckmässige Führung der Rechnungen im concreten Falle durch die allgemeine Theorie noch nicht gewährleistet. Endlich hat Herr *Vessiot* den Gegenstand gar nicht berührt, der im Mittelpunkte unserer Untersuchung steht: den Zusammenhang gewisser linearer Differentialgleichungen mit der Theorie der natürlichen Gleichungen analytischer Curven, und die Zurückführung des einen Integrationsproblems auf das andere. In den hier betrachteten besonderen Fällen hat daher unsere elementare Methode den reicheren Inhalt.

Wir erwähnen schliesslich noch, dass der bekannte Zusammenhang der in § 2 gelegentlich betrachteten linearen Differentialgleichungen 3. Ordnung mit der Theorie der linearen Differentialgleichungen 2. Ordnung (R. Gl., § 12) unschwer auf die Gleichungen

$$y''' + 3Ay'' + 3By' + Cy \equiv 0$$

des in § 6–§ 8 betrachteten Typus ausgedehnt werden kann, wenn auch das Resultat minder einfach zu sein scheint.

Nachdem nämlich auf die beschriebene Art die Functionen Φ, Ψ und aus diesen die Functionen ϕ, ψ bestimmt worden sind, kann man die mit $\sigma(s)$, $\tau(s)$ bezeichneten Functionen in die Form

$$\sigma \equiv \frac{v_1}{w_1}, \qquad \tau \equiv \frac{v_2}{w_2}$$

setzen, wo v_1, w_1 und v_2, w_2 passend gewählte Lösungen von je einer der

*Annales de l'École Normale Supérieure, ser. III, t. 9 (1892).

Gleichungen

$$u_1'' + \frac{\phi}{4} u_1 \equiv 0, \qquad u_2'' + \frac{\psi}{4} u_2 \equiv 0$$

sind. (R. Gl., § 10). Wie die Gleichungen (2) in § 4 zeigen, hat man dann in den Quotienten

$$\frac{v_1 v_2}{v_1 w_2 - v_2 w_1}, \qquad \frac{w_1 w_2}{v_1 w_2 - v_2 w_1}, \qquad \frac{v_1 w_2 + v_2 w_1}{v_1 w_2 - v_2 w_1}$$

drei linear-unabhängige Lösungen der vorgelegten Gleichung 3. Ordnung.

Ob zwischen der Riccati'schen Gleichung

$$\frac{d\Omega}{ds} \equiv \frac{i}{2} \left\{ \frac{1}{T} - 2 \frac{\Omega}{R} - \frac{\Omega^2}{T} \right\},$$

zu der die classische Theorie der natürlichen Gleichungen führt, und den von uns benutzten unter einander äquivalenten Gleichungen

$$\frac{d\mu}{ds} \equiv \frac{\phi}{4} + \mu^2, \qquad \frac{d\nu}{ds} = \frac{\psi}{4} + \nu^2$$

ein ähnlich-einfacher Zusammenhang stattfindet, wie der, den wir in § 2 in einem Grenzfall nachgewiesen hatten, das wissen wir nicht zu sagen.

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TWO-DIMENSIONAL CHAINS AND THE ASSOCIATED COLLINEATIONS IN A COMPLEX PLANE*

BY

JOHN WESLEY YOUNG

Introduction.

The recognition of the abstract identity of geometry and analysis, which results from the notion of coördinates on the one hand and from the classical work of VON STAUDT † on the algebra of throws on the other, and which recent work on the foundations of geometry has fully established, has brought with it a broader conception of the content of geometry. It has meant not only the introduction of imaginary elements and the resulting conception of a complex space (of any number of dimensions), but it has also led to the consideration of geometries with respect to any number-system (finite or infinite), i. e., of spaces the elements of which may be determined by sets of numbers (coördinates) belonging to a given number-system. ‡ An important result of the recognition of the identity referred to is the emphasis it places on the possibility of using geometric or synthetic methods in the solution of analytic problems. It seems likely that the fact that such methods have received comparatively little attention hitherto has resulted in a loss of power. The present paper, it is hoped, will tend to substantiate this assertion.

We are here concerned with certain fundamental problems in the projective geometry on a *complex* plane, i. e., a plane the points of which are determined by sets of homogeneous coördinates (x_1, x_2, x_3), where the x_i are any ordinary complex numbers. not all zero. Though the investigation is essentially geometric and the results are susceptible of immediate application to problems of importance in geometry, these results are of even greater interest in the theory of functions of two complex variables.

To make this clear we shall glance briefly at the corresponding problems on a complex line, the results of which are well-known in the theory of functions

* Presented to the Society, April 25, 1908, under a slightly different title.

† VON STAUDT, *Beiträge zur Geometrie der Lage*, Nürnberg, 1856–60, Heft 2, p. 261.

‡ Cf., e. g., § 2 of the paper by Professor VEBLEN and myself cited below ; and for the finite cases, O. VEBLEN and W. H. BUSSEY, *Finite projective geometries*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 241–59.

of one complex variable and which I have recently considered from the point of view of projective geometry.* The notion of a *linear* (or one-dimensional) *chain* on a complex line was introduced by VON STAUDT † in connection with his introduction of imaginary elements into geometry and has since been made fundamental in recent work on the foundations of geometry.‡ Such a chain may be defined as any class of points on a line which is projective with the class of real points on a line. There is on a line one and only one chain containing any three distinct points of the line.§ The place of this notion in the theory of functions of a complex variable becomes apparent, if we adopt the usual representation of complex numbers (i. e., of the points of our line) by the real points of a plane or sphere. The chains on the line are then represented by the real circles (and straight lines) of the plane or sphere, and the study of the projectivities on the line (i. e., of the linear fractional transformations on the complex variable) with reference to their behavior toward the chains on the line is fundamental in the projective geometry on the line as well as in the theory of functions. This study leads to the classification of the projectivities into hyperbolic, elliptic, parabolic and loxodromic and to the well-known systems of chains (circles) associated with the first three of these types of projectivities.

In the present paper we are concerned with the corresponding problems in the plane. Here the notion of a *planar* (or two-dimensional) *chain* is fundamental. A planar chain may be conveniently described as any class of points and lines in the plane which may be obtained from the class of real points and lines by a projective collineation. A planar chain is then a two-dimensional spread of points and lines within the four-dimensional spread of all points and lines in our complex plane.‖ It follows readily from the description given that there is one and only one planar chain through any four points of the plane, no three of which are collinear.¶ The notion of a planar chain was first introduced

* J. W. YOUNG, *The geometry of chains on a complex line*, Annals of Mathematics, series 2, vol. 11 (1909), No. 1, pp. 33–48. This paper is referred to in the sequel by the letter *A*.

† VON STAUDT, loc. cit., p. 137.

‡ Cf. PIERI, *Nuovi principii di geometria projettiva complessa*, Memorie della R. Accademia delle Scienze di Torino, series II, vol. 45 (1905), pp. 189–235, in which the chain is chosen as one of the primitive notions; also O. VEBLEN and J. W. YOUNG, *A set of assumptions for projective geometry*, American Journal of Mathematics, vol. 30 (1908), pp. 347–380, where the chain is defined in terms of order and continuity relations.

§ The coördinate system on a line is determined as soon as 0, 1, ∞ are assigned to three distinct points of the line, which are entirely arbitrary. The chain determined by the three points is then simply the class of all points on the line that have real coördinates.

‖ In this paper we use the word "dimension" throughout in the sense of *real* dimension, i. e., a spread of n dimensions is one whose elements may be made to depend on n independent *real* parameters. The plane is, of course, a spread of two *complex* dimensions, but we shall not use the word here in this sense.

¶ We may indeed establish a coördinate system in the plane by choosing any three points

by JUEL and SEGRE* in connection with their study of anti-projective transformations.† These planar chains are of value in those problems in the plane in which the distinction is maintained between real and imaginary elements. The projective point of view, however, emphasizes the fact that this distinction is relative, since any planar chain may be taken as the real planar chain; the points and lines not belonging to this chain are then "imaginary" with reference to the points and lines of the chain.

In § 1 below we define a planar chain geometrically in terms of the notion of a linear chain and derive certain properties of these chains which are needed in the later developments. In § 2 we determine necessary and sufficient conditions that a collineation in the plane leave a planar chain invariant; this is equivalent to the determination of the conditions that a collineation may be represented (transformed into one) with real coefficients. We are thus led to certain characteristic *systems of planar chains* which form a generalization of the systems of circles referred to above. In the Conclusion these systems are briefly considered and reference is made to certain immediate applications of the results obtained. Here also is outlined the important problem of the order relations in a complex plane, to which I expect to return on a future occasion.

§ 1. *Definition and fundamental properties of planar chains.*

We assume the theorems of alignment in the ordinary complex projective geometry of the plane.‡ All points and lines considered are coplanar. We assume further the notion and fundamental properties of (linear) chains on a line. We then define a planar chain as follows:

(which are not collinear) as the vertices of the triangle of reference and assigning to any fourth point which is not on a side of this triangle the coördinates (1, 1, 1). The planar chain determined by these four points then consists of all points and lines the ratios of whose coördinates are real numbers.

* JUEL, *Über einige Grundgebilde der projectiven Geometrie*, Acta Mathematica, vol. 14 (1890), pp. 1–30; SEGRE, *Un nuovo campo di ricerche geometriche*, Atti della R. Accademia delle Scienze di Torino, vol. 25 (1890), pp. 276–301, 430–457; vol. 26 (1890), pp. 35–71, 592–612. Though closely related in subject matter, the present paper has little direct contact with either of the two papers cited.

† The term "anti-projective" is due to SEGRE and seems to have been generally adopted. JUEL calls the anti-projective transformations in the plane "Symmetralitäten." He describes a planar chain in his introduction substantially as we have described it above (though in his description the planar chain consists of a class of points only). SEGRE on the other hand obtains them as the class of invariant elements of certain anti-projective transformations. It may be noted in passing that the property of a planar chain to the effect that the point of intersection of two lines joining two pairs of points of the chain is a point of the chain, which JUEL apparently takes as his definition of a planar chain (loc. cit., p. 3) is not sufficient to characterize it. The property is characteristic of any *planar field*, i. e., of any set of points and lines in a plane whose coördinates are numbers of a given field. The notion of a general field in a geometry of any number of dimensions I defined in a paper presented to the Society (Chicago section), January 2, 1909.

‡ Our developments are subject, e. g., to Assumptions A, E, H, C and I of the paper by Professor VEBLEN and myself quoted above.

DEFINITION. Given two linear chains $\mathfrak{C}_1$, $\mathfrak{C}_2$, on distinct lines and having a point O in common, a point is said to be *linearly related* to $\mathfrak{C}_1$, $\mathfrak{C}_2$, if it is the intersection of two lines, each of which joins a point of $\mathfrak{C}_1$ to a distinct point of $\mathfrak{C}_2$. A line is said to be *linearly related* to $\mathfrak{C}_1$, $\mathfrak{C}_2$, if it joins two points linearly related to $\mathfrak{C}_1$, $\mathfrak{C}_2$. The class of all points and lines linearly related to $\mathfrak{C}_1$, $\mathfrak{C}_2$ is called the *two dimensional chain* or the *planar chain* determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$.

If a scale * be established on the line of $\mathfrak{C}_1$ by choosing the point O as 0 and any other two distinct points of $\mathfrak{C}_1$ as 1 and ∞ respectively, the points of $\mathfrak{C}_1$ correspond to the set of all ordinary real numbers (and ∞) and only these. If then a scale be established in a similar manner on the line of $\mathfrak{C}_2$, the coördinate system in the plane determined by these two scales will assign to every point (or line) of the planar chain $\mathfrak{C}^2$ determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$ a set of real coördinates; and, conversely, to every set of real coördinates will correspond a point (or line) of $\mathfrak{C}^2$. Since a projective collineation in the plane is equivalent to a change in the system of coördinates, this justifies the analytic definition suggested in the introduction:

Any class of points and lines which can be obtained from the real points and lines of the plane by a projective collineation is a planar chain.

Since a projective transformation transforms linear chains into linear chains, it follows at once from the definition given above (and also from the analytic characterization just obtained), that

I. *Any class of points and lines of a plane which is projective with a planar chain is a planar chain.* †

The following fundamental *internal* properties of a planar chain $\mathfrak{C}^2$ may be deduced readily (they are simply properties of the points and lines of a *real* projective plane):

II. *There is one and only one planar chain containing four points (lines) of a plane no three of which are collinear (concurrent).*

III. *Any two lines of a planar chain meet in a point of the chain.*

IV. *Any line of a planar chain has the points of a linear chain in common with the planar chain, and only these.*

We should note also the *principle of duality* in a planar chain, where the elements "point" and "linear chain" are dual elements.

The following *external* properties are of fundamental importance:

VI. *Every line of the plane which is not a line of a planar chain $\mathfrak{C}^2$ in the plane has one and only one point in common with $\mathfrak{C}^2$.*

This theorem is most readily proved analytically. Let a system of homo-

* VEBLEN and YOUNG, loc. cit., § 2, p. 353.

† In this theorem the projective transformation may be either a collineation or a reciprocity. The theorem is still true, moreover, if the transformation in question is *anti-projective* (cf. SEGRE, *Un nuovo campo etc.*, loc. cit.).

geneous coördinates be established in such a way that the points of $\mathfrak{C}^2$ correspond to *real* ratios $x_1 : x_2 : x_3$ of the coördinates. The equation of any line in the plane which is not a line of $\mathfrak{C}^2$ is then $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$, where the ratios $u_1 : u_2 : u_3$ are not all real. Placing $u_j = u_j' + i u_j''$ ($j = 1, 2, 3$; u_j', u_j'' real) the equations

$$u_1' x_1 + u_2' x_2 + u_3' x_3 = 0, \qquad u_1'' x_1 + u_2'' x_2 + u_3'' x_3 = 0,$$

determine a real point $x_1 : x_2 : x_3$ of the line. The line cannot have more than one point in common with $\mathfrak{C}^2$, for otherwise it would be a line of $\mathfrak{C}^2$.

The argument dual to this gives at once:

VII. *Through every point of the plane which is not a point of a planar chain $\mathfrak{C}^2$ in the plane passes one and only one line of $\mathfrak{C}^2$.*

These theorems lead at once to the notion of pairs of points (lines) *inverse* or *conjugate* with respect to a planar chain. Let P be any point not on a given planar chain $\mathfrak{C}^2$ and let l be the line of $\mathfrak{C}^2$ through P (VII). This line l has a linear chain in common with $\mathfrak{C}^2$ (IV); the point $\bar{P}$ which is the inverse of P with respect to this linear chain is called the *inverse* or *conjugate* of P with respect to the planar chain $\mathfrak{C}^2$; P, $\bar{P}$ form a pair of *conjugate points* with respect to $\mathfrak{C}^2$. The latter phrase is justified by the evident fact that either of two such points is the conjugate of the other. Analytically two conjugate points with respect to the planar chain all of whose elements have real coördinates are determined by coördinates of which one set consists of the conjugate complex numbers of those forming the other set. Every point of a planar chain $\mathfrak{C}^2$ is, by definition, its own conjugate with respect to $\mathfrak{C}^2$.*

The process dual to the one described gives rise to the notion of pairs of *conjugate lines* with respect to a planar chain. It follows further that *the line joining two points conjugate with respect to a planar chain $\mathfrak{C}^2$ is a line of $\mathfrak{C}^2$; and the point of intersection of two lines conjugate with respect to $\mathfrak{C}^2$ is a point of $\mathfrak{C}^2$.* It is then at once evident analytically that

VIII. *The line joining two points A, B is conjugate with respect to a planar chain with the line joining the two points $\bar{A}$, $\bar{B}$ which are conjugate with A, B respectively; and every point on a line l is conjugate with a point on the line conjugate with l.*

Theorems I–VIII just given are sufficient to derive the fundamental properties of the geometry of chains in a plane. In preparation for the application of these properties to the classification of collineations in the plane with reference to their behavior toward the planar chains of the plane and the derivation of the resulting characteristic systems of planar chains, we call attention to a few general results of importance.

* The transformation obtained by replacing each point of the plane by its conjugate with respect to a planar chain (wherein the points of the planar chain are self-conjugate) is called by SEGRE (loc. cit.) an *antinvolution* with double elements. Cf. the footnote, p. 282.

We note first, that a triangle in the plane determines an infinite system of chains containing the vertices and sides of this triangle; there is, in fact, through every point of the plane which is not on a side of the triangle one and only one chain of this system, and one and only one through every line which does not meet one of the vertices (II). In general, two planar chains of this system have only the vertices and sides of the triangle in common; certain pairs of chains, however, have more than these elements in common, as the following considerations will show. A linear chain $\mathfrak{C}$ of points and a point not on $\mathfrak{C}$ determine an infinite system of planar chains each of which contains P and the points of $\mathfrak{C}$; viz., there is one such chain through any point (distinct from P and not in $\mathfrak{C}$) of a line joining P to any point of $\mathfrak{C}$. For there are thus determined two linear chains ($\mathfrak{C}$ and one on the line through P and containing P) which determine a planar chain (Definition). *Two distinct planar chains, however, can not have more points in common than a linear chain $\mathfrak{C}$ of points and a point P not on $\mathfrak{C}$.* For, let Q be another common point; there exist then under the hypothesis two points A, B of $\mathfrak{C}$ such that no three of the points A, B, P, Q are collinear. Hence, the two planar chains would coincide (II).* In other words *two planar chains which have the points of a linear chain and two other points in common coincide.*

It should be noted, however, that a linear chain of points and two points not on this chain do not in general determine a planar chain in which they lie. They will determine a planar chain only if the line joining the two points meets the linear chain in question (IV).

Another important system of chains is determined as follows:

Definition. Any planar chain is said to be *about* two points B, C, if B, C are conjugate points with respect to the planar chain.

Now let A, B, C be the vertices of a triangle. The class of all planar chains through A and about B, C is the system in question. There is one and only one chain of this system through any point not on a side of the triangle ABC. For, let P be such a point; the line AP meets the line BC in a point Q. Let $\mathfrak{C}$ be the linear chain through Q and about B, C. The chain $\mathfrak{C}$ and the two points P, A determine a unique planar chain with the specified property. Two chains of this system have in general only the point A and the line BC in common; certain pairs of these chains have, however, also a linear chain of points on BC and a linear chain of lines through A in common.

* Two distinct chains through the points of a linear chain C and a point P not on C have also a linear chain of lines through P in common (the linear chains of points on these lines are, of course, different for the two planar chains).

§ 2. *Collineations with invariant planar chains.*

We now seek necessary and sufficient conditions that a collineation in the plane leave a planar chain invariant.* Suppose π is a collineation leaving a planar chain $\mathfrak{C}^2$ invariant, and let a system of homogeneous coördinates be determined so that the elements of $\mathfrak{C}^2$ are determined by real ratios of the coördinates. It is then clear that a necessary and sufficient condition that π leave $\mathfrak{C}^2$ invariant is that when π is represented analytically by equations

$$x_i' = \Sigma a_{ij} x_j \qquad (i, j = 1, 2, 3),$$

the coefficients a_{ij} shall be real numbers (or at least that they may be rendered real by multiplication with a suitably chosen factor of proportionality). We seek, however, a simpler and geometric condition. The five well-known types of collineations in the plane will be treated successively.

Type I. *Three (and only three) double points.* If the collineation be represented analytically as described above, the characteristic equation will have real coefficients and three distinct roots. There are then two cases to consider:

h) The roots of the characteristic equation are all real;

e) One of the roots is real and the other two conjugate complex.†

Type Ih. If the roots of the characteristic equation are all real the double points are all real. In other words, any invariant planar chain $\mathfrak{C}^2$ contains all three double points, A, B, C. Each of the sides of the triangle ABC then has a linear chain in common with $\mathfrak{C}^2$ (IV) and, since these lines are invariant, these linear chains are invariant under the projectivities on the invariant lines. Since these linear chains contain the (distinct) double points of the projectivities mentioned, *the projectivities on the sides of the invariant triangle must all be hyperbolic.*‡ This necessary condition is also sufficient. For let π be any collineation of Type I such that the projectivities on two sides AB and BC of the invariant triangle are hyperbolic, and let $\mathfrak{C}^2$ be *any* planar chain through A, B, C. The linear chains $\mathfrak{C}_1$, $\mathfrak{C}_2$ on AB, BC respectively are then invariant, since a hyperbolic projectivity on a line leaves every chain through the double points invariant (A, Theorem 14). Since $\mathfrak{C}^2$ is determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$, it follows that $\mathfrak{C}^2$ is invariant. That the linear chain of $\mathfrak{C}^2$ on CA is likewise invariant then follows almost immediately. Hence,

* That such collineations exist is obvious. Let A, B, C, D be the vertices of any quadrangle, and let $\mathfrak{C}^2$ be the planar chain determined by A, B, C, D. Let A', B', C', D' be any other four points of $\mathfrak{C}^2$ no three of which are collinear. The collineation determined by the homologous pairs A, A'; B, B'; C, C'; D, D' then clearly leaves $\mathfrak{C}^2$ invariant, since it transforms $\mathfrak{C}^2$ into the planar chain determined by A', B', C', D', which is identical with $\mathfrak{C}^2$ (II).

† These two cases are also readily obtained by a purely synthetic argument. Cf. footnote on next page.

‡ In accordance with the definition given in *A*, p. 42, the involutions are to be considered as both hyperbolic and elliptic.

THEOREM 1. *A necessary and sufficient condition that a collineation of Type I leave a planar chain through the double points invariant is that the projectivities on two sides of the invariant triangle be hyperbolic. The projectivity on the third side is then likewise hyperbolic, and the collineation leaves every planar chain through the double points invariant.*

A collineation satisfying the conditions of this theorem we will call *hyperbolic of Type I*, or *of Type Ih.*

Moreover, a hyperbolic collineation of Type I leaves invariant no other planar chains than those mentioned in the theorem. For, if A is a double point, there is at least one line through A which meets a given invariant planar chain in a linear chain $\mathfrak{C}_1$ (VII). If this linear chain is invariant, A must be a point of C_1 and hence of the planar chain, since the projectivity in this chain is hyperbolic. If $\mathfrak{C}_1$ is not invariant, it is transformed into another linear chain of the invariant planar chain, and the lines of these two linear chains determine A as a point of the invariant planar chain. Hence,

THEOREM 2. *The system of invariant planar chains of a hyperbolic collineation of Type I consists of all the planar chains through the double points and only these. There is one chain of the system, and only one, through every point of the plane which is not on a side of the invariant triangle.*

Type Ie. If one of the roots of the characteristic equation is real and the other two are conjugate complex, any invariant planar chain $\mathfrak{C}^2$ contains one double point, say A, and the other two double points B, C are conjugate with respect to the planar chain.* The line BC is then a line of the planar chain (it also contains one invariant line, therefore; so that the invariant figure is self-dual within the planar chain) and has an invariant linear chain $\mathfrak{C}_1$ in common with the invariant planar chain (IV). Since $\mathfrak{C}_1$ does not contain the double points of the projectivity on BC, this projectivity must be elliptic.

This condition is not, however, sufficient. A necessary and sufficient condition is readily obtained by considering the projectivities on the two conjugate lines AB and AC. If P is any point of $\mathfrak{C}^2$ not on a side of the invariant triangle, and the line PC meets AB in a point M, the line PB meets AC in a point N which is the conjugate of M with respect to $\mathfrak{C}^2$; for the line conjugate with MC passes through B (the conjugate of C) and P, and must contain the conjugate of M which is on AC (VIII). Let P' be the point homologous with P and let the points M', N' on AB, AC be constructed. If $\mathfrak{C}^2$ is invariant,

* That any invariant planar chain of a collineation of Type I either passes through all the double points or passes through one and about the other two may be seen synthetically as follows. Suppose an invariant planar chain $\mathfrak{C}^2$ does not contain the double point B. Through B passes a line l of $\mathfrak{C}^2$. The line l must then be invariant; for otherwise B would be the intersection of two lines of $\mathfrak{C}^2$ and would hence be a point of $\mathfrak{C}^2$. The invariant linear chain of l is therefore about the two double points B and C (say) on l. No other line of $\mathfrak{C}^2$ can pass through B or C. Through the third double point A, however, passes a line of $\mathfrak{C}^2$, and since (as just noted) this line cannot be invariant, A is a point of $\mathfrak{C}^2$.

the projectivities on AB and AC must then be so related that if a pair of points are homologous under the projectivity on AB, their conjugates with respect to $\mathfrak{C}^2$ must be homologous under the projectivity on AC. Two projectivities which are related in this way with respect to a planar chain we will call *conjugate*. A necessary condition that $\mathfrak{C}^2$ be invariant under a collineation with double points A, B, C is then that the projectivities on AB and AC be conjugate with respect to $\mathfrak{C}^2$. But this condition is at once seen to be also sufficient. Hence,

THEOREM 3. *A necessary and sufficient condition that a collineation of Type I leave a planar chain through one double point A and about the other two B, C invariant is that the projectivities on the two conjugate invariant lines be conjugate. The projectivity on the third side of the invariant triangle is then elliptic. The collineation leaves every planar chain through A and about B, C invariant.**

A collineation satisfying the condition of Theorem 3 we will call *elliptic of Type I* or *of Type Ie*. The double points on the side of the invariant triangle on which the projectivity is elliptic we will call the *conjugate* double points. We then have:

THEOREM 4. *The system of invariant planar chains of an elliptic collineation of Type I consists of all the planar chains about the conjugate double points and through the remaining double point, and only these. There is one and only one chain of this system through every point which lies in the plane but not on a side of the invariant triangle.*

Type II. *Two and only two double points.* The invariant figure consists of two points A, B, the line $l = AB$, and a line l' through A. The collineation is completely determined by the projectivity on the line l and the parabolic projectivity on l'. We note first that the characteristic equation of any collineation of this type has a double root, and that hence all the roots of this equation must be real. It then follows at once that *every invariant planar chain of such a collineation must contain the two points A and B and a linear chain of l'.*†

Let $\mathfrak{C}^2$ be any invariant planar chain. The linear chain of $\mathfrak{C}^2$ on AB is

* The theorem that *if a collineation of Type I leaves one planar chain* $\begin{matrix}\textit{through the double points } A, B, C\\ \textit{through one double point } A \textit{ and}\end{matrix}$ *about the other double points B, C invariant, it leaves every chain* $\begin{matrix}\textit{through } A, B, C\\ \textit{through } A \textit{ and about } B, C\end{matrix}$ *invariant*, may also be proved directly as follows: Let π be the given collineation and $\mathfrak{C}^2$ a planar chain which it leaves invariant. Let P be any point of $\mathfrak{C}^2$ which is not on a side of the invariant triangle of π and let P_1 be any other point in the plane not on a side of this invariant triangle. Let π_1 be the collineation which has the same invariant triangle as π and which transforms P into P_1. π_1 then clearly transforms $\mathfrak{C}^2$ into a planar chain $\mathfrak{C}_1^2$ through P_1. Moreover, if $\mathfrak{C}^2$ contains the double points A, B, C of π, so also will $\mathfrak{C}_1^2$; and if $\mathfrak{C}^2$ is through A and about B, C, so also is $\mathfrak{C}_1^2$. Now the transformation $\pi_1\pi\pi_1^{-1}$ clearly leaves $\mathfrak{C}_1^2$ invariant. But since π, π_1 have the same invariant triangle, they are commutative and we have $\pi_1\pi\pi_1^{-1} = \pi$; i. e., π leaves $\mathfrak{C}_1^2$ invariant.

† This result, as well as the corresponding one under Type III below, may also be established readily by a purely synthetic argument.

invariant under the projectivity on AB; hence this projectivity must be *hyperbolic.* Now let P be any point not on l or l'. We show that one and only one invariant planar chain of the collineation contains P. Let the line BP meet l' in a point Q and let Q' be the point homologous with Q under the parabolic projectivity on l'. Any invariant planar chain through P must contain Q (III) and hence Q'. The points P, B, Q', A determine uniquely a planar chain; hence, there is no more than one invariant planar chain through P. That the planar chain thus determined is indeed invariant follows from the fact that it is uniquely determined by its two linear chains on the line $l = AB$ and l' (Def.) and each of these linear chains is invariant; the one on l because it contains the two double points of a hyperbolic projectivity (A, Theorem 14), the one on l' because it contains the double point and one pair of homologous points of a parabolic projectivity (A, Theorem 15). We have then the following:

THEOREM 5. *A necessary and sufficient condition that a collineation of Type II leave a planar chain invariant is that the projectivity on one of the invariant lines be hyperbolic. The collineation then leaves every planar chain through the point B and an invariant linear chain on l' invariant.*

THEOREM 6. *The system of invariant planar chains is determined by the point B and the system of invariant chains of the parabolic projectivity on l'. There is one and only one chain of this system through any point not on l or l'.*

A collineation of Type II satisfying the condition of Theorem 5 may be called *hyperbolic of Type II* or *of Type IIh.*

Type III. *One and only one double point.* The invariant figure consists of a point A and a line l through A. The collineation is completely determined when the parabolic projectivity (with double point A) on l and one pair of homologous points (not on l) are given. As before, by reference to the characteristic equation which in this case has a triple root, we see that every invariant planar chain of such a collineation must contain A and a linear chain on l. Now let P, P' (not on l) be any pair of points homologous under the collineation in question. The line PP' then meets l in a point Q distinct from A. Let Q' be the point homologous with Q. Let $\mathfrak{C}^2$ be the planar chain determined by the linear chain $\mathfrak{C}$ through A, Q, Q' and the points P, P'. We show that $\mathfrak{C}^2$ is invariant under the collineation. We note first that the chain $\mathfrak{C}$ is invariant (A, Theorem 15). Similarly, the linear chain of lines at A determined by the lines l, AP, AP' is invariant, since it contains the double line and a pair of homologous lines of the parabolic projectivity in the pencil of lines at A. The line AP' is therefore transformed by the collineation into a line AM where M is a point of $\mathfrak{C}^2$ (III). The point P'' which is homologous with P' is now determined as the intersection of the lines $P'Q'$, AM, and is therefore a point of $\mathfrak{C}^2$ (III). Hence $\mathfrak{C}^2$ is invariant. Clearly also $\mathfrak{C}^2$ is the only invariant planar chain through P. We have then

THEOREM 6. *Every collineation of Type III leaves invariant one and only one planar chain through each point not on l. Every invariant planar chain contains an invariant linear chain of the parabolic projectivity on the double line; through every invariant linear chain of this parabolic projectivity there pass an infinite number of invariant planar chains of the collineation.*

A collineation of Type III may conveniently be called a *parabolic* collineation.

Type IV. *A point O and every point of a line l not through O is invariant.* A collineation of this type is called a *homology*, the point O and the line l being the *center* and *axis* respectively of the homology. A homology is completely determined when the center, axis, and one pair of homologous points (collinear with O) are given. Any invariant planar chain of a homology contains O and a linear chain on l. For let P, P' and Q, Q' be any two pairs of homologous points on any invariant planar chain (P, P', Q not collinear). The lines PP', QQ' meet in O which is therefore a point of the planar chain (III). The dual argument shows similarly that l contains two points and hence a linear chain of any invariant planar chain. Let $\mathfrak{C}^2$ be any planar chain through O and a linear chain $\mathfrak{C}$ on l. Any line joining O to a point of $\mathfrak{C}$ has a linear chain in common with $\mathfrak{C}^2$, and this linear chain must be invariant if $\mathfrak{C}^2$ is invariant. Hence, if a homology leaves a planar chain invariant, the projectivities along the invariant lines through O are *hyperbolic*. This necessary condition is also clearly sufficient, since the projectivities on all lines through O are hyperbolic, if one is, and a hyperbolic projectivity leaves every linear chain through its double points invariant. We have then

THEOREM 7. *A necessary and sufficient condition that a homology leave a planar chain invariant is that the projectivity on an invariant line through the center be hyperbolic. If this condition is satisfied, the homology leaves every planar chain through the center and any linear chain on the axis invariant.*

A homology satisfying the condition of this theorem we call a *hyperbolic homology* or a collineation of *Type IVh*.

THEOREM 8. *The system of invariant chains of a hyperbolic homology consists of all planar chains through the center and any linear chain on the axis, and of these only. Through any point not on the axis and distinct from the center there pass an infinite number of chains of this system. Through any two such points not collinear with the center passes one and only one chain of this system.**

*On every line through O there is a projectivity which has the center O and a point on the axis l for double points. It is well known, moreover, that any two projectivities on such lines are equivalent, i. e., can be transformed into each other (or have the same characteristic cross ratio). We may therefore classify homologies with reference to these projectivities into *hyperbolic, elliptic*, and *loxodromic*. The hyperbolic homologies alone leave a planar chain invariant. The elliptic homologies, however, leave invariant every linear chain about the double points on any line through A. The loxodromic homologies leave no planar chain invariant and no linear chain on any line except l. The elliptic and loxodromic projectivities leave invariant, however, every *system* of planar chains through A and a linear chain on l.

Type V. *All points of a line l and all lines through a point O on l are invariant.* A collineation of this type is called an *elation*, the line l and point O being the *axis* and *center* respectively. An elation is completely determined when the center, axis, and one pair of homologous points are given. If P, P' and Q, Q' are any two pairs of homologous points (not all collinear), the lines PP' and QQ' meet in O and the lines PQ and $P'Q'$ meet in a point of l. It follows at once that any invariant planar chain must contain O and a linear chain on l (III). The construction for the elation shows also at once that every planar chain containing a pair of homologous points and a linear chain on l through O is invariant. We have then

THEOREM 9. *Every elation leaves invariant every planar chain containing a pair of homologous points and a linear chain on the axis and through the center; and only these. Through any point not on the axis pass an infinite number of invariant planar chains; through two such points not collinear with the center passes one and only one.*

We note finally the following theorem which is an immediate consequence of the results of the preceding discussion:

THEOREM 10. *If a collineation leaves invariant more than one planar chain through a point which is not on a double line it is a perspective collineation, i. e., either a homology or an elation.*

§ 3. *Conclusion.*

Our discussion has led to the definition of six fundamental systems of planar chains in the plane, which form a natural generalization of the three well-known systems of linear chains on a line; i. e., of the three systems of circles (in the Argand representation of a complex variable) associated with the hyperbolic, elliptic, and parabolic transformations (linear fractional) on one complex variable. Each of the first four of these six systems, i. e., those associated with a hyperbolic collineation of Type I, an elliptic collineation of Type I, a hyperbolic collineation of Type II, or a parabolic collineation (Type III), consists of ∞^2* planar chains, one and only one through each point which is not a double point nor on a double line. Each of the last two systems, i. e., those associated with a hyperbolic homology or an elation, consists of ∞^4 planar chains, one and only one passing through two points neither of which is a double point and both of which are not on the same invariant line. In view of their importance, we shall describe these systems independently of the collineations by means of which they were obtained. They are as follows:

System Ih. Such a system is defined by any three points which are not collinear, and consists of all the planar chains through these points. Associated

* By ∞^2 is meant that the system may be made to depend on two independent *real* parameters; similarly for the ∞^4 in the next sentence, and similar expressions following.

with any such system is a two-parameter group of hyperbolic collineations having the three points as double points, each collineation of which leaves every chain of the system invariant.

System Ie. Three such systems are defined by any three points which are not collinear; each of these systems consists of all the chains through one of the points and about the other two. Associated with any system of this type is a two-parameter group of elliptic collineations having the three points as double points; each collineation of the group leaves every chain of the associated system invariant.

System II. Such a system consists of all the planar chains through two given points A, B and meeting a given line l' through A in a system of linear chains all mutually tangent at A. Associated with any such system is a two-parameter group of hyperbolic collineations, each collineation of which leaves every chain of the system invariant.

System III. Such a system meets a given line in a system of linear chains all mutually tangent at a given point A of the line, and meets the pencil of lines at A in a system of linear chains (of lines) all mutually tangent at the given line. Associated with any such system is a three-parameter group of parabolic collineations, each collineation of which leaves every chain of the system invariant.

System IV. Such a system consists of all the planar chains through a given line and a given point not on that line. Associated with any such system is a one-parameter group of hyperbolic homologies, each homology of which leaves every chain of the system invariant.

System V. Such a system consists of all the planar chains through a given line and through a given point not on that line. Associated with any such system is a one-parameter group of elations, each elation of which leaves every chain of the system invariant.

The results of this paper may be applied at once to the enumeration and geometric definition of *all* the continuous groups of collineations in the plane which leave a planar chain invariant. This is equivalent to the enumeration of all continuous groups of collineations in the plane which can be represented with real coefficients. No complete list of the groups of *real* collineations in a plane, in which the groups are characterized geometrically or in which the finite equations of the collineations are given, has ever been published, so far as the writer is aware.* The enumeration in both of these forms by the methods developed above is very simple, and offers a good example of a problem apparently analytic which yields readily to synthetic treatment.†

* LIE has given the infinitesimal equations of these groups. Cf. LIE-ENGEL, *Theorie der Transformationgruppen*, Leipzig (1893), vol. 2, pp. 78–109 and pp. 380–384.

† The results of this enumeration were presented by the writer to the Society September 11, 1908, and will be published in the near future.

We close with a word regarding the relation of our developments to the theory of functions of two complex variables. In the theory of functions of one complex variable the circles in the Argand plane owe their importance in part to the fact that the necessary and sufficient condition that a linear fractional transformation on the variable may be transformed into one with real coefficients is that it leave a circle invariant. In all analogous problems in the theory of functions of two complex variables the planar chains will play a role similar to that of the circles (linear chains) in the one-dimensional case. On the other hand, the circles owe their importance in the theory of functions of one complex variable also in part to the fact that they divide the points of the Argand plane into two regions. This aspect of the linear chain does not find its generalization in the planar chain, since a two-dimensional spread of points can not divide the four-dimensional spread of points in the complex plane into two regions. The generalization here in question may be found by combining a system of ∞^1 planar chains (or a system of ∞^2 linear chains) into a spread of three dimensions with the desired property. A three-dimensional spread of points dividing the complex plane into two regions is obtained very simply, moreover, as the class of all the points on the lines of a linear chain of lines. To this aspect of the problem and in general to the question of the order relations in a complex plane, I shall return on a future occasion. To this problem the present paper forms a preliminary investigation.

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GROUPS OF RATIONAL TRANSFORMATIONS IN A GENERAL FIELD*

BY

LEWIS IRVING NEIKIRK

Introduction.

Groups of linear transformations of a single variable of both finite and infinite orders are well known, but the only known examples of non-linear rational transformation groups in one variable are those given by the following writers: HERMITE, BETTI, and others have investigated special quantics, known as substitution quantics, with coefficients taken with respect to a prime modulus (p), which define substitutions on a set of residues (mod p) and generate finite groups (mod p). Substitution quantics with coefficients in a Galois field have been investigated by DICKSON in his dissertation,† where the reader will find a complete bibliography of the subject.

The object of this paper is to find all *non-linear* groups of rational transformations of a single variable. It is proved in § 1 that these groups of transformations define substitution groups on the roots of an equation $f(x) = 0$. They are a two-fold generalization of substitution quantics and form finite groups $(\bmod f(x))$. Section 2 is devoted to finding these transformations and section 3 to the conditions for the existence of such transformations in a general field F. The other articles apply and extend these results.

§1. *General developments.*

Consider a group G of rational integral transformations

$$T_i \equiv [x : \phi_i(x)],$$

$$\phi_i(x) = \sum_{j=0}^{j=m_i} \alpha_{ij} x^{m_i - j} \qquad (\alpha_{i0} \neq 0),$$

where the coefficients α_{ij} are elements of a general field F and the quantity x belongs to a set X_i in a field F' containing F. It is assumed that at least one m_i exceeds unity, so that the group is not linear.

* Presented to the Society (Chicago), April and December, 1909.

† L. E. DICKSON, *The analytical representation of substitutions on a power of a prime number of letters*, etc., Annals of Mathematics, ser. 1., vol. 11 (1896), pp. 65–120, 161–183.

Let $T_i(X_i) = X'_i$. Then *

(a) $$X_i \equiv X'_i, \text{ for every } i.$$

(b) $$X_i \equiv X_{i'} \equiv X, \text{ for every } i \text{ and } i'.$$

(a) Since T_i^2 is in G, X'_i is a subset of X_i, and since T_i^{-2} is in G, X_i is a subset of X'_i. Therefore $X_i \equiv X'_i$.

(b) X_i must be a subset of $X_{i'}$ since $T_{i'}T_i$ is in G, and $X_{i'}$ must be a subset of X_i since $T_iT_{i'}$ is in G. Therefore $X_i \equiv X_{i'} \equiv X$.

Since T_i, of degree $m_i > 1$, has an inverse in G, let $T_i^{-1} = T_{i'}$. Then

$$T_iT_{i'} \equiv [x : x] = [x : \phi_i\{\phi_{i'}(x)\}],$$

whence

(1) $$\phi_i\{\phi_{i'}(x)\} = x,$$

so that x satisfies an equation of degree $m_im_{i'} > 1$, the leading coefficient being $\alpha_{i0}\alpha_{i'0} \neq 0$.

Therefore the elements of the set X are roots of an equation rational in F.

Let $X = (x_1, x_2, x_3, \cdots, x_n)$ be a set whose elements are the roots of an equation,

$$f(x) = \sum_{r=0}^{r=n} a_r x^{n-r} = 0,$$

with the coefficients in F and having no double root.

All the transformations reduce (mod $f(x)$) to degree $n-1$ or less.†

Let T_i change X according to the scheme

$$\begin{pmatrix} x_1 x_2 \cdots x_n \\ x_{i_1} x_{i_2} \cdots x_{i_n} \end{pmatrix}.$$

If any root is repeated in the lower line, T_i will not have an inverse in the group G. Therefore the lower line is a permutation of the upper line and T_i defines a substitution on the roots of $f(x) = 0$. Hence we have proved

THEOREM I. *The only non-linear groups of rational integral transformations on one variable are finite groups taken modulo $f(x)$ which define substitution groups on the roots of the equation $f(x) = 0$.*‡

§ 2. *Determination of the transformation corresponding to a given substitution.* §

Given a substitution on the roots of $f(x) = 0$,

$$S_i = \begin{pmatrix} x_1 x_2 \cdots x_n \\ x_{i_1} x_{i_2} \cdots x_{i_n} \end{pmatrix},$$

* BURNSIDE (*Theory of Groups*, p. 12) makes use of property (a) without explicit mention in the proof that if A_{-1} is the inverse of A, then A is the inverse of A_{-1}.

† H. WEBER, *Lehrbuch der Algebra*, vol. I, p. 170.

‡ The actual existence of these groups will be established in the next two articles.

§ L. E. DICKSON, Dissertation, l. c.

we seek the corresponding transformation T_i. We have the n linear equations

$$x_{i_t} = \phi_i(x_t) = \sum_{j=0}^{j=n-1} \alpha_{ij} x_t^{n-1-j} \qquad (t=1, 2, \cdots, n)$$

between the n coefficients α_{ij}. From these

$$(2) \qquad \alpha_{ij} = \frac{\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_{i_1} & x_1^{n-j-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_{i_2} & x_2^{n-j-2} & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_{i_n} & x_n^{n-j-2} & \cdots & 1 \end{vmatrix}}{\pm\sqrt{\Delta}} \qquad (j=0, 1, 2, 3, \cdots, n-1)$$

where Δ is the discriminant of $f(x)$, so that

$$\pm\sqrt{\Delta} = \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_n^{n-1} & x_n^{n-2} & \cdots & 1 \end{vmatrix} \neq 0.$$

We can also determine T_i by the Lagrangian interpolation formula

$$\phi_i(x) = \sum_{t=1}^{t=n} \frac{x_{i_t} f(x)}{(x-x_t) f'(x_t)}, \qquad f(x) = (x-x_1)(x-x_2)\cdots(x-x_n).$$

The coefficients of ϕ_i determined by either of these two methods are not necessarily contained in the general field F.

§ 3. *Condition for transformations with coefficients in F.*

THEOREM II. *The necessary and sufficient condition for the existence of the transformation T with coefficients in the field F on the roots of the equation $f(x)=0$ with coefficients in F is that the substitution S be permutable with every substitution of the Galois group of $f(x)=0$ for F.*

Let

$$S = \begin{pmatrix} x_t \\ x_{tS} \end{pmatrix} \qquad (t=1, 2, \cdots, n).$$

Determine $\phi(x)$ by means of one of the two methods given in section 2. We have the equations

$$(3) \qquad x_{tS} = \phi(x_t) \qquad (t=1, 2, 3, \cdots, n).$$

(1) Proof that condition is necessary. The coefficients of ϕ are in F by hypothesis. Hence we may apply to (3) the substitutions R of the Galoisian group.* Hence

$$x_{tSR} = \phi(x_{tR}).$$

But, by (3),

$$x_{tRS} = \phi(x_{tR}).$$

Hence $x_{tRS} = x_{tSR}$ for every t, and thus $RS = SR$.

(2) Proof that the condition is sufficient. By hypothesis, $RS = SR$ for every R in the Galoisian group.

Let $x_{tR} = x_p$. Then $x_{tSR} = x_{tRS} = x_{pS}$. Hence if R replaces x_t by x_p it replaces x_{tS} by x_{pS}. In § 2, $x_{1S}, \cdots, x_{nS}$ were denoted by $x_{i_1}, \cdots, x_{i_n}$. Hence if R replaces x_t by x_p, it replaces x_{i_t} by x_{i_p}. Hence the coefficients of ϕ given by equation (2) are unaltered by R and thus belong to F.

§ 4. *The representation of substitutions.*

The substitution

$$S_i \equiv \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{i_1} & x_{i_2} & \cdots & x_{i_n} \end{pmatrix}$$

can be represented by the transformation

$$T_i \equiv [x_t : x_{\phi_i(t)}],$$

where

$$\phi_i(t) = \sum_{j=0}^{j=n} \frac{i_j f(t)}{(t-j)f'(j)}, \qquad f(t) = (t-1)(t-2)\cdots(t-n).$$

We may also determine the coefficients of

$$\phi_i(t) = \sum_{j=0}^{j=n-1} \alpha_{ij} t^{n-1-j}$$

from the n linear equations

$$\phi_i(t) = i_t \qquad (t = 1, 2, 3, \cdots, n).$$

The results are

$$\alpha_{ij} = \frac{\begin{vmatrix} 1^{n-1} & 1^{n-2} & \cdots & i_1 & 1^{n-2-j} & \cdots & 1 & 1 \\ 2^{n-1} & 2^{n-2} & \cdots & i_2 & 2^{n-2-j} & \cdots & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n^{n-1} & n^{n-2} & \cdots & i_n & n^{n-2-j} & \cdots & n & 1 \end{vmatrix}}{\pm\sqrt{\Delta}} \qquad (j = 0, 1, 2, 3, \cdots, n-1),$$

* The theorems used here are known as properties A and B of the Galois group. See DICKSON, *Introduction to the theory of algebraic equations*, p. 53.

where

$$\pm\sqrt{\Delta} = \begin{vmatrix} 1^{n-1} & 1^{n-2} & \cdots & 1 & 1 \\ 2^{n-1} & 2^{n-2} & \cdots & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ n^{n-1} & n^{n-2} & \cdots & n & 1 \end{vmatrix} = \prod_{r=1}^{r=n-1} (n-r)!.$$

§ 5. *Special examples.*

1. Let $n = 3$ and

$$S_1 = (x_1 x_2 x_3), \qquad S_2 = (x_1 x_2).$$

Then

$$f(t) = t^3 - 6t^2 + 11t - 6, \qquad \phi_1(t) = -\tfrac{3}{2}t^2 + \tfrac{11}{2}t - 2,$$

$$\phi_2(t) = \tfrac{3}{2}t^2 - \tfrac{11}{2}t + 6.$$

These define the symmetric group on three letters.

2. Let $n = 4$ and

$$S_1 = (x_1 x_2 x_3 x_4), \qquad S_2 = (x_1 x_2)(x_3 x_4), \qquad S_3 = (x_1 x_2 x_3).$$

Then

$$f(t) = t^4 - 10t^3 + 35t^2 - 50t + 24, \qquad \phi_1(t) = -\tfrac{2}{3}t^3 + 4t^2 - \tfrac{19}{3}t + 5,$$

$$\phi_2(t) = -\tfrac{4}{3}t^3 + 10t^2 - \tfrac{65}{3}t + 15, \qquad \phi_3(t) = \tfrac{4}{3}t^3 - \tfrac{19}{2}t^2 + \tfrac{121}{6}t - 10.$$

These define the symmetric group on four letters.

§ 6. *Rational fractional transformations.*

The results of the previous articles can be extended to rational fractional transformations.

Consider a group G of transformations

$$T_i \equiv [x : \psi_i(x)],$$

where

$$\psi_i(x) = \frac{\phi_i(x)}{\theta_i(x)}, \qquad \phi_i(x) = \sum_{j=0}^{j=m_i} \alpha_{ij} x^{m_i - j}, \qquad \theta_i(x) = \sum_{j=0}^{j=n_i} \beta_{ij} x^{n_i - j}.$$

$\alpha_{i0} \neq 0$, $\beta_{i0} \neq 0$, while $\phi_i(x)$ and $\theta_i(x)$ have no common factor and at least one of the degrees m_i, n_i exceeds unity.

The coefficients α_{ij} and β_{ij} are elements of a general field F and the quantity x belongs to a set X in a field F'. As before, these transformations are associative and have the closure property. If T_i and $T_{i'}$ are inverses

$$T_i T_{i'} \equiv [x : x] = [x : \psi_i\{\psi_{i'}(x)\}]$$

and we have

$$\psi_i\{\psi_{i'}(x)\} = x \qquad (m_i n_i > 1).$$

This is either (*a*) *an equation of condition*, $f(x) = 0$, or (*b*) *an identity*.

(*a*) In this case the transformations reduce $[\bmod f(x)]$ to the integral form considered in the first part of the paper.*

(*b*) In this case,

$$y = \frac{\phi_i(x)}{\theta_i(x)} \quad \text{gives} \quad x = \frac{\phi_{i'}(y)}{\theta_{i'}(y)},$$

therefore to each y there is only one x and therefore $\phi_i(x)$ and $\theta_i(x)$ are linear. Case (*b*) is therefore excluded.

§ 7. *Representation of products of substitutions.*

Consider any k substitutions R_j of order r_j $(j = 1, 2, \cdots, k)$ on the n roots of $f(x) = 0$.

Take the products of powers of these substitutions of the form †

$$S_i = R_1^{y_1^{(i)}} R_2^{y_2^{(i)}} \cdots R_k^{y_k^{(i)}} \equiv \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_{i_1} & x_{i_2} & \cdots & x_{i_n} \end{bmatrix}.$$

The number of these products is

$$r = \prod_{j=1}^{j=k} r_j$$

and i will have the range $1, 2, \cdots, r$.

When the basic substitutions $R_j (j = 1, 2, \cdots, k)$ are given, S_i will be determined by the exponents $y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)}$.

It is possible to represent all these substitutions by the transformations

$$T_i \equiv [x : \phi(x; y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)})] \tag{4}$$

where ϕ is determined by the generalized Lagrangian interpolation formula

$$\phi(x; y_1, y_2, \cdots, y_k) = \sum_{l=1}^{l=n} \sum_{j=1}^{j=r} \frac{x_{j_l} f(x)}{(x - x_l) f'(x_l)} \prod_{p=1}^{p=k} \frac{\theta_p(y_p)}{(y_p - y_p^{(j)}) \theta_p'(y_p^{(j)})}$$

and

$$f(x) = \prod_{t=1}^{t=n} (x - x_t), \qquad \theta_p(y_p) = \prod_{s=1}^{s=r} (y_p - y_p^{(s)}).$$

When any particular set of y's as $(y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)})$ are substituted in the above it reduces to the regular Lagrangian formula and gives the $\phi_i(x)$ used in first part of this paper and therefore T_i. The function ϕ is a rational integral

* H. Weber, *Lehrbuch der Algebra*, vol. 1, p. 170.

† No two sets $(y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)})$ are alike but no assumption is made concerning the corresponding S_i.

function of x whose coefficients are rational integral functions of the k parameters $y_1, y_2, \cdots, y_k$. The numerical coefficients will be contained in the field F when S_i fulfills the conditions in Theorem II for every value of i.

Any set of substitutions S_i $(i = 1, 2, \cdots, r)$ where each substitution is characterized by a particular set of values $y_1^{(i)}, y_2^{(i)}, \cdots, y_k^{(i)}$ of the k parameters $y_1, y_2, \cdots, y_k$ can be represented by transformations T_i determined as above. *It is therefore possible to represent* an entire group of transformations by a single formula* (4).

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* Some of the transformations may be repeated.

ON OSCULATING ELEMENT-BANDS ASSOCIATED WITH LOCI OF SURFACE-ELEMENTS*

BY

PERCEY F. SMITH

In a memoir by SCHEFFERS, entitled *Isogonalcurven, Aequitangentialcurven und complexe Zahlen*, which appeared in volume 60 of the Mathematische Annalen (1905), p. 491, a number of theorems were demonstrated concerning properties of the osculating circles of certain loci of line-elements in the plane. Two of these may be cited here. Given a differential equation of the first order,

$$f(x, y, p) = 0, \tag{1}$$

in which $p = dy/dx$, and the one-parameter group

$$x = x, \qquad y = y, \qquad \operatorname{arc\,tan} p = \operatorname{arc\,tan} p' + t, \tag{2}$$

by which each line-element is turned about its point through the same angle, then the transformed elements satisfy the equation

$$f\left(x, y, \frac{p' + \tan\tau}{1 - p'\tan\tau}\right) = 0, \tag{3}$$

the integral curves of which are isogonal trajectories of the integral curves of the original equation (1). The first theorem in question is now this: The osculating circles of all the isogonal trajectories through a fixed point pass also through a second fixed point, that is, form a pencil. Furthermore, these circles will osculate at the second point another system of isogonal trajectories.

The second theorem is similar in character if the group (2) is replaced by

$$x = \bar{x} + \frac{t}{\sqrt{1 + p^2}}, \qquad y = \bar{y} + \frac{pt}{\sqrt{1 + p^2}}, \qquad p = p, \tag{4}$$

which has the effect of *sliding* each line-element along its line the same distance t. The integral curves of the transformed equation

$$f\left(\bar{x} + \frac{t}{\sqrt{1 + p^2}}, \quad \bar{y} + \frac{pt}{\sqrt{1 + p^2}}, \quad p\right) = 0 \tag{5}$$

are called by SCHEFFERS *aequitangential curves* of the original system (1). Then

* Presented to the Society, September 15, 1909, and February 26, 1910.

the theorem holds: The osculating circles of the aequitangential curves which touch a given line will touch also a second line. Furthermore, these circles will osculate on the second line a second system of aequitangential curves.

The discussion of the present paper leads to similar theorems for loci of surface-elements in space. Such loci are regarded as associations of *element-bands*, and either of two species of simple bands — parabolic and cubic — appear in the same role as osculating circles in the plane. The investigation is much simplified by means of transformations which make the problem depend upon projective geometry in space of five dimensions. This method has been employed to advantage by EIESLAND in a paper referred to below.

As far as the author is aware, the question of osculating element-bands is discussed for the first time in this paper.

§ 1. *Preliminary Theorems.*

Two united line-elements in the plane determine a unique curvature-element (x, y, y', y''). For, given two such elements (x, y, y') and $(x + dx, y + dy, y' + dy')$, where $dy = y'dx$, then if y'' is defined by the equation $dy' = y''dx$, it is clear that the ratios $dx : dy : dy'$ determine y'' uniquely.

The case is different, however, with two united surface-elements in space. For convenience we may adopt the notation (P, E) for such an element, P being the point and E the plane of the element. Assuming two united elements (P, E) and (P', E') whose coördinates are (x, y, z, p, q) and $(x + dx, y + dy, z + dz, p + dp, q + dq)$, where $dz = pdx + qdy$, then of the ratios

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{p\alpha + q\beta} = \frac{dp}{\delta} = \frac{dq}{\epsilon}, \tag{1}$$

three are independent. On introducing the numbers r, s and t defined by the equations

$$dp = rdx + sdy, \qquad dq = sdx + tdy, \tag{2}$$

or also, from (1),

$$\delta = r\alpha + s\beta, \qquad \epsilon = s\alpha + t\beta, \tag{3}$$

it is obvious that r, s, t are *not* determined uniquely when the ratios (1) are given, but that, in fact, we have the result: * *Two united surface-elements in space determine* ∞^1 *curvature-elements* (x, y, z, p, q, r, s, t).

Consider now any surface to which the united elements under discussion belong. The curvature of normal sections at P is, by the well-known formula, given by

$$\frac{1}{R} = \frac{1}{\sqrt{1 + p^2 + q^2}} \frac{rdx^2 + 2sdxdy + tdy^2}{ds^2}, \tag{4}$$

* ENGEL, *Die höheren Differentialquotienten*, Leipziger Berichte (1893), p. 475.

where ds is the linear element. If we substitute r and t in terms of s from (3), then (4) becomes

$$(5) \qquad \frac{1}{R} = \frac{1}{\sqrt{1+p^2+q^2}} \cdot \frac{\beta\delta dx^2 + \alpha\epsilon dy^2 - s(\beta dx - \alpha dy)^2}{\alpha\beta ds^2},$$

in which of course, for any given value of $dx : dy$, s is arbitrary. We observe, however, that the curvature is the same for all values of s when the normal section is such that $\beta dx - \alpha dy = 0$, that is, such that (1) is satisfied. The result may be expressed as follows: *The ∞^1 curvature-elements determined by two united surface-elements (P, E) and (P', E') hang together along the direction PP' in such a manner that the curvature at P of all normal sections containing PP' is the same.*

From equations (3), we readily find that

$$(6) \qquad \alpha\beta(rt - s^2) = -(\alpha\delta + \beta\epsilon)s + \delta\epsilon.$$

If, then, $\alpha\delta + \beta\epsilon \neq 0$, one of the curvature-elements has zero total curvature. On the other hand, when $\alpha\delta + \beta\epsilon = 0$, the total curvature of all the ∞^1 curvature-elements is the same.

Finally, denoting by $\delta x : \delta y$ the direction conjugate to $dx : dy$ on any surface containing the pair of elements under consideration, then the usual formula $rdx\delta x + s(dx\delta y + dy\delta x) + tdy\delta y = 0$ gives by substitution from (3), and on solving for $\delta x : \delta y$,

$$(7) \qquad \frac{\delta x}{\delta y} = -\frac{\alpha}{\beta}\,\frac{\epsilon dy + s(\beta dx - \alpha dy)}{\delta dx - s(\beta dx - \alpha dy)}.$$

Hence if $\beta dx - \alpha dy = 0$, the conjugate direction is the same for all values of s, namely, $\delta\delta x + \epsilon\delta y = 0$. In words: *On all surfaces containing two united surface-elements (P, E) and (P', E') the direction conjugate to PP' is the same.*

This fact is obvious geometrically. For this common conjugate direction is determined by the line of intersection of the planes E and E' of the united elements.

Evidently, if $\alpha\delta + \beta\epsilon = 0$, the conjugate direction is identical with PP', and the latter direction is therefore an asymptotic direction. This fact when combined with the conclusion found in discussing (6) gives the result: *If the total curvature of the ∞^1 curvature-elements (3) is the same, the direction along which they are united is an asymptotic direction.*

§ 2. *Parabolic and Cubic Bands.*

In the Differential Geometry of the plane, the circle plays a conspicuous rôle. This fact, when metric properties are left out of consideration, is explained by remarking that a circle is uniquely determined by two of its united elements. Plainly, a curve of any family depending upon three parameters will have this

last property — vertical parabolas, for example. The point is that we consider *unions of line-elements which are determined when two united elements are given.*

An analogous problem for space now arises. We wish to consider unions of ∞^1 surface-elements each one of which is determined if two united elements are given. Let the term *element-band* be used for a union of ∞^1 surface-elements. Analytically an element-band is defined by

$$(1) \quad x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad p = p(t), \qquad q = q(t),$$

provided that $z' = px' + qy'$, where accents indicate derivatives with respect to the parameter t.

Two species of element-bands, now to be described, appear in this paper. The points of the elements lie on a curve, which may be called the *point-locus,* while the planes form a developable, called the *plane-locus.* The two species of element-bands may be respectively named *parabolic bands* and *cubic bands.*

The point-locus of a parabolic band is a parabola whose axis is parallel to the Z-axis. The plane-locus is a parabolic cylinder. To conceive such a band we may think of such a parabola, which may be called a *vertical parabola* then imagine a line meeting the parabola to generate a cylinder, and finally picture to ourselves a narrow band of the cylindrical surface along the parabola. Clearly, ∞^2 parabolic bands may have a common point-locus or dually, the same plane-locus. To dwell for a moment on the last point: — the parabolic cylinders in question obviously contain the point at infinity on the Z-axis. Given such a cylinder, then any plane through the point referred to determines on the cylinder a parabolic band.

The point-locus of a cubic band is a skew cubic. The plane-locus is a quadric cone whose vertex lies on the cubic. Clearly there are ∞^1 cubic bands with a common point-locus. On the other hand, the multiplicity of the system of cubic bands belonging to a quadric cone is the same as that of the system of skew cubics lying on it, and is of no special importance at this moment. The concept of a cubic band is readily visualized if we think of a skew cubic, picture any quadric cone standing on it and think of a narrow band of the conical surface along the cubic.

The fact is now readily established that a parabolic band is determined by two of its united elements, (P, E) and (P', E'). For pass through PP' the plane parallel to the Z-axis. The united line-elements in this plane determined by its intersection with E and E' determine a vertical parabola. Further, the line of intersection of E and E' gives an element of the parabolic cylinder. The multiplicity of all parabolic bands is obviously seven-fold. For there are ∞^8 pairs of united elements and ∞^1 elements determine the same parabolic band. The important exceptional case already noted — when E and E' intersect along PP' — leads to a *twisted band.* Namely, the point-locus is now the

line PP' and the planes of the plane-locus pass through PP'. The points and planes of the elements of the band are in a projective relation, such that P and E, P' and E' are corresponding elements, and moreover, the point at infinity on PP' corresponds to the vertical plane through PP'.

Turn next to the question of the determination of a cubic band by a pair of united surface-elements. From the results of § 1, it is clear that the vertex of any cone to which the united elements (P, E) and (P', E') belong, must lie on the line of intersection of E and E'. For this direction is conjugate to PP', and for the present we assume these directions are not identical. Denote the line of intersection of E and E' by (E, E'). Let us now confine ourselves to quadric cones. Consider the normal section of such a cone through the direction PP'. Then by § 1, the curvature of this conic at P is given. Consequently three conditions are imposed upon this conic. It is therefore clear that ∞^3 quadric cones may contain the same pair of united elements. The cones touch along a common element (E, E') and in addition, their normal sections through PP' osculate at P. To obtain a unique cone of this system we may introduce the three fixed conditions — *the cones shall pass through three non-collinear fixed points A, B and C.* It is easy to see that the required cone is now uniquely determined. For, in the first place, its trace c' on the plane of A, B and C is determined. Secondly, the normal section through PP' now has *two* points given on this trace c', and hence is a unique conic c''. Only one cone with a *given* element can contain two conics.

We see, therefore, that a pair of united surface-elements (P, E), (P', E') will determine a unique quadric cone K standing on three fixed points A, B, C. Let the vertex of this cone be V. Consider the quadric cone whose vertex is A which stands on the five points P, P', V, B and C. Its intersection with the first cone is a skew cubic C_3 and the element VA. Similarly each of the fixed points B and C is the vertex of a quadric cone which will contain the *same* cubic. The cubic band lying on K along C_3 contains (P, E) and (P', E') and furthermore is uniquely determined by them.

The multiplicity of skew cubics passing through three non-collinear fixed points is six-fold. There are accordingly ∞^7 cubic bands, ∞^1 on each of the skew cubics of the system. On the other hand, on each quadric cone standing on the fixed points A, B and C lie ∞^2 cubic bands, and the multiplicity of such cones is five-fold.

If the line of intersection of the planes E and E' is identical with PP', the cubic band is constructed by assuming E as the osculating plane at P, and PP', of course, as the tangent. The cubic satisfying this condition and moreover passing through A, B, and C is unique. The cone K then has its vertex at P.

§ 3. *Osculating Bands.*

Two element-bands having in common two united elements may be said to *osculate.* The analytical conditions are easily derived. For given two bands

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad p = \phi(t), \qquad q = \chi(t),$$

and

$$x = f_1(t), \qquad y = g_1(t), \qquad z = h_1(t), \qquad p = \phi_1(t), \qquad q = \chi_1(t),$$

then osculation demands the equality of f and f_1, g and g_1, etc., and their first derivatives for some value of t. In these conditions, however, the relation $h' = h_1'$ depends upon the others from the conditions for bands, (1), § 2.

Geometrically, two element-bands osculate when their point-loci touch and their plane-loci are tangent along a common generator.

The immediate subject of investigation concerns properties of surface-element loci with respect to osculating parabolic or cubic bands. It is clear that there are ∞^1 bands of this character associated with any element-band. Further, given any surface or union of ∞^2 elements, then ∞^3 osculating parabolic or cubic bands are determined. Of particular interest, however, is the case when the surface-element locus is defined by a partial differential equation of the first order

$$f(x, y, z, p, q) = 0. \tag{1}$$

The ∞^4 elements satisfying (1) are arranged on ∞^3 bands, the characteristic bands. If the coördinates of two united elements of a characteristic band are (x, y, z, p, q) and $(x + dx, y + dy, z + dz, p + dp, q + dq)$, then these coördinates of course satisfy $f = 0$ and further the well-known equations due to Lie,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}. \tag{2}$$

Each element satisfying (1) is associated with a united element determined by (2) such that the pair of united elements lie on a characteristic band. Now, clearly, the parabolic or cubic band determined by this pair of elements will osculate the characteristic band. Thus at each element of the locus (1), there is determined a parabolic or cubic band which osculates the characteristic band containing the given element. There are ∞^4 osculating parabolic or cubic bands. If, now, we look upon the locus of (1) not as a set of integral surfaces, but as an association of ∞^3 element-bands, the characteristic bands, then the properties of osculating parabolic or cubic bands appear a natural subject for investigation. The ∞^4 parabolic or cubic bands which osculate the characteristic bands of a given differential equation of the first order we shall refer to briefly as the osculating parabolic or cubic bands of the locus of the equation.

§ 4. *Transformation of the Problem.*

The investigation of the question proposed in the preceding section is greatly facilitated by the aid of transformations by which the geometry of surface-elements is made to depend upon projective geometry in a space of five dimensions R_5. Such a transformation has been given by LIE.* A second transformation is exhibited in § 7 of this paper. The capital point is that parabolic or cubic bands correspond by these respective transformations to straight lines in R_5. To disclose the details, we begin by examining the transformation already referred to as due to Lie. This is defined by the equations

$$(1)\quad X_1 = x, \qquad X_2 = y, \qquad Y_1 = \tfrac{1}{2}p, \qquad Y_2 = \tfrac{1}{2}q, \qquad Z = z - \tfrac{1}{2}px - \tfrac{1}{2}qy,$$

by which identically

$$(2)\qquad dZ + X_1 dY_1 - Y_1 dX_1 + X_2 dY_2 - Y_2 dX_2 = dz - pdx - qdy.$$

If we interpret (X_1, X_2, Y_1, Y_2, Z) as cartesian point coördinates in R_5, then clearly united surface-elements in ordinary space R_3 transform into *line-elements* in R_5 satisfying the equation

$$(3)\qquad dZ + X_1 dY_1 - Y_1 dX_1 + X_2 dY_2 - Y_2 dX_2 = 0.$$

Now equation (3) is the differential equation of the *null-system*,†

$$(4)\qquad Z - Z' + Y_1 X_1' - X_1 Y_1' + Y_2 X_2' - X_2 Y_2' = 0,$$

that is, of a correlation in R_5 such that point and corresponding linear M_4 are *incident.*

A line in R_5,

$$(5)\quad X_1 = X_1^0 + \lambda_1 t,\ X_2 = X_2^0 + \lambda_2 t,\ Y_1 = Y_1^0 + \mu_1 t,\ Y_2 = Y_2^0 + \mu_2 t,\ Z = Z^0 + \nu t,$$

whose elements satisfy (3), that is, such that

$$(5')\qquad \nu + \mu_1 X_1 - \lambda_1 Y_1 + \mu_2 X_2 - \lambda_2 Y_2 = 0,$$

is a *null-line.* There are plainly ∞^7 null-lines.

By this transformation (1), the null-line in R_5 becomes in R_3 an element-band. This is a parabolic band, as shall now appear. Transforming (5), we obtain for the equations of the corresponding band, using (5′),

$$(6)\quad \begin{gathered} x = x^0 + \lambda_1 t, \qquad y = y^0 + \lambda_2 t, \qquad z = z^0 + (\lambda_1 p^0 + \lambda_2 q^0)t + (\lambda_1\mu_1 + \lambda_2\mu_2)t^2, \\ p = p^0 + 2\mu_1 t, \qquad q = q^0 + 2\mu_2 t. \end{gathered}$$

The point-locus of this band is, in the general case, a vertical parabola.‡ The

* LIE-ENGEL, *Transformationsgruppen*, vol. 2, p. 521.

† For details, see EIESLAND, *On Null-Systems in Space of Five Dimensions*, etc., American Journal of Mathematics, vol. 26 (1904), p. 103.

‡ EIESLAND, l. c., p. 120.

plane-locus is the envelope of

$$z' - z = p(x' - x) + q(y' - y), \tag{7}$$

in which x, y, z, p, q, have the values in (6). On substituting their values in (7), the equation takes the form

$$(\lambda_1\mu_1+\lambda_2\mu_2)t^2-2[\mu_1(x'-x^0)+\mu_2(y'-y^0)]t+z'-z^0-p^0(x'-x^0)-q^0(y'-y^0)=0. \tag{8}$$

The envelope of (8) is clearly, in the general case, a parabolic cylinder. The band (6) is accordingly a parabolic band. Certain exceptional cases are necessary for the following discussion.

1. If $\lambda_1 = \lambda_2 = 0$, then (6) becomes $x = x^0$, $y = y^0$, $z = z^0$. That is, the point-locus is a point. The ∞^1 surface-elements now form a *hinge*, for the planes (8) form a pencil.

2. If $\lambda_1\mu_1 + \lambda_2\mu_2 = 0$ and $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, the point-locus is a line and the plane-locus a plane-pencil. The band is now a *twisted band*, the points and planes of the element forming a projective series.

3. If $\mu_1 = \mu_2 = 0$, the band just discussed becomes *flat*, that is, a straight band on a plane.

It is clear from these results that the contact geometry of parabolic bands in R_3 becomes under the transformation (1) the *projective* geometry of a null-system. The group of the latter is easily found to depend upon 21 parameters.*

In later sections it will be shown that the null-lines (5) are by another transformation changed into cubic bands, and finally that parabolic and cubic bands correspond under a contact transformation.

§ 5. *Theorems on the Null-System in R_5.*

Certain theorems regarding the null-system

$$Z - Z' + Y_1X'_1 - X_1Y'_1 + Y_2X'_2 - X_2Y'_2 = 0, \tag{1}$$

necessary in the sequel, are discussed in this section.

The plane in R_5

$$Y_1 = a_1X_1+a_2X_2+a_3, \qquad Y_2 = b_1X_1+b_2X_2+b_3, \qquad Z = c_1X_1+c_2X_2+c_3 \tag{2}$$

will contain the null-line (5), § 4, if in addition to the obvious conditions that the point (X^0_1, X^0_2, etc.) lies on (2), we have

$$\mu_1 = a_1\lambda_1 + a_2\lambda_2, \qquad \mu_2 = b_1\lambda_1 + b_2\lambda_2, \qquad \nu = c_1\lambda_1 + c_2\lambda_2. \tag{3}$$

On substituting in the condition (5′), § 4, for a null-line, the single condition is obtained

$$[c_1 - a_3 + (b_1 - a_2)X^0_2]\lambda_1 + [c_2 - b_3 + (a_2 - b_1)X^0_1]\lambda_2 = 0. \tag{4}$$

* EIESLAND, l. c., p. 142.

The following conclusions are now apparent:

1. Through an arbitrary point in an arbitrary plane in R_5 there is one null-line.

2. *All* lines in the plane (2) through the point determined by

$$X_1^0 = \frac{b_3 - c_2}{a_2 - b_1}, \qquad X_2^0 = \frac{c_1 - a_3}{a_2 - b_1}, \tag{5}$$

are null-lines. This point is the *null-point* of the plane.

3. If $a_2 = b_1$, $b_3 = c_2$, $c_1 = a_3$, then *every* line in the plane is a null-line. The plane (2) is now a *null-plane.* The conditions obviously demand *symmetry* in the determinant of the right-hand members in (2). Explicitly, the equations (2) take the form, if the plane is a null-plane,

$$Y_1 = a_1 X_1 + a_2 X_2 + a_3, \qquad Y_2 = a_2 X_1 + b_2 X_2 + c_2, \qquad Z = a_3 X_1 + c_2 X_2 + c_3. \tag{6}$$

If we next inquire how many null-planes contain a given null-line, the answer is immediate, namely, *each null-line is the axis of a pencil of* ∞^1 *null-planes.*

Clearly, all planes in R_5 cannot be represented in the form (2). The two exceptional cases are, for null-planes, given by

$$X_1 = a, \qquad X_2 = b, \qquad Z = c_3 - aY_1 - bY_2, \tag{7}$$

and

$$X_1 = aX_2 + b, \qquad Y_2 = cX_2 - aY_1 + e, \qquad Z = eX_2 - bY_1 + h. \tag{8}$$

We may, in general, define a null-surface* in R_5 as one whose tangent planes are null-planes.

Turning now to the locus of *one* equation

$$F(X_1, X_2, Y_1, Y_2, Z) = 0, \tag{9}$$

defining in R_5 a manifoldness M_4 of four dimensions, the linear *tangent* M_4^1 is given as usual by

$$\begin{aligned} F_{X_1}(X_1' - X_1) + F_{X_2}(X_2' - X_2) + F_{Y_1}(Y_1' - Y_1) \\ + F_{Y_2}(Y_2' - Y_2) + F_Z(Z' - Z) = 0. \end{aligned} \tag{10}$$

The null-plane (6) will pass through the point of contact (X_1, X_2, etc.) and lie entirely within the locus of (10) under these conditions:

$$a_3 = Y_1 - a_1 X_1 - a_2 X_2, \qquad c_2 = Y_2 - a_2 X_1 - b_2 X_2, \qquad c_3 = Z - a_3 X_1 - c_2 X_2, \tag{11}$$

$$F_{X_1} + a_1 F_{Y_1} + a_2 F_{Y_2} + a_3 F_Z = 0, \qquad F_{X_2} + a_2 F_{Y_1} + b_2 F_{Y_2} + c_2 F_Z = 0. \tag{12}$$

There is no difficulty in showing that the ∞^1 null-planes satisfying (11) and (12) form a *pencil* whose axis is the null-line for which

$$\frac{dX_1}{F_{Y_1} - X_1 F_Z} = \frac{dX_2}{F_{Y_2} - X_2 F_Z} = \frac{dY_1}{-F_{X_1} - Y_1 F_Z} = \frac{dY_2}{-F_{Y_2} - Y_2 F_Z} = \frac{dZ}{\Sigma X F_X}, \tag{13}$$

* For a development of this idea, see EIESLAND, l. c., p. 128.

the denominator $\Sigma X F_X$ containing the four terms corresponding to X_1, X_2, Y_1, Y_2. We may express this important result as follows:

At each point of an M_4 in R_5 there is a pencil of tangent null-planes whose axis is the null-line (13).

The null-M_4^1 of P contains every null-plane through P. Hence the result just found may be formulated thus:

The tangent M_4^1 and null-M_4^1 of P intersect in a pencil of null-planes.

We shall call the axis of this pencil the *tangent null-axis at P*.

The result just stated is fundamental for the sequel. It is clear that the tangent null-axis at any point on the locus of (9) is distinguished from all other tangent null-lines, of which there are of course ∞^2.

Returning to the plane (2), we may represent this for the moment by the symbol (abc) of the determinant on the right-hand members. As already pointed out, if (abc) is symmetrical, the plane is a null-plane. In the null-system (1), let the point (X_1, X_2, etc.) describe the plane (abc). Then it appears at once that the ∞^2 corresponding M_4^1's intersect in a plane *whose symbol is the conjugate of* (abc). Such planes we may call *conjugate planes* in the null-system. Clearly, a null-plane is *self-conjugate.*

In the null-system (4), to every point corresponds a linear M_4. Dually, to a given linear M_4 will correspond a point, the null-point of the M_4. This fact leads to the determination of the locus of the null-point of the linear tangent M_4 given by (10). To find this null-point $P_0(X_1^0, X_2^0, Y_1^0, Y_2^0, Z^0)$, we merely have to compare equation (10) with the equation of the null-system

$$Z - Z^0 + Y_1 X_1^0 - X_1 Y_1^0 + Y_2 X_2^0 - X_2 Y_2^0 = 0. \tag{14}$$

The result is readily found to be

$$\begin{gathered} X_1^0 = \frac{F_{Y_1}}{F_Z}, \qquad X_2^0 = \frac{F_{Y_2}}{F_Z}, \qquad Y_1^0 = -\frac{F_{X_1}}{F_Z}, \qquad Y_2^0 = -\frac{F_{X_2}}{F_Z}, \\ Z^0 = Z + \frac{X_1 F_{X_1} + X_2 F_{X_2} + Y_1 F_{Y_1} + Y_2 F_{Y_2}}{F_Z}. \end{gathered} \tag{15}$$

When the point $P(X_1, X_2, Y_1, Y_2, Z)$ describes the locus of (9), the point P^0, which we call the *conjugate point*, will describe a locus, which may be named the *conjugate locus.* In the general case, the conjugate locus will be of four dimensions, say $F^0 = 0$. Evidently, from the definition of the conjugate locus, it is derived from the original locus by the correlation established in R_5 by the null-system. In this correlation each null-plane is invariant. Hence the pencil of null-planes tangent to $F = 0$ at P must have the same relation to the conjugate locus at P^0. In other words, the relation between two conjugate loci is *mutual.* The linear tangent M_4 and the null-M_4^1 at P to $F = 0$ are respectively

the null-M_4^1 and linear tangent M_4 to $F^0 = 0$ at P^0. From the theorem on tangent null-planes at P, we now derive the important

THEOREM. *If the loci of $F = 0$ and $F^0 = 0$ are conjugate under a null-system in R_5, and if P and P^0 are corresponding points on these loci, then the ∞^1 null-planes tangent to $F = 0$ at P are tangent also to $F^0 = 0$ at P^0. Further, the tangent null-axes at P and P^0 are identical.*

Analytically, the general case arises when the tangent M_4^1 (10) contains *four* independent parameters. The null-point P_0 will then describe a four dimensional locus. If, then, the number of independent parameters in (10) is less than four, the locus of P_0 will be of three dimensions, or of two (a surface) or of one (a curve). All these cases will be covered if we consider the "envelope" of the linear M_4,

$$E \equiv A_1 X_1 + A_2 X_2 + B_1 Y_1 + B_2 Y_2 + CZ = 0, \tag{17}$$

in which the coefficients are functions of certain parameters in number less than four. For example, consider the case of two parameters u and v. Taking the first partial derivatives of (17) with respect to these parameters,

$$E_u = 0 \quad \text{and} \quad E_v = 0, \tag{18}$$

we see that the tangent null-axes at all the points in the plane $E = E_u = E_v = 0$ pass through a common point, the null-point of $E = 0$. Thus the ∞^4 tangent null-axes form in this case ∞^2 bundles. The two other special cases mentioned are clearly those in which the tangent null-axes form ∞^3 flat pencils or ∞^1 three-dimensional *fascicules*. In the former of these two cases, that of three parameters, the points of tangency of the axes of a pencil lie on a line.

§ 6. *Geometry of Osculating Parabolic Bands.*

The results found in the previous section are now to be interpreted in space of three dimensions by means of Lie's transformation used in § 4,

$$X_1 = x, \qquad X_2 = y, \qquad Y_1 = \tfrac{1}{2}p, \qquad Y_2 = \tfrac{1}{2}q, \qquad Z = z - \tfrac{1}{2}px - \tfrac{1}{2}qy. \tag{1}$$

We obtain first some preliminary theorems. To *any* line in R_5 not a null-line, will correspond in R_3 a surface-element locus, for which

(a) the point-locus is a vertical parabola,

(b) the plane-locus is a parabolic cylinder.

These results follow directly as in § 4, the distinction being that the condition for a null-line simply implies that the point-locus is on the plane-locus.

To any plane in R_5 [(2), § 5] will correspond in R_3 a locus of ∞^2 elements, with the properties

(a) the point-locus is a vertical paraboloid,

(b) the plane-locus is a second vertical paraboloid.

In fact, the plane

$$(2)\quad Y_1 = a_1 X_1 + a_2 X_2 + a_3, \qquad Y_2 = b_1 X_1 + b_2 X_2 + b_3, \qquad Z = c_1 X_1 + c_2 X_2 + c_3,$$

gives under (1) as the point-locus,

$$(3)\qquad z = a_1 x^2 + (a_2 + b_1)xy + b_2 y^2 + (a_3 + c_1)x + (b_3 + c_2)y + c_3.$$

The statement (b) leads to a simple envelope problem.

The paraboloid (3) is the point-locus of ∞^3 assemblages of ∞^2 surface elements, namely, for all such that $a_2 + b_1$, $a_3 + c_1$, and $b_3 + c_2$ are constant. One and only one of these assemblages is a union, namely, when $a_2 = b_1$, $a_3 = c_1$, $b_3 = c_2$, and in this case, the union consists of the elements of the paraboloid. Remarks of like character apply to the plane-locus.

The theorems 1 and 2 on page 309 enable us to state these properties of the point-locus or the plane-locus: *The ∞^2 surface-elements are arranged in ∞^1 parabolic bands having a common element which belongs to both point-locus and plane-locus. The point-locus and the plane-locus touch on this element. Two conjugate planes in R_5 have the same point-loci and the same plane-loci.*

To the *null-plane*

$$(4)\quad Y_1 = a_1 X_1 + a_2 X_2 + a_3, \qquad Y_2 = a_2 X_1 + b_2 X_2 + c_2, \qquad Z = a_3 X_1 + c_2 X_2 + c_3$$

correspond the ∞^2 surface-elements of the paraboloid

$$(5)\qquad z = a_1 x^2 + 2a_2 xy + b_2 y^2 + 2a_3 x + 2c_2 y + c_3.$$

If $a_1 b_2 = a_2^2$, the locus of (5) is a cylinder. That is, the ∞^2 parabolic bands of the locus have now the *same plane-locus.* Obviously, equation (5) gives also all planes as a special case.

The special null-planes (7) and (8) of § 6 yield in R_3 the following results. The null-plane

$$(6)\qquad X_1 = a, \qquad X_2 = b, \qquad Z = aX_1 + bX_2 + c,$$

gives the ∞^2 elements of the point (a, b, c). To the null-plane

$$(7)\quad X_1 = aX_2 + b, \qquad Y_2 = cX_2 - aY_1 + e, \qquad Z = eX_2 - bY_1 + h,$$

correspond the ∞^2 elements of the vertical parabola *

$$(8)\qquad x = ay + b, \qquad z = cy^2 + 2ey + h.$$

The discussion shows that the ∞^2 surface elements of a point or a plane, or of a vertical parabola, or of a parabolic cylinder, correspond to the points of a plane in R_5.

In the previous section it has been shown that a null-line is the axis of a pencil of null-planes. The corresponding result for space of three dimensions is this:

* EIESLAND, l. c., p. 121.

A parabolic band lies on ∞^1 vertical paraboloids.

The connection of these facts with the discussion of § 1 is simple. The point (X_1, X_2, etc.) in R_5 gives a surface-element in R_3. A consecutive point on a null-line in R_3 gives a united element. Through the point-pair in R_5 pass ∞^1 null-planes. The point (X_1, X_2, etc.) and each null-plane determine a curvature-element in R_5. For clearly, in addition to x, y, z, p, q determined by (1), we now have given in (5), $r = \frac{1}{2}a_1$, $s = \frac{1}{2}a_2$, $t = \frac{1}{2}b_2$. Thus the ∞^1 null-surface-elements in R_5 having a common null-axis correspond to ∞^1 curvature-elements with a common pair of united surface-elements.

The null-plane in R_5 transforms into an element union in R_3. More generally, any null-surface will transform into an element union — surface or curve.

The locus of

$$F(X_1, X_2, Y_1, Y_2, Z) = 0 \tag{9}$$

transforms into the partial differential equation of the first order

$$f(x, y, z, p, q) = 0. \tag{10}$$

The partial derivatives of (9) transform as follows:

$$\begin{gathered} F_{X_1} = f_x + \tfrac{1}{2}pf_z, \qquad F_{X_2} = f_y + \tfrac{1}{2}qf_z, \qquad F_Z = f_z, \\ F_{Y_1} = 2f_p + xf_z, \qquad F_{Y_2} = 2f_q + yf_z. \end{gathered} \tag{11}$$

Any null-surface lying entirely in (9) transforms into an integral surface of (10).

Now consider the pencil of tangent null-planes to (9) at any point, and the corresponding null-axis. Each of the ∞^1 corresponding vertical paraboloids will have in common with (10) a curvature-element, and the ∞^1 curvature elements thus arising will hang together on two united surface-elements. Hence *the null-axis transforms into a parabolic band which osculates a characteristic band.* This result is verified by transformation of the equations (13), § 5. For by the equations of the transformation (1) and (11) we obtain immediately,

$$\frac{dx}{2f_p} = \frac{dy}{2f_q} = \frac{\frac{1}{2}dp}{-f_x - pf_z} = \frac{\frac{1}{2}dq}{-f_y - qf_z} = \frac{dz}{2(pf_p + qf_q)},$$

that is, the equations (2) of § 3.

We may at once write down the equations of the parabolic band osculating the characteristic band of (10) at (x, y, z, p, q). They are ((6), § 4), since λ_1, λ_2, μ_1, μ_2 now satisfy (13) of § 5,

$$\begin{aligned} &x' = x + 2f_p t, && y' = y + 2f_q t, \\ &p' = p - 2(f_x + pf_z)t, && q' = q - 2(f_y + qf_z)t, \\ &z' = z + 2(pf_p + qf_q)t - 2[f_x f_p + f_y f_q + f_z(pf_p + qf_q)]t^2. \end{aligned} \tag{12}$$

The exceptional cases noted on page 308 lead to the following results.

1. If $f_p = f_q = 0$, the osculating parabolic bands are *hinges*. The differential equation now contains no partial derivatives and its locus consists of all surface-elements of the points of the surface $f(x, y, z) = 0$.

2. If $f_x f_p + f_y f_q + f_z(p f_p + q f_q) = 0$ by virtue of $f = 0$, the osculating bands are twisted bands. The characteristic curves are now asymptotic lines on all integral surfaces of $f(x, y, z, p, q) = 0$. In fact, as explained in § 1, the direction PP' of the united elements of the band is now an asymptotic direction.

3. If $f_x + p f_z = f_y + q f_z = 0$, that is, if the differential equation is of the form $f(z - px - qy, p, q) = 0$, the osculating bands are flat.

Returning to 2, we see that the fact that the osculating bands are twisted states a *necessary and sufficient condition* that the characteristic curves shall be asymptotic lines on all integral surfaces. In this case, Lie has shown that the point-loci of the ∞^4 bands reduce to ∞^3 lines, that is, ∞^1 bands have the *same* point-loci. The general question may be asked:— *when do the ∞^4 osculating parabolic bands arrange themselves in ∞^3 systems of ∞^1 bands having a common point-locus?* This general problem includes the case just discussed.

If we write the equations (12) in the form

$$x = ay + b, \qquad z = cy^2 + 2ey + h,$$

then

$$(13) \quad \begin{aligned} &a \equiv f_p \div f_q, \qquad b = x - ay, \qquad c = -\tfrac{1}{2}\frac{f_x f_p + f_y f_q + f_z(p f_q + q f_q)}{f_q^2}, \\ &e = \tfrac{1}{2}(pa + q), \qquad h = z - cy^2 - 2ey. \end{aligned}$$

The ∞^4 elements satisfying $f = 0$ lead by these formulas to the point-loci of the osculating parabolic bands. The analytic question is, therefore, when does the elimination of x, y, z, p, q from (13) and $f = 0$ lead to more than one equation in the a, b, c, e, h? Without answering this question in all generality, it is not difficult to show that any one of the following is a *sufficient* condition:—

1. Either a or c constant;
2. A functional relation between a and c.

An interesting case under 2 arises when

$$(13a) \qquad c = k(1 + a^2).$$

The ∞^4 osculating parabolic bands now lie on ∞^3 *congruent* vertical parabolas. The characteristic curves on all integral surfaces have the property expressed by the equation

$$(13b) \qquad dp\,dx + dq\,dy = k(dx^2 + dy^2).$$

Conversely, the condition (13b) is sufficient for the property of the osculating vertical bands (13a). Evidently $k = 0$ gives the case 2.

The equations (15) of the previous section become in space of three dimensions the transformation

$$
(14)\qquad \begin{aligned} x' &= x + 2\frac{f_p}{f_z}, \qquad y' = y + 2\frac{f_q}{f_z}, \qquad p' = -p - 2\frac{f_x}{f_z}, \\ q' &= -q - 2\frac{f_y}{f_z}, \qquad z' = z - 2\frac{f_p f_x + f_q f_y}{f_z^2}. \end{aligned}
$$

Given any element of the locus of $f = 0$, equations (14) define an element (x', y', z', p', q'), which we call the *conjugate element.* Elimination of x, y, z, p and q from $f = 0$ by means of (14) will lead to the locus of the conjugate element—called the *conjugate locus.* In the general case, this elimination will give a single equation

$$
(15)\qquad f'(x', y', z', p', q') = 0.
$$

The theorem of the preceding section now gives the interesting result.

THEOREM. *Given a partial differential equation of the first order. The* ∞^4 *parabolic bands which osculate the characteristic bands will in general osculate also the characteristic bands of a second partial differential equation of the first order which may be found by differentiation and elimination.*

The geometric relation between conjugate equations is simply this: The elements are grouped in conjugate pairs which belong to a parabolic band osculating the characteristic bands of both equations. The given equation may, of course, be *self-conjugate,* that is, $f = 0$ and $f' = 0$ may be identical. The equation $ff' = 0$, where $f = 0$ and $f' = 0$ are conjugate, offers a case of this kind.

If the characteristic curves are asymptotic lines on all integral surfaces of $f = 0$, that is, if the osculating parabolic bands are *twisted bands,* then the characteristics of $f' = 0$ have the same property. More generally, if the property (15) is possessed by the characteristics of $f = 0$, it holds also for those of $f' = 0$.

Referring to the equations of the osculating band (12) we see that the conjugate element (14) is obtained by setting the parameter t equal to $1 \div f_z$.

The preceding discussion assumes that $f_z \neq 0$, that is, that the given equation (10) contains z. If this is not the case and if one of the other variables x or y appears in the equation, say x, we merely need to write in (10) $p = 1 \div \partial x/\partial z$, $q = -\partial x/\partial y \div \partial x/\partial z$, and then interchange x and z. The case when neither x nor y nor z appears is excluded.

The exceptions to the theorem, p. 311, are readily characterized by the aid of the results given at the close of the preceding section. Transforming (16) of that section by (1) gives an equation which may be written

$$
(16)\qquad A_1 x + A_2 y + B_1 p + B_2 q + C(z - \tfrac{1}{2}px - \tfrac{1}{2}qy) + D = 0.
$$

in which the coefficients A_1, A_2, etc., are functions of one, two or three parameters. The exceptional differential equations result, then, by forming first partial derivatives of (16) with respect to the parameters and then eliminating these, using (16). The following properties of the osculating parabolic bands characterize these differential equations.

Three parameters. The ∞^4 parabolic bands osculating the characteristic bands form ∞^3 families of ∞^1 each. The ∞^1 bands of a family osculate the characteristic bands at points lying on a vertical parabola and have a common surface-element. The latter, ∞^3 in all, constitute the conjugate locus.

Two parameters. The ∞^4 osculating parabolic bands now form ∞^2 families of ∞^2 each of this sort: the ∞^2 bands of a family have a common surface-element and osculate the characteristic bands at points lying on a vertical paraboloid. The conjugate locus consists of the ∞^2 common surface-elements of each family.

One parameter. The ∞^4 osculating parabolic bands are arranged in ∞^1 families of ∞^3 each such that all of one family have a common surface-element. The conjugate locus is made up of these latter elements, ∞^1 in all.

The facts here assembled follow at once from the conclusions at the end of § 5 and the results established in the present section.

An interesting theorem in regard to any locus of ∞^2 surface-elements

$$(17) \qquad x = x(t, v), \qquad y = y(t, v), \qquad \cdots, \qquad q = q(t, v)$$

is obtained by the following considerations. The corresponding locus in R_5 is a surface S. Consider the tangent plane to this surface S at a point P. Through P and the null-point [(5), § 5] P' of the tangent plane passes a null-line. Thus at each point of S we determine a tangent null-line, and hence on S a system of ∞^1 *null-curves.* Each null-curve corresponds to an element band satisfying (17). Hence the

THEOREM. *Every locus of ∞^2 surface-elements in space may be arranged in ∞^1 bands.*

The locus of the null-point P' is a surface S' (if S is not a developable). Moreover, it is easily seen that the osculating plane of the null-curve at P is the tangent plane to S' at P'. Further P is the null-point of this plane. Hence the relation between S and S' is mutual. *The ∞^2 null-lines touching the null-curves on S touch also the null-curves on S'.* This result gives for space the

THEOREM. *Given an assemblage A of ∞^2 surface-elements in space. The ∞^1 bands in which these may be arranged determine ∞^2 osculating parabolic bands which in general osculate also the ∞^1 bands of a second assemblage A' of ∞^2 surface-elements.*

It appears, then, that assemblages of ∞^2 surface-elements in space are arranged in *conjugate pairs*, as the relation of A and A' may be named. If A is a union, then A' is identical with A.

The equations of A' are given directly by (17) when we substitute from (17) in the right-hand members. An exception to the theorem arises when S is a developable. The osculating parabolic bands are now arranged in ∞^1 families of ∞^1 each, the bands of a family having a common element and osculating the bands of the assemblage A in points lying on a vertical parabola. The assemblage A' consists of a locus of ∞^1 elements.

In preparation for certain final theorems in this section we now change the surface-element coördinates as follows. Write the equation of the plane of the element thus:

$$Z = uX + vY - w, \tag{18}$$

and take for the new coördinates of (P, E), u, v, w, α, β, the latter satisfying $dw - \alpha du - \beta dv = 0$. Then we readily find the relations

$$u = p, \qquad v = q, \qquad w = z - px - qy, \qquad \alpha = -x, \qquad \beta = -y, \tag{19}$$

between the old and new element-coördinates. The equations (12) of the parabolic band now become, if we assume that by (19),

$$f(x, y, z, p, q) = g(u, v, w, \alpha, \beta), \tag{20}$$

as follows:

$$\begin{aligned} u' &= u + 2g_\alpha t, & v' &= v + 2g_\beta t, \\ \alpha' &= \alpha - 2(g_u + \alpha g_w)t, & \beta' &= \beta - 2(g_v + \beta g_w)t, \\ w' &= w + 2(\alpha g_\alpha + \beta g_\beta)t - 2[g_u g_\alpha + g_v g_\beta + g_w(\alpha g_\alpha + \beta g_\beta)]t^2. \end{aligned} \tag{21}$$

The equations of the osculating parabolic band have, therefore, precisely the same form in the new coördinates. Furthermore, the same remark applies to equations (17) giving the coördinates of the conjugate element.

Let us now turn our attention to the question of *trajectory-bands* of a given differential equation

$$f(x, y, z, p, q) = 0. \tag{22}$$

Given a two-parameter group

$$x = x, \qquad y = y, \qquad z = z, \qquad p = \phi(x, y, z, \bar{p}, \bar{q}; t_1, t_2),$$
$$q = \chi(x, y, z, \bar{p}, \bar{q}; t_1, t_2),$$

by which the element (P, E) is turned about its point, then substitution in (22) leads to the transformed equation

$$\bar{f}(x, y, z, \bar{p}, \bar{q}; t_1, t_2) = 0, \tag{23}$$

the characteristic bands of which for constant values of t_1 and t_2 may be called *trajectory-bands* of $f = 0$.

The case of interest in this connection is associated with the simple transformation of the above form when

$$p = \bar{p} + t_1, \qquad q = \bar{q} + t_2. \tag{24}$$

Referring to the formulas (14) for the conjugate element, we easily see that the conjugate elements of (x, y, z, p, q) and all its transformed elements have also a common point (x', y', z'). For the coördinates x', y' and z' depend upon the partial derivatives of $f = 0$ and these are invariant under (24). We see further from (14) that the conjugate elements are transformed by

$$p' = \bar{p}' - t_1, \qquad q' = \bar{q}' - t_2. \tag{25}$$

Hence if $f(x, y, z, p, q) = 0$ and $f'(x', y', z', p', q') = 0$ are conjugate equations, so also are

$$f(x, y, z, p + t_1, q + t_2) = 0 \quad \text{and} \quad f'(x', y', z', p' - t_1, q' - t_2) = 0.$$

It may be permitted to enlarge upon the geometrical aspect of the facts just presented. Each element (P, E) of $f = 0$ is turned about its point P by the transformation (24), and each of the ∞^2 systems of elements thus obtained is then arranged in the characteristic bands of the corresponding differential equation (23). It is to be noted that the characteristic bands of $f = 0$ do not transform into those of the new equation, for (24) is *not* a contact transformation. If, now, we consider the osculating parabolic bands at (P, E) and at each of its ∞^2 transformed elements, these bands will have in common a *second* point which is common to all the conjugate elements.

A discussion precisely similar applies when the other element coördinates u, v, w, α, β of (19) are employed. Namely, each element is now displaced in its own plane by the transformation

$$\alpha = \bar{\alpha} + \sigma_1, \qquad \beta = \bar{\beta} + \sigma_2, \tag{26}$$

the other coördinates u, v, w remaining constant. Then the conjugate elements of $(u, v, w, \bar{\alpha}, \bar{\beta})$ in the transformed equation, for every σ_1 and σ_2 will also lie in a common plane, namely, the plane of the element $(u', v', w', \alpha', \beta')$ which is conjugate to the initial element (u, v, w, α, β). Furthermore, the conjugate elements satisfy the equations $\alpha' = \bar{\alpha}' - \sigma_1$, $\beta' = \bar{\beta}' - \sigma_2$.

The transformation (26) in the usual coördinates (x, y, z, p, q) takes the form

$$x = \bar{x} - \sigma_1, \qquad y = \bar{y} - \sigma_2, \qquad z = \bar{z} - p\sigma_1 - q\sigma_2, \tag{27}$$

the coördinates p and q remaining constant. Evidently each element is displaced in its own plane in such a manner that the displacement parallel to the XY-plane is the *same* for all elements.

Recapitulation of the results found may be made in the following form:

THEOREM. *Let each surface-element* $(P, E)(x, y, z, p, q)$ *satisfying the differential equation*

$$f(x, y, z, p, q) = 0, \tag{10}$$

be turned about its point in such a manner that it becomes the element (P, E') $(x, y, z, p - t_1, q - t_2)$, *where* t_1 *and* t_2 *are arbitrary constants. Then the transformed elements satisfy the equation*

$$f(x, y, z, p + t_1, q + t_2) = 0,$$

and the osculating parabolic bands of (P, E') *for every* t_1 *and* t_2 *pass through a second common point,*

$$x' = x + \frac{2f_p}{f_z}, \qquad y' = y + \frac{2f_y}{f_z}, \qquad z' = z - 2\,\frac{f_p f_x + f_y f_z}{f_{z^2}},$$

namely, the point of the conjugate element of (P, E) *in* (10).

Again, let each element (P, E) *be moved in its plane so that the transformed element is* $(P', E)(x + \sigma_1, y + \sigma_2, z + p\sigma_1 + q\sigma_2, p, q)$, *where* σ_1 *and* σ_2 *are arbitrary constants. The new elements satisfy the equation*

$$f(x - \sigma_1, y - \sigma_2, z - p\sigma_1 - q\sigma_2, p, q) = 0.$$

The osculating parabolic bands of (P', E) *for every* σ_1 *and* σ_2 *touch a second common plane, namely the plane of the conjugate element of* (P, E) *in* (10).

We may if we so wish consider this theorem as establishing on the parabolic band which osculates a characteristic band at (P, E) the conjugate element. Clearly, as far as the theorem is concerned, the relation of any equation to its conjugate is a mutual one, that is, the new parabolic bands introduced are the *same* for both equations.

Attention may be called to the fact that the ∞^2 parabolic bands issuing from the point (x, y, z) which arise in the first part of the theorem, lie on ∞^1 parabolas, namely, the parabolas through (x, y, z) and the point of the conjugate element. Similarly, the ∞^2 bands referred to in the second part lie on ∞^1 parabolic cylinders.

One final question may be answered. If we start with any element (x, y, z, p, q) satisfying (10), the transformation of the theorem leads to ∞^2 elements at the point (x, y, z). Further, each of these determines a parabolic band, that is, with each is associated a united element. Hence from the discussion of § 1, since each band determines at (x, y, z) ∞^1 curvature elements, we see that from the element (x, y, z, p, q) are derived ∞^3 curvature elements $(x, y, z, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t})$. Also there are ∞^1 elements with the same (x, y, z) satisfying (10). Hence for all points we derive a configuration of ∞^7 curvature elements. These must, of course, satisfy a single partial differential equation of the second order. Obviously, we obtain this equation by forming the first partial derivatives of (23) with respect to t_1 and t_2, and then eliminating these parameters.

It should be said that the facts disclosed in the above theorem are easily foreseen in simple properties of the null-system in R_5.

The theorem states, for surface-element loci in space, facts entirely analogous to those established by Scheffers for the plane. The osculating parabolic bands belong also to a second system of differential equations obtained from the conjugate equation by the transformation (25).

The theorem given is stated without great difficulty in a form invariant within the group G_{21} of all contact-transformations under which the system of parabolic bands remains invariant. The ∞^3 points or planes of space transform into a system σ of ∞^3 vertical paraboloids each of which has a parabolic band in common with a fixed vertical paraboloid. Each element of space determines a σ-paraboloid to which it belongs. Then a two parameter group

$$x = \phi(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, t_1, t_2), \cdots, \qquad q = Q(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, t_1, t_2),$$

exists such that each element is displaced upon the corresponding σ-paraboloid. This transformation turns any differential equation

$$f(x, y, z, p, q) = 0$$

into

$$F(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q};\ t_1, t_2) = 0.$$

The point now is this. If the conjugate element of (x, y, z, p, q) in $f = 0$ is (x', y', z', p', q') and the corresponding paraboloids are σ and σ', then the conjugate element of $\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}$ in $F = 0$ for every t_1, t_2 will lie upon σ'.

§ 7. *Geometry of Osculating Cubic Bands.*

We now turn our attention to the second type of bands described in § 2. The transformation of Lie used in the previous section is here replaced by the following:

$$(1) \qquad x = \frac{X_1}{Y_1}, \qquad y = \frac{X_2}{Y_2}, \qquad z = Z, \qquad p = Y_1^2, \qquad q = Y_2^2,$$

by which we pass from ordinary space to space of five dimensions R_5. Moreover we verify immediately,

$$(2) \qquad dz - pdx - qdy = dZ + X_1 dY_1 - Y_1 dX_1 + X_2 dY_2 - Y_2 dX_2,$$

and the null-system of § 4 bears the same relation as before to the geometry of cubic bands. We verify easily that the null-line in R_5,

$$(3) \quad X_1 = X_1^0 + \lambda_1 t,\ X_2 = X_2^0 + \lambda_2 t,\ Y_1 = Y_1^0 + \mu_1 t,\ Y_2 = Y_2^0 + \mu_2 t,\ Z = Z^0 + \nu t,$$

wherein

$$\nu + \mu_1 X_1 - \lambda_1 Y_1 + \mu_2 X_2 - \lambda_2 Y_2 = 0,$$

gives in R_3 a cubic band. For, substituting in (1), we obtain for the equations

of the band corresponding to (3),

$$
(4)\quad
\begin{aligned}
&x = x^0 + \frac{(\lambda_1 - \mu_1 x^0)\cdot t}{\sqrt{p^0} + \mu_1 t}, \qquad y = y^0 + \frac{(\lambda_2 - \mu_2 y^0)t}{\sqrt{q^0} + \mu_2 t},\\
&z = z^0 + [(\lambda_1 - \mu_1 x^0)\sqrt{p^0} + (\lambda_2 - \mu_2 y^0)\sqrt{q^0}]t, \qquad p = p^0 + \mu_1 t(2\sqrt{p^0} + \mu_1 t),\\
&q = q^0 + \mu_2 t(2\sqrt{q^0} + \mu_2 t),
\end{aligned}
$$

in which the signs of the radicals are unambiguous. Then clearly, the point locus is a cubic which passes through the points at infinity on the axes of x, y and z. Call these points for convenience A, B and C, respectively. The plane locus is the envelope of

$$
(5)\qquad Z - z = p(X - x) + q(Y - y),
$$

in which x, y, z, p, q have the values in (4). The result of this substitution may be written

$$
(6)\quad
\begin{aligned}
&(\mu_1^2 X + \mu_2^2 Y - \lambda_1\mu_1 - \lambda_2\mu_2)t^2 + 2[\mu_1\sqrt{p^0}(X - x^0) + \mu_2\sqrt{q^0}(Y - y^0)]t\\
&\qquad - Z + z^0 + p^0(X - x_0) + q^0(Y - y^0) = 0.
\end{aligned}
$$

The envelope of (6) is a quadric cone. The band (4) has therefore the characteristic described in § 4. The vertex of this cone must lie upon the point-locus, the cubic. It is readily verified that the value of the parameter t which determines the vertex is

$$
(7)\qquad t_v = \sqrt{p^0 q^0}\,\frac{(\lambda_1 - \mu_1 x^0)\mu_1 + (\lambda_2 - \mu_2 y^0)\mu_2}{\mu_1\mu_2\nu}.
$$

The vertex lies at the initial point (x^0, y^0, z^0) when

$$
(8)\qquad (\lambda_1 - \mu_1 x^0)\mu_1 + (\lambda_2 - \mu_2 y^0)\mu_2 = 0.
$$

The cubic band is now the one referred to at the close of § 4.

Pass next to the consideration of what will correspond in R_3 to the general null-plane in R_5,

$$
(9)\quad X_1 = aY_1 + bY_2 + c, \qquad X_2 = bY_1 + dY_2 + e, \qquad Z = -cY_1 - eY_2 + f.
$$

Eliminating Y_1 and Y_2 from (9) and the first three equations of (1) we obtain as the corresponding surface in R_3

$$
(10)\qquad \begin{vmatrix} a - x & b & c \\ b & d - y & e \\ -c & -e & f - z \end{vmatrix} = 0.
$$

This is the equation of a cubic surface with nodes at the fixed points A, B

and C. Furthermore, it is easily seen that a fourth node exists at the point $(a - bc/e, d - be/c, f + ce/b)$.

To the special null-plane

$$X_1 = aY_1, \qquad X_2 = bY_2, \qquad Z = c \tag{11}$$

correspond all the elements of the point (a, b, c).

Two cases of null-planes not representable in the form (9) are

$$Y_1 = aY_2 + b, \quad X_2 = cY_2 - aX_1 + e, \qquad Z = -eY_2 + bX_1 + h, \tag{12}$$

to which will correspond in R_3 a quadric cone through the fixed points A, B and C; and

$$Y_1 = b, \qquad Y_2 = c, \qquad Z = bX_1 + cX_2 + d, \tag{13}$$

which gives in R_3 the plane $z = b^2x + c^2y + d$.

It may be remarked that (12) is the same as (4), § 6, if $a_1b_2 = a_2^2$, a case to which attention was called at the time, while (13) is also a special case of the same equations. That is, the null-planes in R_5 which under Lie's transformation gave parabolic cylinders now go over into quadric cones. Further, the null-planes (13) in R_5 give planes in R_3 under both transformations.

To the general plane not a null-plane in R_5 will correspond in R_3 an assemblage of ∞^2 elements whose points lie upon a cubic surface having A, B and C as nodes, while the planes envelop a cubic with four nodes, of which the fixed points A, B and C constitute three. This difference in character of plane-locus and point-locus is noteworthy and is readily accounted for by considerations in R_5 which may be omitted here. The three-nodal cubic referred to is entirely *general.*

We may develop as in the previous section the analysis for the theory of cubic bands which osculate the characteristic bands of a given partial differential equation of the first order. To avoid the details, we limit ourselves to the statement of results.

The equations of the cubic band which osculates the characteristic band of

$$f(x, y, z, p, q) = 0$$

at (x, y, z, p, q) are

$$x' = x + \frac{2pf_p t}{p - (f_x + pf_z)t}, \qquad y' = y + \frac{2qf_q t}{q - (f_y + qf_z)t},$$

$$z' = z + 2(pf_p + qf_q)t, \qquad p' = \frac{[p - (f_x + pf_z)t]^2}{p}, \tag{13'}$$

$$q' = \frac{[q - (f_y + qf_z)t]^2}{q}.$$

The osculating cubic bands of the equation

$$f(x, y, z, p, q) = 0 \tag{14}$$

belong also to a second equation obtained by elimination of x, y, z, p, q from (14) and

$$\begin{gathered} x' = x - 2p\frac{f_p}{f_x}, \qquad y' = y - 2q\frac{f_q}{f_y}, \qquad z' = z + \frac{2pf_p + 2qf_q}{f_z}, \\ p' = \frac{1}{p}\frac{f_x^2}{f_z^2}, \qquad q' = \frac{1}{q}\frac{f_y^2}{f_z^2}. \end{gathered} \tag{15}$$

The conjugate element (x', y', z', p', q') is given by (13′) if $t = 1 \div f_z$.

Finally, we may state in a manner analogous to the theorem at the close of the preceding section the

THEOREM. *Let each surface element* (P, E) (x, y, z, p, q) *of the locus of*

$$f(x, y, z, p, q) = 0 \tag{16}$$

be turned about its point in such a manner that it becomes the element (P, E') $(x, y, z, pe^{-2t_1}, qe^{-2t_2})$, *where* t_1 *and* t_2 *are arbitrary constants. Then the transformed elements satisfy the equation*

$$f(x, y, z, pe^{2t_1}, qe^{2t_2}) = 0,$$

and the osculating cubic bands of (P, E') *for every* t_1 *and* t_2 *pass through a second common point given by the first three equations of* (15).

Again, let each element (P, E) *be moved in its own plane in such a manner that the transformed element* (P', E) *is*

$$(x + \sigma_1/\sqrt{p},\ y + \sigma_2/\sqrt{q},\ z + \sqrt{p}\sigma_1 + \sqrt{q}\sigma_1,\ p, q);$$

then the new elements satisfy the equation

$$f(x - \sigma_1/\sqrt{p}, y - \sigma_2/\sqrt{q}, z - \sqrt{p}\sigma_1 - \sqrt{q}\sigma_2,\ p, q) = 0.$$

Furthermore, the osculating cubic bands of (P', E) *for every* σ_1 *and* σ_2 *will touch a second plane, namely the plane of the element* x', y', z', p', q' *given by* (15).

The statement for cubic bands of the results which correspond to other theorems in the preceding section may be omitted here.

We may regard cubic bands, projectively at least, as space analogues of circles in the plane. For the circle, as a line-element locus, is determined uniquely by two fixed points (the circular points) and two given united elements, while, as we have seen, a cubic band is determined without ambiguity if we postulate that it shall pass through three fixed points and contain two given united surface-elements.

§ 8. *Concluding Theorems.*

Comparison of the transformations of §§ 6 and 7 gives in space the contact-transformation

$$(1)\quad x = \frac{2x'}{p'}, \qquad y = \frac{2y'}{q'}, \qquad z = z' - \tfrac{1}{2}p'x' - \tfrac{1}{2}q'y', \qquad p = \tfrac{1}{4}p'^2, \qquad q = \tfrac{1}{4}q'^2,$$

which has the cardinal property of transforming parabolic into cubic bands. Further properties of this transformation are readily derived from the preceding discussion.

Planes transform into planes.

Parabolic cylinders passing through the extremity of the z-axis at infinity transform into quadric cones having three fixed points A, B, C in common. These points lie at infinity on the axes. Also, straight lines transform into quadric cones of this family.

Vertical paraboloids and parabolas go over into four nodal cubics, three of the nodes being A, B, C. In special cases, the singularities may change to two nodes and one binode, all at A, B and C.

Elimination of p, q, p', q' from (1) leads to the *aequatio-directrix*

$$(2)\qquad xyz - z'xy + y'^2x + x'^2y = 0,$$

from which it appears that points (x', y', z') are transformed into cubics for which A and B are nodes and C a binode, while points (x, y, z) go over into vertical paraboloids for which the planes $x' = 0$ and $y' = 0$ are principal planes.

We may draw one conclusion from the details outlined above. From the double system of generators on a paraboloid we infer at once that for a four-nodal cubic there exists a double system of quadric cones each of which will pass through three of the nodes and touch the cubic along a skew cubic. Taking the four nodes by threes in the four possible ways, we obtain eight systems of enveloping quadric cones.

SHEFFIELD SCIENTIFIC SCHOOL,
February, 1910.

FIELDS OF EXTREMALS IN SPACE

BY

GILBERT AMES BLISS AND MAX MASON *

The minimizing space curves for the integral

$$J = \int F(x, y, z, x', y', z')\, dt$$

form a family which contains four arbitrary constants. Of special interest are the two-parameter families which pass through a given point or are cut transversally by a given curve or surface. Such families occur when the integral J is to be minimized with respect to curves which join two fixed points or which have one end-point fixed while the other is free to vary on a fixed curve or surface.† In the study of the sufficient conditions which insure a minimum for the integral J under these conditions it is essential to know that such two-parameter families fill out simply a portion of space about the point through which all of the curves of the family pass, or about the curve or surface by which they are cut transversally. The minimizing curves, the extremals, are said to form a field in the region in which this property holds.

The field proofs for the analogous cases in the plane have already been made.‡ The proof given by BOLZA for the existence in the plane of a field about an arc of an extremal which does not contain a pair of conjugate points, can be extended readily to the analogous case in space. This proof includes the field proof mentioned above for the extremals which are cut transversally by a given surface.

It is to be shown here that the extremals through a fixed point § or the extremals to which a given curve is transversal also form fields. In these two cases the functional determinant of the equations to be solved vanishes at the point or along the curve in question. For the former case a suitably selected transformation of the variables reduces the equations to a set whose functional determinant does not vanish. The solution is then obtained from the theory of

* Presented to the Society, Sept. 1, 1908.

† These minimizing problems were considered by the writers in a paper entitled "The properties of curves in space which minimize a definite integral," vol. 9 (1908), p. 440.

‡ See BOLZA, *Vorlesungen über Variationsrechnung*, pp. 164, 249, 271.

§ The construction of a field formed by extremals through a fixed point has been previously discussed by two writers, both of whom used methods quite different from the one given here. See BILL, *The construction of a space field of extremals*, Bulletin of the American Mathematical Society, vol. 15 (1909), p. 374; also HADAMARD, *Leçons sur le calcul des variations*, Note A, p. 497.

implicit functions by the aid of a theorem due to BOLZA,* which is proved in § 1.

A different method is used in § 3 to prove the existence of a field about a curve L, but either this or the method of § 2 is applicable to both problems. In order to apply the method of § 2 to prove the existence of a field about L it is necessary first to construct a system of polar coördinates in the neighborhood of L with properties similar to those of the well-known ones used in § 2.

§ 1. *Bolza's theorem concerning implicit functions.*

Consider a set of equations

$$(1) \qquad f_i(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n) = 0 \qquad (i = 1, 2, \cdots, n),$$

for which the functions f_i satisfy the following conditions:

A) The functions f_i are continous and have continuous first partial derivatives with respect to the y's for all values $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ in a certain region $\mathfrak{A}$.

B) There exists a finite closed set $\mathfrak{C}$ of solutions $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ of equation (1) entirely within the region $\mathfrak{A}$. The projection of $\mathfrak{C}$ in the $(x_1, x_2, \cdots, x_m)$-space will be denoted by $\mathfrak{H}$.

C) No two distinct points in $\mathfrak{C}$ have the same projection in $\mathfrak{H}$: in other words, to any point $(x_1, x_2, \cdots, x_m)$ in $\mathfrak{H}$ there corresponds one and but one solution $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ in $\mathfrak{C}$.

D) In the set $\mathfrak{C}$ the functional determinant $\partial(f_1, f_2, \cdots, f_n)/\partial(y_1, y_2, \cdots, y_n)$ is always different from zero.

Then there exists a constant ϵ such that no two solutions of equations (1) in $\mathfrak{C}_\epsilon$† possess the same projection in $\mathfrak{H}_\epsilon$, and a constant $\delta \leqq \epsilon$ can be determined so that to any point in the neighborhood $\mathfrak{H}_\delta$ there corresponds at least one solution of equations (1) in the neighborhood $\mathfrak{C}_\epsilon$. The functions

$$(2) \qquad y_1(x_1, x_2, \cdots, x_m),\ y_2(x_1, x_2, \cdots, x_m),\ \cdots,\ y_n(x_1, x_2, \cdots, x_m)$$

so defined in $\mathfrak{H}_\delta$ are of class C, and if the function f_i are of class $C^{(n)}$‡ in $\mathfrak{A}$ these functions are also of class $C^{(n)}$ in $\mathfrak{H}_\delta$.

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* Loc. cit., p. 166. BOLZA makes the proof of the general theorem depend upon a special case which he has discussed in the preceding pages. The theorem was proved independently by the writers for the purposes of the present paper, and the proof in § 1 is the direct proof which they derived, the statement of the theorem being slightly different from that of Bolza.

† See BOLZA, loc. cit., p. 154. His notation is $(\varepsilon)_{\mathfrak{C}}$. The neighborhood $\mathfrak{C}_\epsilon$ for any set $\mathfrak{C}$ and any constant ε is the set of points $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ for each of which there exists at least one point $(x_1', x_2', \cdots, x_m' : y_1', y_2', \cdots, y_n')$ in $\mathfrak{C}$ satisfying the relations

$$|x_1 - x_1'| < \varepsilon,\ |x_2 - x_2'| < \varepsilon,\ \cdots,\ |x_m - x_m'| < \varepsilon,$$

with similar inequalities for $y_1, y_2, \cdots, y_n$.

‡ A function is of class $C^{(n)}$ if its derivatives up to and including those of order n are continuous.

As a first step in the proof it can be shown that a constant ϵ exists such that no two distinct solutions of equations (1) in the neighborhood $\mathfrak{C}_\epsilon$ can have the same projection $(x_1, x_2, \cdots, x_m)$ in $\mathfrak{H}_\epsilon$. For if this were not the case two sequences of solutions of equations (1)*

$$(3)\qquad \begin{gathered}\{x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)} : y_1^{(i)}, y_2^{(i)}, \cdots, y_n^{(i)}\},\\ \{x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)} : Y_1^{(i)}, Y_2^{(i)}, \cdots, Y_n^{(i)}\}\end{gathered} \qquad (i=1, 2, \cdots),$$

$$(y_1^{(i)}, y_2^{(i)}, \cdots, y_n^{(i)}) \neq (Y_1^{(i)}, Y_2^{(i)}, \cdots, Y_n^{(i)}),$$

could be found, with limit points

$$(\xi_1, \xi_2, \cdots, \xi_m : \eta_1, \eta_2, \cdots, \eta_n), \qquad (\xi_1, \xi_2, \cdots, \xi_m : H_1, H_2, \cdots, H_n)$$

which lie in the region $\mathfrak{C}$. Since by C) no two points in $\mathfrak{C}$ have the same projection in $\mathfrak{H}$ it follows that $(\eta_1, \eta_2, \cdots, \eta_n)$ and $(H_1, H_2, \cdots, H_n)$ are the same. At the point $(\xi_1, \xi_2, \cdots, \xi_m : \eta_1, \eta_2, \cdots, \eta_n)$ the usual theorems on implicit functions† can be applied to equations (1) on account of A), B), D), and it follows that to any point in the neighborhood of $(\xi_1, \xi_2, \cdots, \xi_m)$ there can correspond but one set of solutions in the neighborhood of $(\xi_1, \xi_2, \cdots, \xi_m : \eta_1, \eta_2, \cdots, \eta_n)$. Thus the existence of the sequences (3) is contradicted.

The constant ϵ can be still further restricted if necessary so that the functional determinant $\partial(f_1, f_2, \cdots, f_n)/\partial(y_1, y_2, \cdots, y_n)$ is different from zero in $\mathfrak{C}_\epsilon$, on account of D).

The next step is to select a constant $\delta \leqq \epsilon$ so that to each point in $\mathfrak{H}_\delta$ there corresponds at least one solution in the region $\mathfrak{C}_\epsilon$. Suppose that P is any point of the set $\mathfrak{C}$, and Q its projection in $\mathfrak{H}$. Then by the theorems on implicit functions referred to above, a constant δ can be found such that to each point $(x_1, x_2, \cdots, x_m)$ in the neighborhood Q_δ there corresponds a solution of equations (1) lying in P_ϵ and therefore in $\mathfrak{C}_\epsilon$. A constant δ can be selected which will be effective in this way uniformly for all the points P of $\mathfrak{C}$. Otherwise a sequence $\{P_n\}$ would exist in $\mathfrak{C}$, with limit point π, for which the corresponding constants δ_n would have the limit zero. This is, however, impossible, since there is a value $\delta_\pi > 0$ for the limit point π.

By combining these results it follows that to any point in the region $\mathfrak{H}_\delta$ there corresponds one and but one solution of equations (1) in the neighborhood $\mathfrak{C}_\epsilon$. Since the functional determinant does not vanish in $\mathfrak{C}_\epsilon$ it follows from the usual theorems concerning implicit functions that the functions (2) so defined over the region $\mathfrak{H}_\delta$ will be surely of class C under the conditions on the functions f_i stated under A), and of class $C^{(n)}$ when the functions f_i are of class $C^{(n)}$ in $\mathfrak{A}$.

* For the details of the construction of these sequences see BOLZA, loc. cit., p. 161.

† See, for example, OSGOOD, *Lehrbuch der Funktionentheorie*, vol. 1, p. 52; or GOURSAT–HEDRICK, *A Course in Analysis*, vol. 1, p. 45 and footnote.

§ 2. *The field about a point.*

The equations of the extremals through a given point $P(x_0, y_0, z_0)$ may be taken in the form *

$$x = \phi(s, \xi, \eta, \zeta), \qquad y = \psi(s, \xi, \eta, \zeta), \qquad z = \chi(s, \xi, \eta, \zeta),$$

where the functions ϕ, ψ, χ and their first partial derivatives with respect to s are of class C' in the neighborhood of the values of s, ξ, η, ζ defined by the conditions

$$(\mathfrak{S}) \qquad s = 0, \qquad \xi^2 + \eta^2 + \zeta^2 = 1.$$

For $s = 0$ the identities

$$(4) \quad \begin{aligned} &\phi(0, \xi, \eta, \zeta) = x_0, \qquad \psi(0, \xi, \eta, \zeta) = y_0, \qquad \chi(0, \xi, \eta, \zeta) = z_0, \\ &\phi_s(0, \xi, \eta, \zeta) = \xi, \qquad \psi_s(0, \xi, \eta, \zeta) = \eta, \qquad \chi_s(0, \xi, \eta, \zeta) = \zeta \end{aligned}$$

hold, and the identity

$$\phi\phi_s + \psi\psi_s + \chi\chi_s = 0$$

is satisfied for all arguments. If

$$(5) \qquad \xi^2 + \eta^2 + \zeta^2 = 1,$$

it follows that s is the length of arc measured along the extremal from the point P, so that

$$(6) \qquad \phi_s^2 + \psi_s^2 + \chi_s^2 = 1,$$

and ξ, η, ζ are the direction cosines of the tangent to the extremal at P.

It is desired to show that *two positive constants d, e exist such that to any point (x, y, z) in the region P'_d,*†

$$(P'_d) \qquad 0 < |x - x_0| < d, \; 0 < |y - y_0| < d, \; 0 < |z - z_0| < d,$$

there exists one and but one solution $(x, y, z : s, \xi, \eta, \zeta)$ of the equations

$$(7) \quad \xi^2 + \eta^2 + \zeta^2 = 1, \; \phi(s, \xi, \eta, \zeta) = x, \; \psi(s, \xi, \eta, \zeta) = y, \; \chi(s, \xi, \eta, \zeta) = z$$

for which (s, ξ, η, ζ) is in the region $\mathfrak{S}'_e$ defined by the conditions

$$(\mathfrak{S}'_e) \qquad 0 < s < e, \qquad \xi^2 + \eta^2 + \zeta^2 = 1.$$

In terms of the polar coördinates R, A, B, C defined by the equations

$$x - x_0 = RA, \; y - y_0 = RB, \; z - z_0 = RC, \; R > 0, \; A^2 + B^2 + C^2 = 1,$$

* See MASON and BLISS, loc. cit., p. 444.

† Note that P'_d is the region P_d as defined in a preceding footnote, with the point (x_0, y_0, z_0) excluded.

the equations (7) become

$$(8)\quad \xi^2+\eta^2+\zeta^2=1,\ \phi(s,\xi,\eta,\zeta)=RA,\ \psi(s,\xi,\eta,\zeta)=RB,\ \chi(s,\xi,\eta,\zeta)=RC.$$

If s is replaced by σ, where $s = R\sigma$, they take the form

$$(9)\quad \begin{aligned} &\xi^2+\eta^2+\zeta^2=1,\\ &\frac{1}{R}[\phi(R\sigma,\xi,\eta,\zeta)-x_0]=A,\\ &\frac{1}{R}[\psi(R\sigma,\xi,\eta,\zeta)-y_0]=B,\\ &\frac{1}{R}[\chi(R\sigma,\xi,\eta,\zeta)-z_0]=C. \end{aligned}$$

By Taylor's formula with the integral form of the remainder term * and the identities (4), it follows that

$$(10)\qquad \phi(s,\xi,\eta,\zeta)-x_0=s\int_0^1\phi_s(ts,\xi,\eta,\zeta)\,dt,$$

with similar equations for ψ and χ, so that when the first members of equations (9) are defined for $R=0$ by their limiting values, these equations reduce, for $R=0$, to

$$(11)\qquad \xi^2+\eta^2+\zeta^2=1,\ \sigma\xi=A,\ \sigma\eta=B,\ \sigma\zeta=C.$$

It is evident then that equations (9) have the solutions

$$(\mathfrak{C})\quad (R,A,B,C:\sigma,\xi,\eta,\zeta)=(0,A,B,C:1,\xi,\eta,\zeta),\quad A^2+B^2+C^2=1,$$

and the projection of $\mathfrak{C}$ in terms of the polar coördinates is the region defined by the conditions

$$(\mathfrak{H})\qquad R=0,\quad A^2+B^2+C^2=1.$$

From equation (10) and its analogues for ψ and χ, it follows that the left members of equations (9) are of class C' in the neighborhood of $\mathfrak{C}$, and in $\mathfrak{C}$ their functional determinant with respect to σ, ξ, η, ζ is calculated from (11) to be $2\sigma^2=2$. Hence Bolza's theorem is applicable, and it follows that a constant ϵ exists such that no two solutions of equations (9) in the region $\mathfrak{C}_\epsilon$ have the same projection in $\mathfrak{H}_\epsilon$, and a second constant $\delta\leqq\epsilon$, can be chosen so that to any (R,A,B,C) in $\mathfrak{H}_\delta$ there corresponds a set of values

$$(12)\quad \sigma(R,A,B,C),\ \xi(R,A,B,C),\ \eta(R,A,B,C),\ \zeta(R,A,B,C)$$

which with (R,A,B,C) satisfy equations (9) and lie in $\mathfrak{C}_\epsilon$. The functions (12) are of class C' and reduce to $1, A, B, C$ for $R=0$.

* See C. JORDAN, *Cours d'Analyse*, vol. 1, p. 245.

With these results in mind it is possible to show the existence of a constant e such that no two sets of values (s, ξ, η, ζ) in $\mathfrak{S}'_e$ can define by means of equations (7) the same point (x, y, z), or what is equivalent, the same values (R, A, B, C). In fact, for any solution of equations (8) the value of R is given in terms of (s, ξ, η, ζ) by means of the formula

$$R^2 = [\phi - x_0]^2 + [\psi - y_0]^2 + [\chi - z_0]^2,$$

and from equations (10) and (4) it is evident therefore that the quotient s/R is a continuous function of s, ξ, η, ζ which reduces to 1 for $s = 0$. Similarly A, B, C in equations (8) are seen to be continuous functions of s, ξ, η, ζ which take the values ξ, η, ζ, respectively, for $s = 0$. Hence if e is sufficiently small, no two sets of values (s, ξ, η, ζ) in $\mathfrak{S}'_e$ can define the same (x, y, z). Otherwise equations (9) would have two solutions $(R, A, B, C : \sigma, \xi, \eta, \zeta)$ in $\mathfrak{C}$ corresponding to the same (R, A, B, C).

Let δ be taken so small that the values

$$(13) \quad s = R\sigma(R, A, B, C),\ \xi(R, A, B, C),\ \eta(R, A, B, C),\ \zeta(R, A, B, C)$$

corresponding to any (R, A, B, C) in $\mathfrak{H}_\delta$, lie in $\mathfrak{S}_e$. If $d = \delta/\sqrt{3}$, then to any point (x, y, z) in P'_d there will correspond values of (R, A, B, C) in $\mathfrak{H}_\delta$, and consequently a solution of equations (7) for which (s, ξ, η, ζ) lies in $\mathfrak{S}'_e$. By the preceding paragraph this solution will be unique in $\mathfrak{S}'_e$.

The functions

$$(14) \qquad s(x, y, z),\ \xi(x, y, z),\ \eta(x, y, z),\ \zeta(x, y, z)$$

so defined have the following properties:

1) *In the neighborhood P'_d all are of class C'.*

2) *In P_d the function $s(x, y, z)$ is continuous and has the value 0 at (x_0, y_0, z_0).*

3) *Let C be an arc of class C',*

$$(C) \qquad x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad (0 \leqq t \leqq T),$$

which passes through $P(x_0, y_0, z_0)$ for $t = 0$. The function $s(t)$ defined by substituting the values (C) in the expressions (14) has at $t = 0$ the derivative

$$s'(0) = \sqrt{f'^2(0) + g'^2(0) + h'^2(0)},$$

and the corresponding functions $\xi(t), \eta(t), \zeta(t)$ are continuous and reduce for $t = 0$ to the direction cosines A_0, B_0, C_0, of the positive tangent to C at P, respectively.

The first two properties follow immediately from the nature of the functions

(2) and of the polar coördinates R, A, B, C as functions of x, y, z. For points on the curve C

$$R^2 = [f - x_0]^2 + [g - y_0]^2 + [h - z_0]^2,$$

$$A = \frac{f - x_0}{R}, \qquad B = \frac{g - y_0}{R}, \qquad C = \frac{h - z_0}{R}.$$

Hence by applying Taylor's formula to the differences $f - x_0$, $g - y_0$, $h - z_0$ it follows that

$$\lim_{t=0} \frac{R}{t} = \sqrt{f'^2(0) + g'^2(0) + h'^2(0)}$$

and also that A, B, C approach the limits A_0, B_0, C_0. From the equations (13) and the properties of the functions (12) it is then easy to prove the statement 3).

The proof which has been made applies to the construction of a field in the neighborhood of a single point (x_0, y_0, z_0). It is important, for the proof of Hilbert's theorem concerning the existence of an extremal joining two given points in space, to know that such fields can be constructed uniformly about the points of a finite closed region $\mathfrak{R}$. The equations of the extremals through points of $\mathfrak{R}$ have the form *

$$\begin{aligned} x &= \phi(s, x_0, y_0, z_0, \xi, \eta, \zeta), \\ y &= \psi(s, x_0, y_0, z_0, \xi, \eta, \zeta), \\ z &= \chi(s, x_0, y_0, z_0, \xi, \eta, \zeta), \end{aligned} \tag{15}$$

where ϕ, ψ, χ and their derivatives for s are of class C' in the neighborhood of the domain defined by the conditions

$$(x_0, y_0, z_0) \text{ in } R, \ s = 0, \ \xi^2 + \eta^2 + \zeta^2 = 1.$$

For $s = 0$ the identities (4) hold with respect to all seven arguments, and the identity (6) is a consequence of equation (5).

The equations corresponding to (9) can be set up in the same manner as before, but they contain now the additional arguments x_0, y_0, z_0. The region corresponding to $\mathfrak{C}$ for these equations is the region

$$(\bar{\mathfrak{C}}) \quad (x_0, y_0, z_0 : 0, A, B, C : 1, A, B, C), \ (x_0, y_0, z_0) \text{ in } R, \ A^2 + B^2 + C^2 = 1,$$

and the region corresponding to $\mathfrak{H}$ is

$$(\bar{\mathfrak{H}}) \quad (x_0, y_0, z_0 : 0, A, B, C), \ (x_0, y_0, z_0) \text{ in } R, \ A^2 + B^2 + C^2 = 1.$$

* Mason and Bliss, loc. cit., p. 444.

The constants δ and ϵ can be determined so that no two solutions of equations (9) in $\bar{\mathfrak{C}}_\epsilon$, have the same projection in $\bar{\mathfrak{H}}_\epsilon$, and so that to each point in $\bar{\mathfrak{H}}_\delta$ there corresponds at least one solution in $\bar{\mathfrak{C}}_\epsilon$, the variables σ, ξ, η, ζ being now functions of class C' of x_0, y_0, z_0, R, A, B, C in $\bar{\mathfrak{H}}_\delta$ which reduce to $(1, A, B, C)$ for points in $\bar{\mathfrak{H}}$.

Each solution $(x_0, y_0, z_0 : R, A, B, C : s, \xi, \eta, \zeta)$ of equations (8) defines as before a solution $(x_0, y_0, z_0 : R, A, B, C : \sigma, \xi, \eta, \zeta)$ of equations (9) whose elements are continuous functions of x_0, y_0, z_0, s, ξ, η, ζ reducing to $(x_0, y_0, z_0 : 0, A, B, C : 1, \xi, \eta, \zeta)$ when (x_0, y_0, z_0) is in R and (s, ξ, η, ζ) is in the region $\mathfrak{S}$. Hence the constant e can be chosen so that the solution of equation of equation (9) always lies in $\bar{\mathfrak{C}}_\epsilon$ when (x_0, y_0, z_0) is in $\mathfrak{R}$ and (s, ξ, η, ζ) in $\mathfrak{S}'_e$, and consequently for any fixed but arbitrarily chosen point (x_0, y_0, z_0) in $\mathfrak{R}$ no two sets of values (s, ξ, η, ζ) in $\mathfrak{S}'_e$ could define the same (R, A, B, C). Otherwise equations (9) would have two solutions in $\bar{\mathfrak{C}}_\epsilon$ corresponding to the same point in $\bar{\mathfrak{H}}_\epsilon$.

The constant $d = \delta/\sqrt{3}$ can similarly be chosen so that for any point P (x_0, y_0, z_0) in $\mathfrak{R}$ the values (R, A, B, C) in the region P'_d about P define solutions $(x, y, z : s, \xi, \eta, \zeta)$ of equations (15) and (5) for which (s, ξ, η, ζ) is unique in $\mathfrak{S}'_e$.

Hence constants d, e can be chosen uniformly for the region $\mathfrak{R}$ so that to any point (x, y, z) in the neighborhood P'_d,

$$0 < |x - x_0| < d, \qquad 0 < |y - y_0| < d, \qquad 0 < |z - z_0| < d,$$

of a point P (x_0, y_0, z_0) of $\mathfrak{R}$ there exists one and but one solution of equations (15) *for which (s, ξ, η, ζ) is in the region*

$$0 < s < e, \qquad \xi^2 + \eta^2 + \zeta^2 = 1.$$

The functions of x_0, y_0, z_0, x, y, z so determined are of class C' for (x_0, y_0, z_0) in $\mathfrak{R}$ and (x, y, z) in P'_d, and have the properties described above when (x_0, y_0, z_0) is fixed.

From this result is derived the following geometrical theorem:

A constant $\delta > 0$ can be determined so that through any point $Q \neq P$ in a sphere of radius δ about P, where P is an arbitrarily selected point of R, there passes one and but one extremal arc (15) *which joins P with Q and lies entirely within the sphere.*

Along an extremal, R is a continuous function of x_0, y_0, z_0, s, ξ, η, ζ and has a derivative dR/ds which reduces to 1 for $s = 0$. To make sure that the extremal arc joining P with Q lies entirely within the sphere it is therefore only necessary to choose e so small that dR/ds remains different from zero for (x_0, y_0, z_0) in R and (s, ξ, η, ζ) in the region S'_e. Then the corresponding constant δ will be the one desired.

§ 3. *The construction of a field of extremals to which a given arc $\mathfrak{L}$ is transversal.*

Consider an arc $\mathfrak{L}$

$$(\mathfrak{L}) \qquad x = X(u),\ y = Y(u),\ z = Z(u),\ u_1 \leqq u \leqq u_2,$$

which has the properties that it is part of a curve of class C'', that it does not intersect itself, and that the parameter u is the length of arc. Let

$$(1) \qquad \alpha, \beta, \gamma:\ l, m, n:\ \lambda, \mu, \nu$$

be the direction cosines of the tangent, principal normal, and binormal, respectively.

In each half-plane through a tangent to $\mathfrak{L}$ and containing the direction

$$l \cos v + \lambda \sin v,\ m \cos v + \mu \sin v,\ n \cos v + \nu \sin v$$

normal to the curve, there is one and but one direction ξ, η, ζ distinct from the

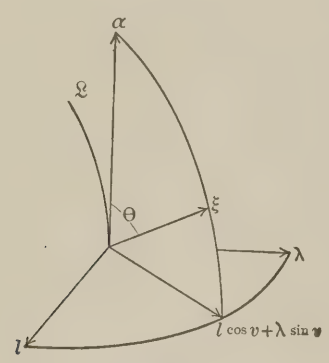

FIG. 1.

tangent, to which the curve $\mathfrak{L}$ is transversal.* The angle $\theta (0 < \theta < \pi)$ (see Fig. 1) determining this direction, is a function of u, v which is periodic in v with period 2π, and the values of ξ, η, ζ are

$$(2) \qquad \begin{aligned} \xi &= \alpha \cos \theta + (l \cos v + \lambda \sin v) \sin \theta, \\ \eta &= \beta \cos \theta + (m \cos v + \mu \sin v) \sin \theta, \\ \zeta &= \gamma \cos \theta + (n \cos v + \nu \sin v) \sin \theta. \end{aligned}$$

The extremals to which $\mathfrak{L}$ is transversal are found by substituting the functions $X(u), Y(u), Z(u)$ defining $\mathfrak{L}$, in place of x_0, y_0, z_0 in equations (15) of

* BLISS AND MASON, loc. cit., p. 463.

§ 2, and the values (2) in place of ξ, η, ζ. Then the equations of the extremals have the form

$$(E) \qquad x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v),$$

where the functions ϕ, ψ, χ can be supposed of class C'' in a certain neighborhood of the region $\mathfrak{S}$ defined by the conditions

$$(\mathfrak{S}) \qquad s = 0, \qquad u_1 \leqq u \leqq u_2, \qquad v \text{ any value},$$

and are periodic in v with period 2π. The parameter s is the length of arc, so that the identity

$$\phi_s^2 + \psi_s^2 + \chi_s^2 \equiv 1$$

is satisfied for all arguments. For $s = 0$,

$$(3) \quad \phi(0, u, v) \equiv X(u), \qquad \psi(0, u, v) \equiv Y(u), \qquad \chi(0, u, v) \equiv Z(u),$$

$$(4) \qquad \begin{aligned} \phi_s(0, u, v) &\equiv \alpha \cos\theta + (l \cos v + \lambda \sin v) \sin\theta, \\ \psi_s(0, u, v) &\equiv \beta \cos\theta + (m \cos v + \mu \sin v) \sin\theta, \\ \chi_s(0, u, v) &\equiv \gamma \cos\theta + (n \cos v + \nu \sin v) \sin\theta, \end{aligned}$$

and since θ is between 0 and π, it follows from the relations which are satisfied by the mutually orthogonal directions (1) that

$$\cos\theta = \alpha\phi_s(0, u, v) + \beta\psi_s(0, u, v) + \gamma\chi_s(0, u, v),$$

$$\sin\theta = (l \cos u + \lambda \sin v)\phi_s(0, u, v) + (m \cos v + \mu \sin v)\psi_s(0, u, v) + (n \cos v + \nu \sin v)\chi_s(0, u, v) > 0.$$

If A, B, C are the cosines of a direction not coincident with the tangent to $\mathfrak{L}$, then there is one and but one value of v (modulo 2π) for which the tangent to the extremal E is coplanar with A, B, C and the tangent to $\mathfrak{L}$, and which lies on the same side of the tangent to $\mathfrak{L}$ with A, B, C. For in order that these conditions may be satisfied the determinant

$$\begin{vmatrix} \phi_s(0, u, v) & \alpha & A \\ \psi_s(0, u, v) & \beta & B \\ \chi_s(0, u, v) & \gamma & C \end{vmatrix}$$

must vanish, and the expression

$$\cos\omega = A(l \cos v + \lambda \sin v) + B(m \cos v + \mu \sin v) + C(n \cos v + \nu \sin v)$$

must be positive. With the help of equations (4) the former condition is seen to determine the ratio of $\cos v$ and $\sin v$, and the latter determines their signs,

uniquely. It follows that if v and v' are not congruent modulo π the determinant

$$(5)\qquad \begin{vmatrix} \phi_s(0, u, v) & \alpha & \phi_s(0, u, v') \\ \psi_s(0, u, v) & \beta & \psi_s(0, u, v') \\ \chi_s(0, u, v) & \gamma & \chi_s(0, u, v') \end{vmatrix} = \sin(v - v')$$

is always different from zero.

By substitution of the values for ϕ_s and ϕ_{sv} from equations (4) a simple computation shows that when $v \equiv v'$ (modulo π)

$$(6)\qquad \begin{vmatrix} \phi_s(0, u, v) & \alpha & \phi_{sv}(0, u, v') \\ \psi_s(0, u, v) & \beta & \psi_{sv}(0, u, v') \\ \chi_s(0, u, v) & \gamma & \chi_{sv}(0, u, v') \end{vmatrix} = \pm \sin\theta \sin\theta' \neq 0.$$

The notation $\mathfrak{L}_\delta$ will be used to denote the δ-neighborhood of the curve $\mathfrak{L}$, and $\mathfrak{L}'_\delta$ will denote the points of this set exclusive of those on the curve of which $\mathfrak{L}$ is a part. In the following pages any two points (s, u, v) and (s', u', v') are regarded as identical when $s = s'$, $u = u'$, and v is congruent to v' modulo 2π $(v \equiv v')$, otherwise distinct. The region $\mathfrak{S}_\epsilon$ is

$$(\mathfrak{S}_\epsilon)\qquad 0 \leqq s \leqq \epsilon, \qquad u_1 - \epsilon \leqq u \leqq u_2 + \epsilon, \qquad v \text{ any value},$$

and $\mathfrak{S}'_\epsilon$ is $\mathfrak{S}_\epsilon$ exclusive of the points for which $s = 0$.

The purpose of the present section is the proof of the following statement:

Two positive constants δ, ϵ can be determined so that to any point (x, y, z) in the neighborhood $\mathfrak{L}'_\delta$ there corresponds one and but one solution $(x, y, z : s, u, v)$ of the equations (E) for which the values (s, u, v) lie in the region $\mathfrak{S}'_\epsilon$.

In the first place ϵ can be chosen so small that no two points (s, u, v), (s', u', v') in the region $\mathfrak{S}_\epsilon$ define the same point (x, y, z) by means of the equations (E), unless $s = s' = 0$, $u = u'$. In other words, no two of the extremals (E) can intersect for values of (s, u, v) in $\mathfrak{S}_\epsilon$ unless they pass through the same point P of the curve of which $\mathfrak{L}$ is a part, in which case P is the only point of intersection, and furthermore no extremal intersects itself. For if these statements were not true, a sequence

$$(7)\qquad \{s_i, u_i, v_i : s'_i, u'_i, v'_i\} \qquad (i = 1, 2, \cdots),$$

$$(s_i, s'_i) \neq (0, 0), \qquad (s_i, u_i, v_i) \neq (s'_i, u'_i, v'_i)$$

of solutions of the equations

$$(8)\qquad \begin{aligned} \phi(s, u, v) &= \phi(s', u', v'), \\ \psi(s, u, v) &= \phi(s', u', v'), \\ \chi(s, u, v) &= \chi(s', u', v'), \end{aligned}$$

could be determined with limiting values of the form

$$(9) \qquad (0, u_0, v_0 : 0, u_0', v_0'), \qquad u_1 \leqq u_0 \leqq u_2, \qquad u_1 \leqq u_0' \leqq u_2,$$

which would also satisfy the equations. Since $\mathfrak{L}$ does not intersect itself, it follows from the identities (3) that u_0 and u_0' must be the same.

With the help of the identities (3) any one of the equations (8), say the first, can be written in the form

$$[\phi(s, u, v) - \phi(0, u', v)] - [\phi(s', u', v') - \phi(0, u', v')] = 0,$$

or by Taylor's formula

$$s\int_0^1 \phi_s[ts, u' + t(u - u'), v]dt + (u - u')\int_0^1 \phi_u[ts, u' + t(u - u'), v]dt$$
$$- s'\int_0^1 \phi_s(ts', u', v')dt = 0.$$

The three equations so found would have solutions s_i, $u_i - u_i'$, s_i' not all zero provided by the sequence (7). Their determinant with respect to the arguments $s, u - u', s'$ outside the integral signs, is a continuous function of s, u, v, s', u', v' which must vanish for the arguments (7), and therefore also for the arguments (9). But for the latter it takes the form (5), showing that v_0 must be congruent to v_0' modulo π, since the determinant (5) vanishes for such values only.

From the identities (3) it appears that the function $\phi_v(0, u, v)$ is identically zero, and therefore

$$(10) \qquad \phi_v(s, u, v) = s\int_0^1 \phi_{vs}(\tau s, u, v)d\tau.$$

By an application of Taylor's formula, and then of equation (10), the first of equations (8) can be written again in the form

$$\phi(s, u, v) - \phi(s', u', v') = \phi(s, u, v) - \phi(s', u', v) + \phi(s', u', v) - \phi(s', u', v')$$
$$= (s - s')\int_0^1 \phi_s[s' + t(s - s'), u' + t(u - u'), v]dt$$
$$+ (u - u')\int_0^1 \phi_u[s' + t(s - s'), u' + t(u - u'), v)]dt$$
$$+ (v - v')s'\int_0^1\int_0^1 \phi_{sv}[\tau s', u', v' + t(v - v')]d\tau dt.$$

The three equations so found have a determinant with respect to the quantities $s - s', u - u', (v - v')s'$ outside the integral signs, which is a continuous function reducing to (6) for the arguments $(0, u, v : 0, u, v')$. At the limiting values (9) this determinant is different from zero since $u_0 = u_0'$, $v_0 \equiv v_0'$ (modulo π).

Hence for a sufficiently large value of i any quantities $s_i - s'_i$, $u_i - u'_i$, $(v_i - v'_i)s'_i$ which satisfy these equations must all vanish, and by interchanging (s, u, v) with (s', u', v') in equation (8) it follows that $(v'_i - v_i)s_i$ must also be zero. If the sequence (7) existed, therefore, the points (s_i, u_i, v_i) and (s'_i, u'_i, v'_i) would have to coincide, or else $s_i = s'_i$, $u_i = u'_i$, and s_i and s'_i would have to vanish simultaneously, for each sufficiently large value of i. But these are the properties which the sequence (7) was assumed not to have, and consequently its existence is impossible.

The constant ϵ can be subjected to a second restriction, if necessary, to the effect that for all values of (s, u, v) in $\mathfrak{S}'_\epsilon$ the functional determinant of equations (E) with respect to s, u, v is different from zero. With the help of equation (10) this determinant takes the form

$$\begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix} = sV(s, u, v),$$

where V is a continuous function in the region $\mathfrak{S}_\epsilon$, different from zero for $s = 0$ on account of the condition (6). Hence if ϵ is properly chosen V will remain different from zero in $\mathfrak{S}'_\epsilon$.

Suppose now that ϵ has been chosen so as to satisfy the two restrictions described above. The points (x, y, z) of the extremals (E) for which $u = u_1 - \epsilon$ or $u = u_2 + \epsilon$, or for which $s = \epsilon$, form a closed set which has no point in common with the arc $\mathfrak{L}$. Let $\delta\sqrt{3}$ be the minimum distance between the points of this set and the points of $\mathfrak{L}$. Then any solution $(x, y, z : s, u, v)$ of equations (E), corresponding to a point $P(x, y, z)$ in $\mathfrak{L}'_\delta$, has values (s, u, v) in the *interior* of $\mathfrak{S}'_\epsilon$. For P is not on $\mathfrak{L}$ and therefore can not correspond to a set (s, u, v) for which $s = 0$, and its distance from $\mathfrak{L}$ is less than $\delta\sqrt{3}$, so that, on account of the way in which δ has just been selected, no one of the three conditions $u = u_1 - \epsilon$, $u = u_3 + \epsilon$, $s = \epsilon$ can be satisfied.

From the first property of the constant ϵ it follows, furthermore, that P can not correspond to more than one set (s, u, v) in $\mathfrak{S}'_\epsilon$.

On the other hand, each point P in $\mathfrak{L}'_\delta$ corresponds to at least one set of values (s, u, v) in the interior of $\mathfrak{S}_\epsilon$. For consider any particular point $P'(x', y', z')$ in $\mathfrak{L}'_\delta$ which has corresponding to it the values (s', u', v'). Join P' with P by an arc D

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad t' \leqq t \leqq T$$

of class C' which lies entirely within $\mathfrak{L}'_\delta$. The functional determinant with respect to s, u, v of the first members of the equations

$$(11) \quad \phi(s, u, v) = x(t), \qquad \psi(s, u, v) = y(t), \qquad \chi(s, u, v) = z(t)$$

is different from zero at (s', u', v') on account of the second property of the constant ϵ, and these equations have the particular solutions (x', y', z', t'). It follows that equations (11) have continuous solutions $s(t)$, $u(t)$, $v(t)$ in the neighborhood of t', which can be continued along the arc D up to the point P, so that P also corresponds to a solution $(x, y, z : s, u, v)$ of equations (E) for which (s, u, v) necessarily lies in $\mathfrak{S}'_\epsilon$. Otherwise there would be an upper bound τ to the values of t which could be reached by continuation, with a corresponding solution $(s_\tau, u_\tau, v_\tau, \tau)$ of equations (11) for which (s_τ, u_τ, v_τ) would lie in $\mathfrak{S}'_\epsilon$. The functional determinant of equations (11) would still be different from zero at (s_τ, u_τ, v_τ) and consequently τ could not be the upper bound described.

It has therefore been shown that to each point (x, y, z) in $\mathfrak{L}'_\delta$ there corresponds a solution $(x, y, z : s, u, v)$ of equations (E) for which (s, u, v) lies in $\mathfrak{S}'_\epsilon$, and on account of the way in which ϵ was chosen this solution is unique.

The functions

$$s(x, y, z),\ u(x, y, z),\ v(x, y, z) \tag{12}$$

which with x, y, z solve equations (E) have the following properties in the region $\mathfrak{L}'_\delta$:

1) *They are all three of class C' in $\mathfrak{L}'_\delta$.*

2) *In $\mathfrak{L}_\delta$ the first two are continuous and for a point (x_0, y_0, z_0) on $\mathfrak{L}$ corresponding to the parameter value u_0,*

$$s(x_0, y_0, z_0) = 0, \qquad u(x_0, y_0, z_0) = u_0.$$

3) *The function $v(x, y, z)$ is infinitely many valued, its values corresponding to a given (x, y, z) being all congruent modulo 2π, but $\cos v$ and $\sin v$ are single valued functions*

$$\cos v = a(x, y, z), \qquad \sin v = b(x, y, z) \tag{13}$$

of class C' in L'_δ.

4) *Let C be a curve*

$$x = f(t), \qquad y = g(t), \qquad z = h(t) \qquad (0 \leqq t \leqq T), \tag{C}$$

with a continuously turning tangent, which intersects $\mathfrak{L}$ at a point P (x_0, y_0, z_0, u_0) for $t = 0$, but is not tangent to $\mathfrak{L}$ at P, and for which t is the length of arc. The tangent to C at P determines uniquely a direction

$$l \cos v_0 + \lambda \sin v_0, \qquad m \cos v_0 + \mu \sin v_0, \qquad n \cos v_0 + \nu \sin v_0, \tag{14}$$

coplanar with the tangents to C and $\mathfrak{L}$ at P, and lying on the same side of the tangent to $\mathfrak{L}$ with the positive tangent to C. As t approaches zero the functions $s(t)$, $u(t)$, $a(t)$, $b(t)$ defined by substituting the values of x, y, z

from (C) *in the functions* (12) *and* (13), *approach the limits* 0, u_0, $\cos v_0$, $\sin v_0$, *and the functions* $s(t)$, $u(t)$ *have derivatives at* $t=0$ *which have the values*

$$s'(0) = \sin\omega \csc\theta, \qquad u'(\theta) = \cos\omega - \sin\omega\cot\theta,$$

where ω *is the angle between the positive tangents to* C *and* $\mathfrak{L}$, *and* θ *is the corresponding angle for* E *and* $\mathfrak{L}$.

The properties 1) and 3) follow readily from the usual theorems concerning implicit functions and the periodicity of ϕ, ψ, χ with respect to v. For at any solution $(x', y', z' : s', u', v')$ of equations (E) for which (x', y', z') is in $\mathfrak{L}'_\delta$ and (s', u', v') is in $\mathfrak{S}'_\epsilon$, the functional determinant of ϕ, ψ, χ with respect to s, u, v is different from zero.

If P_0 is a point (x_0, y_0, z_0) on $\mathfrak{L}$ corresponding to the value u_0, then for any $\eta > 0$ a value $\rho > 0$ can be selected so that $|s(x, y, z)| < \eta$ whenever (x, y, z) is in the ρ-neighborhood of P_0. For the values (s, u, v) in $\mathfrak{S}'_\epsilon$ for which $s \geqq \eta$ define a closed set of points (x, y, z) distinct from P_0, whose minimum distance from the point P_0 is a positive constant $\rho\sqrt{3}$. Hence any (x, y, z) in the ρ-neighborhood of P_0 will have its distance from P_0 less than ρ, and will necessarily correspond to a value of s for which $0 < s < \eta$. In a similar manner the points of $\mathfrak{S}_\epsilon$ for which $|u - u_0| \geqq \eta$ define a closed set of (x, y, z)-points distinct from P_0, and the continuity of $u(x, y, z)$ as stated in 2) can be proved.

It remains to consider the property 4). An application of Taylor's formula shows that the equations

$$\begin{aligned}
&\frac{1}{t}\{\phi(\sigma t,\ u_0 + \tau t,\ v) - x_0\} = \frac{f(t) - x_0}{t},\\
(15)\qquad &\frac{1}{t}\{\psi(\sigma t,\ u_0 + \tau t,\ v) - y_0\} = \frac{g(t) - y_0}{t},\\
&\frac{1}{t}\{\chi(\sigma t,\ u_0 + \tau t,\ v) - z_0\} = \frac{h(t) - z_0}{t},
\end{aligned}$$

reduce for $t = 0$ to

$$(16)\qquad \sigma\phi_s + \tau\alpha = f'(0), \qquad \sigma\psi_s + \tau\beta = g'(0), \qquad \sigma\chi_s + \tau\gamma = h'(0),$$

where $\sigma \geqq 0$ and the arguments of ϕ_s, ψ_s, χ_s are $(0, u_0, v)$. As a result of the statements proved on page 334, there can be but one value v_0 (modulo 2π) for which these equations are satisfied, and the corresponding direction (14) has the properties described above. It follows that equations (15) have for $t = 0$ unique solutions

$$\sigma_0 = \sin\omega\csc\theta, \qquad \tau_0 = \cos\omega - \sin\omega\cot\theta, \qquad v_0$$

determined by equations (16). The functional determinant of equations (15) for the values $(\sigma_0, \tau_0, v_0, t = 0)$ is found with the help of the formula (10) to be the determinant (6). Hence these equations have continuous solutions $\sigma(t)$, $\tau(t)$, $v(t)$ reducing to σ_0, τ_0, v_0 for $t = 0$. The corresponding functions

$$s = \sigma t, \qquad u = u_0 + \tau t, \qquad \cos v = a(t), \qquad \sin v = b(t)$$

define solutions of equations (E), and hence must coincide with the functions $s(t)$, $u(t)$, $a(t)$, $b(t)$ determined under 4). It follows readily that $s'(0)$, $u'(0)$ exist and have the values σ_0, τ_0, while $a(t)$, $b(t)$, approach $\cos v_0$ and $\sin v_0$, respectively, as t approaches zero.

GROUPS GENERATED BY TWO OPERATORS (s_1, s_2) SATISFYING THE EQUATION $s_1 s_2^2 = s_2 s_1^2$.*

BY

G. A. MILLER

§ 1. *Introduction.*

In 1878 CAYLEY considered the groups generated by two operators (s_1, s_2) satisfying the equation $s_1 s_2 = s_2^2 s_1^2$ and observed the interesting theorem that it is not possible to represent all the operators of such a group in the form $s_1^\alpha s_2^\beta$ except in the special trivial case in which the group is cyclic.† This theorem is of historic interest as it is one of the earliest theorems relating to a general category of abstract groups. More recently NETTO published a few additional general results which may be deduced from this equation and he also determined the possible groups when the orders of s_1, s_2 are both less than 6.‡ The results obtained by NETTO were extended by the author of the present paper, mainly by means of special considerations, as regards the possible groups in which the two generating operators are of order 6.§

For convenience the equation under consideration is written, in the present paper, in the form $s_1 s_2^2 = s_2 s_1^2$, and a number of new general results are deduced from it. By means of these the known results are obtained much more easily than in the earlier papers. It is also proved that the two equations $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ imply that $s_2^5 = 1$ and hence either the first or the second of the three generational relations $s_1^5 = 1$, $s_2^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ given by NETTO in the article cited above is redundant. This is of interest since it proves that the non-cyclic group of order 55 is the only non-abelian group which can be generated by two operators satisfying the two conditions $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ and hence it establishes contact between the present paper and the one devoted to the "Finite groups which may be defined by two operators satisfying two conditions." ||

Among the other results the following are perhaps of most interest. There

* Presented to the Society, December 29, 1909.

† CAYLEY, Messenger of Mathematics, vol. 7 (1878), p. 188.

‡ NETTO, Crelle's Journal, vol. 128 (1905), p. 243.

§ Quarterly Journal of Mathematics, vol. 40 (1909), p. 197.

|| American Journal of Mathematics, vol. 31 (1909), p. 167.

is an infinite system of solvable groups each of which may be generated by two operators which satisfy the three conditions $s_1^6 = 1$, $s_2^6 = 1$, $s_1 s_2^2 = s_2 s_1^2$; but there are only two groups which may be generated by two operators satisfying the three conditions $s_1^7 = 1$, $s_2^7 = 1$, $s_1 s_2^2 = s_2 s_1^2$. Combining these results with those known earlier we may say that the only groups which can be generated by two operators satisfying the three conditions,

$$s_1^\alpha = 1, \qquad s_2^\alpha = 1, \qquad s_1 s_2^2 = s_2 s_1^2,$$

are: The group of order 2 when $\alpha = 2$; the group of order 3 and the tetrahedral group when $\alpha = 3$; the cyclic groups of orders 2 and 4, and the holomorph of the group of order 5 when $\alpha = 4$; the group of order 5 and the non-cyclic group of order 55 when $\alpha = 5$; the cyclic groups of orders 2, 3 and 6, and an infinite system of solvable non-abelian groups when $\alpha = 6$; the group of order 7 and the non-cyclic group of order 203 when $\alpha = 7$. The fact that for $\alpha = 6$ there is an infinite system of possible groups, while there are only two for the larger value 7 of α, is to be emphasized.

§ 2. *General considerations.*

By writing the equation $s_1 s_2^2 = s_2 s_1^2$ in the form $s_1 s_2^2 s_1^{-1} = s_2 s_1$, and raising both members to the same power, there results the formula

$$s_1 s_2^{2n} s_1^{-1} = (s_2 s_1)^n.$$

In a similar manner we obtain the formula

$$s_2 s_1^{2n} s_2^{-1} = (s_1 s_2)^n.$$

From these two formulas we see directly that the three operators s_1^2, s_2^2, $s_1 s_2$ are of the same order, and that either s_1, s_2 have the same order or else the order of one of these operators is an odd number and the order of the other is twice this odd number. These results may be expressed in the form of a theorem as follows: *If two operators satisfy the equation $s_1 s_2^2 = s_2 s_1^2$, their squares are of the same order as their product.*

By transforming the given formulas with respect to s_1^{-1}, s_2^{-1}, respectively, we obtain

$$s_1^2 s_2^{2n} s_1^{-2} = (s_1 s_2)^n \qquad s_2^2 s_2^{2n} s_1^{-2} = (s_2 s_1)^n,$$

and hence

$$(A) \qquad s_1 s_2^{2n} s_1^{-1} s_2^2 s_1^{-2n} s_2^{-2} = s_2 s_1^{2n} s_2^{-1} s_1^2 s_2^{-2n} s_1^{-2} = 1.$$

The commutator of s_1, s_2 is $s_1^{-1} s_2^{-1} s_1 s_2 = s_1 \cdot s_1^{-2} s_2^{-1} \cdot s_1 s_2 = s_1 s_2^{-1}$, and hence

$$(B) \qquad (s_1^{-1} s_2^{-1} s_1 s_2)^2 = (s_1 s_2^{-1})^2 = s_1^3 \cdot s_1^{-2} s_2^{-1} \cdot s_1 s_2^{-1} = s_1^3 s_2^{-3}.$$

The group generated by the two commutators $s_1 s_2^{-1}$, $s_2^{-1} s_1$ is invariant under

the group G generated by s_1, s_2 since

$$s_1^{-1} s_2^{-1} s_1^2 = s_1 \cdot s_1^{-2} s_2^{-1} s_1^2 = s_1 s_2^{-2} s_1 = s_1 s_2^{-1} \cdot s_2^{-1} s_1,$$

$$s_2^{-1} \cdot s_2^{-1} s_1 \cdot s_2 = s_2^{-1} s_1^2 s_2^{-1} = s_2^{-1} s_1 \cdot s_1 s_2^{-1}.$$

As the quotient group of G with respect to this group is cyclic it results that this invariant subgroup is the commutator subgroup of G. That is, *if two operators satisfy the equation $s_1 s_1^2 = s_2 s_1^2$, they generate a group whose commutator subgroup is generated by two conjugate commutators and whose commutator quotient is cyclic.*

The commutator $s_1 s_2^{-1}$ is transformed into $s_2^{-1} s_1$ by each of the two operators s_1, s_2. It is also transformed into the same operator by each of the two operators s_1^3, s_2^3, since $(s_1 s_2^{-1})^2 = s_1^3 s_2^{-3}$. The two operators $s_1 s_2 s_1$, $s_2 s_1 s_2$ must have the same order since they may be obtained by transforming $s_2 s_1^2$, $s_1 s_2^2$ by s_1^{-1}, s_2^{-1} respectively. Hence $s_2^{-2} s_1 s_2 s_1 s_2^2$, $s_1^{-2} s_2 s_1 s_2 s_1^2$ must also have the same order. The last two operators are equal respectively to $s_2^{-1} s_1^4$, $s_1^{-1} s_2^4$, as may be seen by employing the relation $s_1 s_2^2 = s_2 s_1^2$. This fact was proved by Netto by means of more lengthy considerations. For convenience in the following applications the equation $s_1 s_2^2 = s_2 s_1^2$ is written also in the following forms:

$$s_2^{-2} s_1^{-1} = s_1^{-2} s_2^{-1}, \qquad s_1^{-1} s_2 = s_2^2 s_1^{-2}, \qquad s_2^{-1} s_1 = s_1^2 s_2^{-2}.$$

Throughout the following two paragraphs it will be assumed that s_1 and s_2 are both of odd order. Hence formula (A) may be written in the simpler form

$$(A') \qquad s_1 s_2^m s_1^{-1} s_2^2 s_1^{-m} s_2^{-2} = s_2 s_1^m s_2^{-1} s_1^2 s_2^{-m} s_1^{-2} = 1,$$

where m is an arbitrary integer. If we let $m = 1$ there results

$$(A'') \qquad s_1 s_2 s_1^{-1} s_2^2 s_1^{-1} s_2^{-2} = s_1 s_2 s_1^{-2} s_2 s_1 s_2^{-2} = 1.$$

From the last equation we have

$$s_2 (s_1 s_2^{-1})^m s_2^{-1} = s_1 (s_1 s_2^{-1})^m s_1^{-1}.$$

Hence $s_1 s_2^{-1}$, $s_2^{-1} s_1$ are commutative. That is, *if two operators of odd order satisfy the equation $s_1 s_2^2 = s_2 s_1^2$ they generate a solvable group whose commutator subgroup is either cyclic or the direct product of two cyclic groups.*

All the exponents of the second member of (A'') may be multiplied by 2 without changing its value, since $s_1 s_2 = s_1^2 s_2^2 s_1^{-2}$ and $s_2 s_1 = s_2^2 s_1^2 s_2^{-2}$. Hence the equation

$$s_1^2 s_2^2 s_1^{-4} s_2^2 s_1^2 s_2^{-4} = 1.$$

By employing the given relations between s_1, s_2 and especially those which can be directly obtained from $s_1 s_2^n s_1^{-1} = s_2^2 s_1^n s_2^{-2}$, $s_2 s_1^n s_2^{-1} = s_1^2 s_2^n s_1^{-2}$, it is not difficult

to verify the following equations:

$$(s_1 s_2^{-1})^5 = s_1^3 s_2^{-3} \cdot s_1 s_2^{-1} \cdot s_1^3 s_2^{-3} = s_1^3 s_2^{-4} s_1^2 s_2 s_1 s_2^{-3} = s_1^3 s_2^{-4} s_1^3 s_2^2 s_1^{-1} s_2^{-3}$$
$$= s_1^3 s_2^{-4} s_1^3 s_2^4 s_1^{-2} s_2^{-4} = s_1^3 s_2^{-4} s_1 s_2 s_1^4 s_2^{-5} = s_1^3 s_2^{-3} s_1^2 s_2^{-1} s_1^4 s_2^{-5}$$
$$= s_1^3 s_2^{-2} s_1^{-1} s_2^{-1} s_1^6 s_2^{-5} = s_1 s_2^{-2} s_1^6 s_2^{-5}.$$

Hence it results directly that

$$(B') \qquad (s_1 s_2^{-1})^{4n} = s_2^{-1} (s_1^6 s_2^{-6})^n s_2.$$

From the equations

$$1 = s_2 s_1 s_2^{-1} s_1^2 s_2^{-1} s_1^{-2} = s_2 s_1 s_2^{-2} s_1 s_2 s_1^{-2} = s_2 s_1 s_2^{-1} s_1^2 s_2^{-1} s_1^{-2} = s_2 s_1^3 s_2^{-2} s_1^3 s_2^{-2} s_1^{-3},$$

it follows that $s_2^2 \cdot s_2^{-1} s_1^3 \cdot s_2^{-2} = s_1^3 s_2^2 s_1^{-3}$, and hence the two operators $s_2^{-1} s_1^3$, s_2^2 have the same order. In a similar manner we observe from the equations

$$1 = s_1 s_2^{-1} s_1^{-1} s_2^2 s_1 s_2^{-2} = s_2^{-2} s_1 s_2^{-1} s_1^{-1} s_2^2 s_1 = s_2^{-2} s_1 s_2^{-1} s_1^{-2} s_2 s_1^3$$
$$= s_2^{-2} s_1^3 s_2^{-2} s_1^{-3} s_2 s_1^3 = s_2^2 \cdot s_2^{-4} s_1^3 \cdot s_2^{-2} \cdot s_1^{-3} s_2 s_1^3$$

that $s_2^{-4} s_1^3$ is of the same order as s_2. That is, *when s_1, s_2 are both of odd order all of the following five operators: s_1, s_2, $s_1 s_2$, $s_2^{-1} s_1^3$, $s_2^{-4} s_1^3$ are of the same order.*

§ 3. *Several Special Cases.*

If we add to the equation $s_1 s_2^2 = s_2 s_1^2$ the additional equation $s_1^n = 1$, it is not difficult to determine all the possible groups for the different values of n from 2 to 5. If $n = 2$ it results directly from the former of these equations that $s_1 = s_2^{-1}$ and hence the group (G) generated by s_1, s_2 is of order 2. If $n = 3$ it results from the preceding section that the order of s_2 is either 3 or 6 and that the order of the commutator $s_1 s_2^{-1}$ is either 2 or 4, since $(s_1 s_2^{-1})^2 = s_1^3 s^{-3}$. If s_2 is of order 3 the order of $s_1 s_2^{-1}$ is 2 and hence G is generated by two operators of order 3 whose product is of order 2. It is well known that every pair of operators which satisfy these conditions generate the tetrahedral group. On the other hand, if s_2 is of order 6 when $n = 3$, the order of $s_1 s_2^{-1}$ is 4 and $(s_1 s_2^{-1})^2 = s_2^3$. It is known that the group of order 24 which does not involve a subgroup of order 12 is the only group that can be generated by two operators which fulfil these conditions.*

When $n = 4$ it results from the preceding section that s_2 is also of order 4 and that $s_1 s_2$ is of order 2. Hence, $s_2 s_1 = s_1^{-1} s_2^{-1}$, and $s_1^{-2} \cdot s_1 s_2^{-1} \cdot s_1^2 = s_1^{-1} s_2^{-1} \cdot s_1^2 = s_2 s_1^3 = s_2 s_1^{-1} = (s_1 s_2^{-1})^{-1}$. On the other hand, $s_1^{-1} \cdot s_1 s_2^{-1} \cdot s_1 = s_2^{-1} s_1 = s_2^3 s_1^{-3} = (s_1 s_2^{-1})^{-2}$. That is, s_1^2 transforms $s_1 s_2^{-1}$ both into its 4th power and also into its

*Transactions of the American Mathematical Society, vol. 8 (1907), p. 4. An operator is said to fulfil the condition $s_1^n = 1$ if it is of order n; it is said to satisfy this condition if its order is a divisor of n.

inverse, and hence the order of $s_1 s_2^{-1}$ is 5. Since s_1 transforms $s_1 s_2^{-1}$ into the inverse of its square, it has been proved that *G is the holomorph of the group of order* 5 *when* $n = 4$. From the preceding section it results that the order of s_2 is either 5 or 10 when $n = 5$. We shall now prove that it cannot be 10. This will be done by assuming that s_1, s_2 fulfil the conditions $s_1^5 = 1$, $s_2^{10} = 1$, $s_1 s_2^2 = s_2 s_1^2$ and by showing that this assumption leads to a contradiction.

Suppose that s_1, s_2 fulfill the three conditions

$$s_1^5 = 1, \qquad s_2^{10} = 1, \qquad s_1 s_2^2 = s_2 s_1^2.$$

From formula (A) of the preceding section we obtain

$$s_1 s_2^6 s_1^{-1} s_2^2 s_1^{-1} s_2^{-2} = 1.$$

Hence

$$s_1 s_2^5 \cdot s_2 s_1^2 \cdot s_1^2 s_2^2 s_1^{-1} \cdot s_2^{-2} = s_1 s_2^5 s_1 s_2^2 s_1 s_2 s_1 s_2^{-2} = s_1 s_2^5 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^{-4}$$
$$= s_1 s_2^5 s_1 s_2^3 s_1^{-1} s_2^{-4} = s_1 s_2^5 s_1 s_2^5 s_1^{-2} s_2^5 = 1.$$

From the last of these equations it is evident that the conjugates of s_2^5 with respect to s_1^{-1}, s_2^{-2} are commutative, since the product of two operators of order 2 $(s_1 s_2^5 s_1^{-1}, s_1^2 s_2^5 s_1^{-2})$ is of order 2. Transforming this last equation by s_1^{-1} leads to the equation

$$s_1^3 s_2^5 s_1^{-3} = s_1 s_2^5 s_1^{-1} \cdot s_1^2 s_2^5 s_1^{-2} = s_2^5.$$

Since s_1^3 is commutative with s_2^5 and since s_1^3 generates s_1 it results that s_1 is commutative with s_2^5, and hence we have $s_2^{10} = s_2^5 = 1$. As this is contrary to the hypothesis, it has been proved that *the two equations* $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ *imply that* $s_2^5 = 1$.

We proceed to determine all the groups which can be generated by two operators satisfying the two conditions

$$s_1^5 = 1, \qquad s_1 s_2^2 = s_2 s_1^2.$$

As $s_1^{-3} \cdot s_1 s_2^{-1} \cdot s_1^3 = s_1^{-2} s_2^{-1} s_1^3 = s_2^{-2} s_1^2 = s_2^3 s_1^{-3} = (s_1 s_2^{-1})^{-2}$ according to formula (B) of the preceding section, it results that $s_1^{-15}(s_1 s_2^{-1}) s_1^{15} = (s_1 s_2^{-1})^{-32}$ and hence the order of $s_1 s_2^{-1}$ must divide 33. To prove that this order is 11 or 1 we may proceed as follows: $s_1^{-6} \cdot s_1 s_2^{-1} \cdot s_1^6 = s_1^{-1} s_1 s_2^{-1} s_1 = s_2^{-1} s_1 = (s_1 s_2^{-1})^4$. Multiplying both members of the last equation by $s_1^3 s_2^{-3} = (s_1 s_2^{-1})^2$, we find that $(s_1 s_2^{-1})^6 = s_1^3 s_2 s_1$; hence $(s_1 s_2^{-1})^{11} = s_2 s_1^{-1} \cdot s_1^3 s_2 s_1^{-1} s_2 s_1 = s_2 s_1^2 s_2 s_1^{-1} s_2 s_1 = s_1 s_2^3 s_1^{-1} s_2 s_1$ $= s_1 \cdot s_2^{-2} s_1^{-1} \cdot s_2 s_1 = s_1 \cdot s_1^{-2} s_2^{-1} \cdot s_2 s_1 = 1$. As $s_1 s_2^{-1}$ is of order 11 or 1 it results that G is either the non-cyclic group of order 55 or the group of order 5. That is, *the group of order* 5 *and the non-cyclic group of order* 55 *are the only two groups which can be generated by two operators which satisfy both of the conditions* $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$.

§ 4. *Groups whose Two Generators satisfy the Three Conditions* $s_1^6 = 1,\ s_2^6 = 1,\ s_1 s_2^2 = s_2 s_1^2.$

In the preceding section we considered all the possible groups when the order of at least one of the operators s_1, s_2 is less than 6. Hence we shall assume throughout the present section that each of these operators is actually of order 6. We shall first prove that each of the two commutators $s_1 s_2^{-1}$, $s_2^{-1} s_1$ which generate the commutator subgroup of G must transform the square of the other into its inverse. This interesting fact may be established as follows:

$$s_2^{-1} s_1 \cdot s_1 s_2^{-1} \cdot s_1^{-1} s_2 = s_2^{-1} s_1^2 s_2^{-1} \cdot s_1^{-1} s_2 = s_2^{-1} s_1^2 s_2 s_1^{-2}.$$

Hence it results that

$$s_2^{-1} s_1 (s_1 s_2^{-1})^2 s_1^{-1} s_2 = s_2^{-1} s_1^2 s_2 \cdot s_1^{-2} s_2^{-1} \cdot s_1^2 s_2 s_1^{-2} = s_2^{-1} s_1^2 \cdot s_2^{-1} s_1 \cdot s_2 s_1^{-2}$$
$$= s_2^{-1} \cdot s_1^{-2} s_2^{-1} \cdot s_1^{-2} = s_2^{-3} s_1^{-3} = (s_1 s_2^{-1})^{-2}.$$

In a similar manner we may prove that $s_1 s_2^{-1}$ transforms $s_2^{-1} s_1$ into its inverse, as follows:

$$s_1 s_2^{-1} \cdot s_2^{-1} s_1 \cdot s_2 s_1^{-1} = s_1 s_2^{-2} s_1 s_2 s_1^{-1} = s_1 s_2^{-1} s_1^2 s_2^{-1} s_1^{-1},$$

$$s_1 s_2^{-1} (s_2^{-1} s_1)^2 s_2 s_1^{-1} = s_1 s_2^{-1} \cdot s_1^2 s_2^{-2} \cdot s_1^2 s_2^{-1} s_1^{-1} = s_1 s_2^{-2} s_1^3 s_2^{-1} s_1^{-1}$$
$$= s_1^{-1} s_2^{-1} \cdot s_1^{-2} s_2^{-1} \cdot s_1^{-1} = s_1^{-1} s_2^{-3} s_1^{-2} = (s_2^{-1} s_1)^{-2}.$$

Having proved that the commutator subgroup of G is generated by two operators each of which transforms the square of the other into its inverse we proceed to consider this general category of groups. That is, we shall consider the category of groups generated by t_1, t_2 where no restrictions are imposed on t_1, t_2 except that they satisfy the two equations

$$t_1^{-1} t_2^2 t_1 = t_2^{-2}, \qquad t_2^{-1} t_1^2 t_2 = t_1^{-2}.$$

These conditions imply that $(t_1 t_2)^2 = (t_2 t_1)^{-2}$, and hence each of the three cyclic groups generated by the operators t_1^2, t_2^2, $(t_1 t_2)^2$ respectively is invariant under the group K generated by t_1, t_2. Any two of these three cyclic subgroups have at most two common operators, since each one of the three operators t_1, t_2, $t_1 t_2$ transforms into their inverses all the operators of two of these three cyclic groups and those of the third into themselves. That is, all the operators of the three groups $\{t_2^2, (t_1 t_2)^2\}$, $\{t_1^2, (t_1 t_2)^2\}$, $\{t_1^2, t_2^2\}$ are transformed into their inverses by t_1, t_2, $t_1 t_2$ respectively.

From the preceding results it is evident that the abelian group $\{t_1^2, t_2^2, (t_1 t_2)^2\}$ is invariant under K. Each of the three groups

$$\{t_1, t_2^2, (t_1 t_2)^2\},\ \{t_1^2, t_2, (t_1 t_2)^2\},\ \{t_1^2, t_2^2, t_1 t_2\}$$

is also invariant under K since $t_2^{-1} t_1 t_2 = t_2^{-2} t_1^{-1} (t_1 t_2)^2$. As a consequence, the quotient group of K with respect to $\{t_1^2, t_2^2, (t_1 t_2)^2\}$ is either the four group or a subgroup of this group, and K is always solvable. Hence the theorem: *If each of two operators transforms the square of the other into its inverse, these operators must also transform the square of their product into its inverse, and these three squares generate an abelian subgroup which is invariant under the group generated by the two operators. The index of this subgroup is a divisor of* 4.

It is not difficult to construct such groups in which the orders of $t_1^2, t_2^2, (t_1 t_2)^2$ are three arbitrary numbers. To do this we may write, in distinct sets of letters, three cyclic substitution groups of the required orders, each generator being composed of two equal cycles when its order is even. We then select t_1, t_2 in such a way that they generate respectively the first two of these cyclic groups and that their components involving the letters of the other cyclic groups are of order 2, transform the operators of these cyclic groups into their inverses and give a product of the required order. It follows directly from the properties of the dihedral group that these substitutions can be so selected that t_1, t_2 have the given properties, and this establishes the existence of K for any arbitrary set of numbers for the orders of $t_1^2, t_2^2, (t_1 t_2)^2$.

Having established some fundamental properties of the general category of groups which may be generated by two operators, each of which transforms the square of the other into its inverse, we return to the operators under consideration. The three operators,

$$(s_1 s_2^{-1})^2 = s_1^3 s_2^{-3}, \qquad (s_2^{-1} s_1)^2 = s_1^2 s_2^{-3} s_1, \qquad (s_1 s_2^{-2} s_1)^2 = s_1 s_2^{-3} s_1^2,$$

generate an invariant subgroup under the commutator group of G and its index under this group divides 4 according to the general theory. It is easy to verify that these three operators together with their inverses constitute a complete set of conjugates under G and hence the group generated by them is also invariant under G.

These results suffice to prove that G can be constructed when the order of $(s_1 s_2^{-1})^2$ is an arbitrary number. The general method can be easily deduced from the following example. Suppose that the order of $(s_1 s_2^{-1})^2$ is 3. Write the three cyclic substitutions abc, def, ghi and select a substitution of order 3

$$t = adg \cdot beh \cdot cfi$$

which merely permutes the corresponding letters of these cycles. Find a substitution of order 2 which transforms each of the cyclic substitutions into its inverse and is commutative with t. In the present case we may select $ab \cdot de \cdot gh$. For s_1 take the continued product of the second cycle, this substitution of order 2, and t. In the present case

$$s_1 = ab \cdot ef \cdot gh \cdot t = aeicfh \cdot bdg.$$

For s_2 we take the continued product of the first cycle, the component of the given substitution of order 2 which involves the letters of this cycle, and t.

$$s_2 = bc \cdot t = adg \cdot bficeh.$$

It is easy to see that any two substitutions selected in the manner in which s_1, s_2 were chosen must be of order 6 and must also satisfy the equation $s_1 s_2^2 = s_2 s_1^2$. The three cyclic substitutions generate a group whose order is the cube of that of one of the cycles. This is the group generated by

$$(s_1 s_2^{-1})^2,\ (s_2^{-1} s_1)^2,\ (s_1 s_2^{-2} s_1)^2.$$

This abelian invariant subgroup is of index 4 under the commutator subgroup and of index 24 under G. In the special case under consideration G is of order 648 and it is the largest group which can be generated by two operators which satisfy the four conditions:

$$s_1^6 = 1,\ s_2^6 = 1,\ (s_1 s_2^{-1})^6 = 1,\ s_1 s_2^2 = s_2 s_1^2.*$$

If the order of $(s_1 s_2^{-1})^2$ had been even we should have proceeded in the same way except that we should have selected cyclic substitutions whose order is twice that of $(s_1 s_2^{-1})^2$ in place of substitutions like *abc*, *def*, *ghi* whose order is equal to that of $(s_1 s_2^{-1})^2$. But the case where $(s_1 s_2^{-1})^2$ is of odd order is sufficient to prove the theorem: *There is an infinite number of distinct groups each of which is generated by two operators satisfying the three conditions* $s_1^6 = s_2^6 = 1$, $s_1 s_2^2 = s_2 s_1^2$. *All such groups are solvable. The commutator subgroup of each of them is generated by two conjugate operators each of which transforms the square of the other into its inverse, and the order of each of the groups generated by* s_1, s_2 *is a divisor of three times the cube of the order of the commutator of* s_1, s_2.

If the commutator $s_1 s_2^{-1}$ of s_1, s_2 were of odd order, $s_2^{-1} s_1$ would also be of odd order. This is clearly impossible since each of these operators transforms the square of the other into its inverse. That is, *if two operators satisfy the three conditions* $s_1^6 = s_2^6 = 1$, $s_1 s_2^2 = s_2 s_1^2$, *their commutator must be of even order.* This theorem was proved by means of prolix considerations in volume 40 of the *Quarterly Journal of Mathematics*, page 201. It may be remarked that the other results of that article relating to the sets of conditions under consideration follow almost immediately from the theorems of the present paper and may be used to illustrate these theorems.

§ 5. *Groups whose two generators satisfy either the three conditions* $s_1^7 = s_2^7 = 1$, $s_1 s_2^2 = s_2 s_1^2$; *or the conditions* $s_1^9 = s_2^9 = 1$, $s_1 s_2^2 = s_2 s_1^2$.

Throughout the first two paragraphs it will be assumed that the first one of these two sets of conditions is satisfied.

* Quarterly Journal of Mathematics, vol. 40 (1909), p. 203.

By making $n = 1$ in formula B' of section 2 there results $(s_1 s_2^{-1})^4 = s_2^{-1} s_1^{-1} s_2^2$. Hence $(s_1 s_2^{-2})^5 = s_1 s_2^{-2} s_1^{-1} s_2^2 = s_1^{-1} s_2 = (s_2^{-1} s_1)^{-1}$. That is, the group generated by $s_1 s_2^{-1}$ involves $s_2^{-1} s_1$ and is therefore invariant under G. Hence the commutator subgroup of G is the cyclic group generated by $s_1 s_2^{-1}$. Since $s_1^{-1} \cdot s_1 s_2^{-1} \cdot s_1 = s_2^{-1} s_1 = (s_1 s_2^{-1})^{-5}$ it results that $(-5)^7 \equiv 1$ modulo the order of $s_1 s_2^{-1}$. Hence the order of $s_1 s_2^{-1}$ divides $6 \cdot 29 \cdot 449$. To verify that this order is 29 we may proceed as follows:

$$(s_1 s_2^{-1})^{29} = (s_1 s_2^{-1})^4 (s_1^{-1} s_2)^5 = (s_1 s_2^{-1})^4 s_1^{-1} s_2 (s_1^{-1} s_2)^4 = (s_1 s_2^{-1})^4 s_1^{-1} s_2 s_1^{-1} (s_1 s_2^{-1})^{-4} s_1$$

$$= s_2^{-1} s_1^{-1} s_2^2 s_1^{-1} s_2 s_1^{-1} s_2^{-2} s_1 s_2 s_1 = s_2^{-1} s_1^{-1} s_2^4 s_1^{-3} s_2^{-2} s_1 s_2 s_1$$

$$= s_2^{-1} s_1^{-1} s_2^4 s_1^{-2} s_2^{-2} s_1^2 = s_2^{-1} s_1^{-1} s_2^2 s_1 s_2^{-2} s_1 = 1.$$

Since the order of $s_1 s_2^{-1}$ is 29 or 1, G must be the non-cyclic group of order 203 or the group of order 7. It is evident that G may be of order 7. That it may also be of order 203 may be proved in the following manner.

Suppose that t is an operator of order 19 and that $s_1^{-1} t s_1 = t^{24}$. Since 24 belongs to exponent 7 modulo 29 we may assume that s_1 is of order 7 and hence $s_1 t$ is also of order 7. The two operators s_1, $s_1 t$ evidently generate the non-cyclic group of order 203 and $s_1 (s_1 t)^2 = s_1^2 t s_1 t = s_1^3 s_1^{-1} t s_1 t = s_1^3 t^{25} = s_1 t s_1^2$ since $24^2 \equiv 25 \bmod 29$. These results prove that there is only one prime number p which satisfies the condition that it is possible to find a number α belonging to exponent 7 and satisfying the equation $\alpha + 1 \equiv \alpha^2 \bmod p$. This prime number is 29 and α is 24. This result may also be stated as follows: *If α belongs to exponent 7 with respect to a prime number p and if $\alpha + 1 \equiv \alpha^2$ mod p then $p = 29$ and $\alpha = 25$.* We are thus led to a characteristic property of the prime 29. Similar characteristic properties of the primes 5 and 11 may be deduced from the results of section 3, and in what follows we shall establish also such a characteristic property of 19.

When $s_1^9 = s_2^9 = 1$, $s_1 s_2^2 = s_2 s_1^2$ formula B' gives rise to the equations

$$(s_1 s_2^{-1})^{4n} = s_2^{-1} (s_1^{-3} s_2^3)^n s_2 = s_2^{-1} s_1^{-3} (s_1 s_2^{-1})^{-2n} s_1^3 s_2.$$

Since $s_1^3 s_2$ transforms $(s_1 s_2^{-1})^2$ into a power of itself we shall first prove that the order of $s_1^3 s_2$ is 9. This fact is established by the following equations:

$$(s_1^3 s_2)^4 = s_1^3 s_2 s_1^3 s_2 s_1^3 s_2 s_1^3 s_2 = s_1^4 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1 s_2$$

$$= s_1^4 s_2^3 s_1^2 s_2^{-1} s_1^4 s_2^2 s_1 s_2 = s_1^4 s_2^4 s_1^{-1} s_2^{-1} s_1^6 s_2^2 s_1 s_2$$

$$= s_1^4 s_2^2 s_1^{-2} s_2^{-2} s_1^6 s_2^2 s_1 s_2 = s_1^4 s_2^6 s_1^{-4} s_2^{-1} s_1^{-2} s_2^2 s_1 s_2$$

$$= s_1^4 s_2^6 s_1^{-6} s_2 s_1^{-1} s_2 s_1 s_2 = s_1^4 s_2^6 s_1^3 s_2^3 s_1^{-1} s_2.$$

Hence $(s_1^3 s_2)^3 = s_1^4 s_2^6 s_1^3 s_2^3 s_1^{-4} = s_1^4 s_2^6 \cdot s_1^3 \cdot (s_1^4 s_2^6)^{-1}$.

Since s_1^3 is of order 3 the order of $s_1^3 s_2$ must be 9, and from the fact that this operator transforms $(s_1 s_2^{-1})^2$ into $(s_1 s_2^{-1})^{-4}$ it results that its 9th power transforms $(s_1 s_2^{-1})^2$ into its -512th power. That is, the order of $s_1 s_2^{-1}$ must divide $1026 = 2 \cdot 19 \cdot 27$. In view of this result we are led to inquire whether the order of $s_1 s_2^{-1}$ could be 19. That this is possible results directly from the fact that if t is an operator of order 19, and if $s_1^{-1} t s_1 = t^5$ then s_1, $s_1 t$ satisfy the equation $s_1 (s_1 t)^2 = s_1 t s_1^2$, as $6 \equiv 25 \mod 19$. It is also easy to see that the group of order 36 generated by $s_1 = abc \cdot t_1$, $s_2 = bdc \cdot t_1$, where t_1 is invariant and of order 9, fulfils the conditions imposed upon the generators at the beginning of this paragraph. Hence it has been proved that *there is more than one non-abelian group which involves two generators satisfying the conditions* $s_1^9 = s_2^9 = 1$, $s_1 s_2^2 = s_1 s_2^2$, *but the only prime numbers which divide the order of such a group are* 2, 3, *and* 19.

CONGRUENCES OF THE ELLIPTIC TYPE*

BY

LUTHER PFAHLER EISENHART

Introduction.

Let L be a line of a rectilinear congruence; X, Y, Z, its direction-cosines; and x, y, z, the coördinates of the point in which L cuts a surface of reference S. These six quantities are functions of two parameters, say u and v, which we assume to be real. As usual we put

$$(1)\qquad \begin{aligned} &E = \Sigma\left(\frac{\partial X}{\partial u}\right)^2, \qquad F = \Sigma\frac{\partial X}{\partial u}\frac{\partial X}{\partial v}, \qquad G = \Sigma\left(\frac{\partial X}{\partial v}\right)^2, \\ &e = \Sigma\frac{\partial x}{\partial u}\frac{\partial X}{\partial u}, \quad f = \Sigma\frac{\partial x}{\partial v}\frac{\partial X}{\partial u}, \quad f' = \Sigma\frac{\partial x}{\partial u}\frac{\partial X}{\partial v}, \quad g = \Sigma\frac{\partial x}{\partial v}\frac{\partial X}{\partial v}. \end{aligned}$$

Of all the ruled surfaces, formed by lines of the congruence, which pass through L two at most are developable. They are defined by the equation

$$(2)\qquad (Ef' - Fe)\,du^2 + (Eg + Ff' - Ff - Ge)\,du\,dv + (Fg - Gf)\,dv^2 = 0.$$

In the case of normal congruences, congruences of Guichard, cyclic congruences and congruences of tangents to a real family of curves on a surface, the integrals of equation (2) are real. But there is a large variety of congruences for which the integrals of this equation are imaginary. We say that a congruence is of the *hyperbolic* or *elliptic type* according as the two values of dv/du given by (2) are real or imaginary. This paper deals with congruences of the latter type, and particularly with pairs of ruled surfaces which are real only in this case and which possess properties analogous to those of the developable surfaces of a congruence of the hyperbolic type.

One of these systems may be defined analytically by means of the following theorem of CIFARELLI: †

Given two quadratic differential forms

$$(3)\qquad a_1\,du^2 + 2a_2\,du\,dv + a_3\,dv^2, \qquad b_1\,du^2 + 2b_2\,du\,dv + b_3\,dv^2,$$

of which the first is definite, that is, $a_1 a_3 - a_2^2 > 0$; if one forms the Jacobian

* Presented to the Society September 13, 1909.

† *Le Congruenze*, Annali di Matematica, ser. 3, vol. 2 (1899), p. 148.

of these forms and equates to zero the Jacobian of the resulting quadratic form and of the first of (3), the solutions of the resulting differential equation define a real transformation, say $u' = \phi(u, v)$, $v' = \psi(u, v)$, which changes the forms (3) into two

$$a_1' du'^2 + 2a_2' du'dv' + a_3' dv'^2, \qquad b_1' du'^2 + 2b_2' du'dv' + b_3' dv'^2,$$

which are such that

$$\frac{a_1'}{b_1'} = \frac{a_3'}{b_3'}, \qquad a_2' = 0. \tag{4}$$

Since the left-hand member of equation (2) is a definite quadratic form, it may be taken as the first of (3). If we take for the second the square of the linear element of the spherical representation, namely

$$d\sigma^2 = Edu^2 + 2Fdu\,dv + Gdv^2, \tag{5}$$

the parametric ruled surfaces $u' =$ const., $v' =$ const. constitute a system which we shall study in detail. We call them the *characteristic ruled surfaces* of the congruence.

In § 1 the equation of the characteristic ruled surfaces is derived, and therefrom we discover properties of the lines of striction of these surfaces, of their spherical representation, and the fact that their parameters of distribution are equal to one another and to the harmonic mean of the maximum and minimum values of the parameter of distribution of all the surfaces through the line.

The determination of a congruence with an assigned spherical representation of its characteristic ruled surfaces is investigated in § 2, and the results are applied in § 3 to the discussion of congruences whose characteristic surfaces meet the surface of reference in its asymptotic lines. These congruences are of the RIBAUCOUR type. The necessary and sufficient conditions for their existence are found, and an example of such congruences is given in § 4. Incidentally a theorem is derived concerning the case where the characteristic lines on a surface correspond to the asymptotic lines on an associate surface.

Since the quadratic form (2) is definite, there exist an infinity of real transformations of variables such that in terms of the new variables the expression (2) is of the form $\lambda(du'^2 + dv'^2)$. We say that such a parametric system is *isothermic*. Section 5 deals with the determination of congruences with an assigned spherical representation of an isothermic system of ruled surfaces. If ρ denotes the distance from the middle point of a line to one of its two conjugate purely imaginary focal points, the surfaces which are the loci of the points at the distances $i\rho$, $-i\rho$ from the middle point are of interest in this theory. We call them the *pseudofocal surfaces* of the congruence.

When the pseudofocal surfaces correspond with parallelism of tangent planes, as in § 6, the congruences are of the kind studied by LILIENTHAL. He took

three functions of $u + iv$, written

$$x_1 + ix_2 = f_1(u + iv), \qquad y_1 + iy_2 = f_2(u + iv), \qquad z_1 + iz_2 = f_3(u + iv),$$

and considered the congruence of lines joining corresponding points of the surfaces which are loci of the points (x_1, y_1, z_1), (x_2, y_2, z_2). We say that two such surfaces are *conjugate-potential*. These *congruences of Lilienthal* are of the RIBAUCOUR type, and the spherical representation of their imaginary developables is similar to that of congruences whose focal surfaces are curves.

Conjugate-potential surfaces are associate to one another. Section 7 deals with congruences which consist of lines joining corresponding points on associate surfaces. Certain congruences of RIBAUCOUR possess this property in an infinity of ways, and of this group are the congruences of LILIENTHAL.

§ 1. *Characteristic Ruled Surfaces.*

Given a congruence of the elliptic type expressed in terms of any parametric system and with an arbitrary surface of reference. If we put

$$(6) \qquad A = \frac{Ef' - Fe}{H}, \qquad B = \frac{Eg + F(f' - f) - Ge}{2H}, \qquad C = \frac{Fg - Gf}{H},$$

where $H = \sqrt{EG - F^2}$, the equation of the developables (2) may be written

$$(7) \qquad A du^2 + 2B du dv + C dv^2 = 0.$$

If we apply the theorem of Cifarelli to the left-hand member of this equation and to the right-hand member of (5), we find that the differential equation of the characteristic surfaces is reducible to

$$(8) \qquad \begin{aligned} &[A(AG - CE) - 2B(AF - BE)]\, du^2 + 2[B(AG + CE) \\ &\qquad - 2ACF]\, du dv + [2B(BG - CF) - C(AG - CE)]\, dv^2 = 0. \end{aligned}$$

In order that the characteristic ruled surfaces be parametric, we must have

$$(9) \qquad \begin{aligned} (AG - CE)A - 2B(AF - BE) &= 0, \\ (AG - CE)C - 2B(BG - CF) &= 0, \end{aligned}$$

which equations are reducible to

$$(10) \qquad \frac{A}{E} = \frac{C}{G}, \qquad B = 0,$$

unless

$$(11) \qquad A(BG - CF) - C(AF - BE) = 0.$$

But in the latter case the middle term also of (8) vanishes, so that the characteristic ruled surfaces are indeterminate in this case. Hence equations (10), or

in other form

$$(12)\qquad \begin{aligned} (f+f') - F\left(\frac{e}{E}+\frac{g}{G}\right) &= 0, \\ F(f'-f) + Eg - Ge &= 0, \end{aligned}$$

constitute a necessary and sufficient condition that the characteristic surfaces be parametric.

If the middle surface of the congruence is taken as the surface of reference, we have*

$$(13)\qquad (f+f')F - EG\left(\frac{e}{E}+\frac{g}{G}\right) = 0.$$

Combining this equation and the first of (12), we have

$$(14)\qquad f+f' = 0 \qquad \frac{e}{E}+\frac{g}{G} = 0,$$

and the second of equations (12) may be given either of the forms

$$(15)\qquad Eg + Ff' = 0, \qquad Ff + Ge = 0.$$

In order to interpret the second of equations (14), we recall that the abscissa r, measured from the surface of reference, of the point P on a line L where the line of striction of the ruled surface defined by a value of dv/du meets L is given by †

$$(16)\qquad r = -\frac{edu^2 + (f+f')dudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2}.$$

From this and (14) it follows that the values of r for the two characteristic surfaces through L differ only in sign. Hence we have

THEOREM 1. *The lines of striction of the two characteristic ruled surfaces through a line L of the congruence meet L in points equidistant from the middle point.*

We call these the *characteristic points.*

Before proceeding we return to the consideration of the particular case for which equation (11) holds as well as (9), that is when the characteristic ruled surfaces are indeterminate. These conditions are equivalent to

$$(17)\qquad \frac{AG-CE}{2B} = \frac{AF-BE}{A} = \frac{BG-CF}{C} = \lambda,$$

where λ denotes the factor of proportionality to be determined. The condition that these equations be consistent is

$$\begin{vmatrix} G & 2\lambda & E \\ F-\lambda & E & 0 \\ 0 & -G & F+\lambda \end{vmatrix} = 2\lambda(EG - F^2 + \lambda^2) = 0.$$

* E., p. 401. A reference in this form is to my *Differential Geometry*, Boston, 1909.

† E., p. 395.

Since the quantities are real, we must have $\lambda = 0$, and consequently equations (17) are reducible to

$$\frac{A}{E} = \frac{B}{F} = \frac{C}{G}.$$

Hence the congruences sought are such that their developables are represented on the sphere by the minimal lines of the latter. This is a characteristic property of isotropic congruences.* Hence we have:

THEOREM 2. *The characteristic ruled surfaces of an elliptic congruence are real and determinate, except when the congruence is isotropic.*

Isotropic congruences will be excluded from the subsequent discussion.

Let $\bar{r}$ and R denote the abscissæ, measured from the middle point of a line L, of a limit point P and of the characteristic point C on the same side of the middle point. If ω denotes the angle which the tangent plane at C to the corresponding characteristic surface makes with the tangent plane to the principal surface at P, we have from Hamilton's equation †

$$R = \bar{r}(\cos^2 \omega - \sin^2 \omega).$$

In like manner for the other characteristic ruled surface we have

$$R = -\bar{r}(\cos^2 \omega' - \sin^2 \omega').$$

From these equations we obtain

$$\cos 2\omega + \cos 2\omega' = 0. \tag{18}$$

Consequently the curves on the sphere which represent the characteristic ruled surfaces either form an orthogonal system, or they are equally inclined to the curves which bisect the angles between the curves representing the principal ruled surfaces.

In the former case, when these surfaces are parametric, we should have $F = 0$ and also $f + f' = 0$, from the first of (12). Moreover, from the second of (12) we should have $Eg = Ge$; that is, the congruence is isotropic. Hence $F \neq 0$, and we have

THEOREM 3. *The curves on the sphere which represent the characteristic ruled surfaces are equally inclined to the curves which bisect the angles between the images of the principal surfaces.*

The developables of a congruence of the hyperbolic type possess the same property.

By means of the preceding results we obtain the quadratic equation which the abscissæ of the characteristic points satisfy, when the parameters and surface of reference are any whatsoever. If R_1 and R_2 denote these abscisssæ

* E. p. 412.

† E, p. 397.

and r_1, r_2, the abscissæ of the limit points, we have from Hamilton's equation

$$R_1 = r_1 \cos^2 \omega + r_2 \sin^2 \omega,$$

$$R_2 = r_1 \cos^2 \omega' + r_2 \sin^2 \omega',$$

and from (18)

$$\cos^2 \omega' = \sin^2 \omega, \qquad \sin^2 \omega' = \cos^2 \omega.$$

From these equations we deduce

$$R_1 + R_2 = r_1 + r_2,$$

$$R_1 R_2 = (r_1 + r_2)^2 \frac{\cos^2 \theta_0}{4} + r_1 r_2 \sin^2 \theta_0,$$

where θ_0 denotes the angle between the spherical images of the characteristic surfaces. If the differential equation of the latter surfaces be written in the form

$$Ldu^2 + 2Mdudv + Ndv^2 = 0,$$

it is readily shown that

$$\cos^2\theta_0, \sin^2\theta_0 = \frac{(EN - 2FM + GL)^2, \quad 4H^2(M^2 - NL)}{E^2N^2 - 4EFMN + 2EG(2M^2 - NL) - 4FGLM + 4F^2LN + G^2L^2}.$$

Since moreover

$$r_1 + r_2 = \frac{(f + f')F - gE - eG}{EG - F^2}, \qquad r_1 r_2 = \frac{4eg - (f + f')^2}{EG - F^2},$$

we can readily determine the expressions of the coefficients of the quadratic equation satisfied by the abscissæ of the characteristic points. On account of the involved form of this equation we will not give it here.

We shall obtain another property of the characteristic ruled surfaces by recalling that the parameter of distribution of the ruled surface determined by a value of dv/du is given by*

$$p = \frac{Adu^2 + 2Bdudv + Cdv^2}{Edu^2 + 2Fdudv + Gdv^2}. \tag{19}$$

The first of equations (10) expresses the fact that the parameters of distribution of the two characteristic surfaces through a line are equal.

The differential equation (8) can be written

$$\frac{AG + CE - 2BF}{2(AC - B^2)} = \frac{Edu^2 + 2Fdudv + Gdv^2}{Adu^2 + 2Bdudv + Cdv^2}. \tag{20}$$

If now p_1 and p_2 denote the maximum and minimum values of p for all the ruled surfaces through a line, and if p_c denotes the value for the characteristic

* E, p. 424.

surfaces, equation (20) may be written, in consequence of (19),

$$\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)=\frac{1}{p_e}.$$

Hence we have

THEOREM 4. *The parameters of distribution of the two characteristic ruled surfaces through a line have the same value, namely the harmonic mean of the maximum and minimum values of the parameter for all the ruled surfaces through the line.*

The first part of this theorem is a consequence of the following theorem which can be readily proved:

THEOREM 5. *If the lines of striction of two ruled surfaces through a line L meet the latter at points equidistant from its middle point, the parameters of distribution of the two surfaces have the same value; and conversely.*

§ 2. *Spherical Representation of Characteristic Ruled Surfaces.*

It can be shown that the first derivatives of the coefficients of the surface of reference of a congruence are expressible in the form *

$$(21)\qquad \begin{aligned} \frac{\partial x}{\partial u} &= \frac{eG-f'F}{H^2}\frac{\partial X}{\partial u}+\frac{f'E-eF}{H^2}\frac{\partial X}{\partial v}+\gamma X,\\ \frac{\partial x}{\partial v} &= \frac{fG-gF}{H^2}\frac{\partial X}{\partial u}+\frac{gE-fF}{H^2}\frac{\partial X}{\partial v}+\gamma_1 X, \end{aligned}$$

and similar equations in y and z, where γ and γ_1 are functions which must satisfy the conditions

$$(22)\qquad \begin{aligned} &\frac{\partial e}{\partial v}-\frac{\partial f}{\partial u}-\left\{{12 \atop 1}\right\}'e+\left\{{11 \atop 1}\right\}'f-\left\{{12 \atop 2}\right\}'f'+\left\{{11 \atop 2}\right\}'g+F\gamma-E\gamma_1=0,\\ &\frac{\partial f'}{\partial v}-\frac{\partial g}{\partial u}-\left\{{22 \atop 1}\right\}'e+\left\{{12 \atop 1}\right\}'f-\left\{{22 \atop 2}\right\}'f'+\left\{{12 \atop 2}\right\}'g+G\gamma-F\gamma_1=0,\\ &\frac{\partial \gamma}{\partial v}-\frac{\partial \gamma_1}{\partial u}+f-f'=0, \end{aligned}$$

the Christoffel symbols being $\left\{{rs \atop t}\right\}'$ are formed with respect to the quadratic form (5).

We consider now a congruence of the elliptic type referred to its characteristic surfaces. We assume that the middle surface is the surface of reference and that R is the abscissa of the characteristic point of the surface $u=$ const. From (14), (15) and (16) we have

$$(23)\qquad R=\frac{e}{E}=\frac{-g}{G}=-\frac{Ff}{EG}=\frac{Ff'}{EG}.$$

* E, pp. 406, 407.

The abscissæ of the limit and focal points are given by the respective equations*

$$\bar{r}^2 = \frac{(f+f')^2 - 4eg}{4H^2}, \qquad \rho^2 = \frac{ff' - eg}{H^2}. \tag{24}$$

From equations (23) and (24) we obtain

$$R = \bar{r} \sin \theta_0, \qquad R = i\rho \cos \theta_0, \tag{25}$$

where θ_0 denotes the angle between the central planes of the two characteristic surfaces.

In consequence of (23) equations (21) are reducible to

$$\frac{\partial x}{\partial u} = \frac{ER}{F}\frac{\partial X}{\partial v} + \gamma X, \qquad \frac{\partial x}{\partial v} = -\frac{GR}{F}\frac{\partial X}{\partial u} + \gamma_1 X. \tag{26}$$

From the first two of (22) we find that

$$\begin{aligned} \gamma &= -\frac{\partial}{\partial v}\left(\frac{ER}{F}\right) - \frac{R}{F}\left(E\left\{{22 \atop 2}\right\}' + G\left\{{11 \atop 2}\right\}'\right), \\ \gamma_1 &= \frac{\partial}{\partial u}\left(\frac{GR}{F}\right) + \frac{R}{F}\left(E\left\{{22 \atop 1}\right\}' + G\left\{{11 \atop 1}\right\}'\right), \end{aligned} \tag{27}$$

and the last of (22) may be written

$$\begin{aligned} \frac{\partial^2}{\partial u^2}\left(\frac{GR}{F}\right) + \frac{\partial^2}{\partial v^2}\left(\frac{ER}{F}\right) + \frac{\partial}{\partial u}\left[\frac{R}{F}\left(E\left\{{22 \atop 1}\right\}' + G\left\{{11 \atop 1}\right\}'\right)\right] \\ + \frac{\partial}{\partial v}\left[\frac{R}{F}\left(E\left\{{22 \atop 2}\right\}' + G\left\{{11 \atop 2}\right\}'\right)\right] + 2\frac{EG}{F}R = 0. \end{aligned} \tag{28}$$

Since equations (22) are sufficient conditions upon the functions e, f, f', g γ and γ_1 that equations (21) define the surface of reference of a congruence with a given spherical representation of linear element (5),† we have

THEOREM 6. *Given any system of curves on the sphere and let E, F, G denote the fundamental coefficients; each solution of equation* (28) *determines a congruence whose characteristic ruled surfaces are represented on the sphere by the given system of curves.*

From (26) we obtain by differentiation with respect to u and v the following

$$\begin{aligned} \frac{\partial^2 x}{\partial u^2} &= \left(\frac{\partial}{\partial u}\log\frac{ER}{F} + \left\{{12 \atop 2}\right\}'\right)\frac{\partial x}{\partial u} - \left(\frac{E}{G}\left\{{12 \atop 1}\right\}' + \frac{F\gamma}{GR}\right)\frac{\partial x}{\partial v} + A_{11}X, \\ \frac{\partial^2 x}{\partial u \partial v} &= -\frac{G}{E}\left\{{11 \atop 2}\right\}'\frac{\partial x}{\partial u} - \frac{E}{G}\left\{{22 \atop 1}\right\}'\frac{\partial x}{\partial v} + A_{12}X, \\ \frac{\partial^2 x}{\partial v^2} &= \left(\frac{F\gamma_1}{ER} - \frac{G}{E}\left\{{12 \atop 2}\right\}'\right)\frac{\partial x}{\partial u} + \left(\frac{\partial}{\partial v}\log\frac{GR}{F} + \left\{{12 \atop 1}\right\}'\right)\frac{\partial x}{\partial v} + A_{22}X, \end{aligned} \tag{29}$$

* E., pp. 396, 399.

† E., p. 407.

where, for the sake of brevity, we have put

$$
(30)\quad
\begin{aligned}
A_{11} &= \gamma\gamma_1 \frac{F}{GR} - \gamma\left(\left\{{12 \atop 2}\right\}' + \frac{\partial}{\partial u}\log\frac{ER}{F}\right) + \gamma_1 \frac{E}{G}\left\{{12 \atop 1}\right\}' - ER + \frac{\partial\gamma}{\partial u},\\
A_{12} &= \frac{E}{G}\left\{{22 \atop 1}\right\}'\gamma_1 + \frac{G}{E}\left\{{11 \atop 2}\right\}'\gamma + \frac{\partial\gamma}{\partial v} - \frac{GE}{F}R,\\
&= \frac{E}{G}\left\{{22 \atop 1}\right\}'\gamma_1 + \frac{G}{E}\left\{{11 \atop 2}\right\}'\gamma + \frac{\partial\gamma_1}{\partial u} + \frac{GE}{F}R,\\
A_{22} &= -\gamma\gamma_1 \frac{F}{ER} - \gamma_1\left(\left\{{12 \atop 1}\right\}' + \frac{\partial}{\partial v}\log\frac{GR}{F}\right) + \gamma\frac{G}{E}\left\{{12 \atop 2}\right\}' + GR + \frac{\partial\gamma_1}{\partial v}.
\end{aligned}
$$

§ 3. *Congruences whose Characteristic Ruled Surfaces meet their Middle Surfaces in Asymptotic Lines.*

We consider in particular the case where the characteristic surfaces cut the middle surface in its asymptotic lines. From (29) and (30) it is seen that a necessary and sufficient condition for this is

$$A_{11} = A_{22} = 0.$$

In this case we have

$$A_{11}G + A_{22}E = G\frac{\partial\gamma}{\partial u} + E\frac{\partial\gamma_1}{\partial v} - \gamma G\frac{\partial}{\partial u}\log\frac{ER}{F} - \gamma_1 E\frac{\partial}{\partial v}\log\frac{GR}{F} = 0.$$

When the values of γ and γ_1 from (27) are substituted in this equation, the result is reducible to

$$(31)\quad \frac{\partial}{\partial u}\left[\frac{G}{E}\left\{{11 \atop 2}\right\}' + \left\{{22 \atop 2}\right\}' + \frac{\partial}{\partial v}\log\sqrt{\frac{E}{G}}\right] = \frac{\partial}{\partial v}\left[\frac{E}{G}\left\{{22 \atop 1}\right\}' + \left\{{11 \atop 1}\right\}' + \frac{\partial}{\partial u}\log\sqrt{\frac{G}{E}}\right].$$

But this is a necessary and sufficient condition that the parametric curves on the sphere be the spherical representation of the characteristic conjugate system on a unique surface Σ.* The coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of Σ are given by the quadratures

$$(32)\quad \frac{\partial\bar{x}}{\partial u} = -\frac{\lambda}{G}\left(G\frac{\partial X}{\partial u} - F\frac{\partial X}{\partial v}\right), \qquad \frac{\partial\bar{x}}{\partial v} = \frac{\lambda}{E}\left(F\frac{\partial X}{\partial u} - E\frac{\partial X}{\partial v}\right),$$

where λ is given by

$$
(33)\quad
\begin{aligned}
\frac{\partial\log\lambda}{\partial u} &= \frac{\partial}{\partial u}\log\sqrt{\frac{E}{G}} - \left\{{22 \atop 1}\right\}'\frac{E}{G} - \left\{{11 \atop 1}\right\}',\\
\frac{\partial\log\lambda}{\partial v} &= \frac{\partial}{\partial v}\log\sqrt{\frac{G}{E}} - \left\{{11 \atop 2}\right\}'\frac{G}{E} - \left\{{22 \atop 2}\right\}'.
\end{aligned}
$$

* EISENHART, *Three particular systems of lines on a surface*, Transactions of the American Mathematical Society, vol. 5 (1904), p. 434.

From (26) and (32) we have

$$(34)\qquad \Sigma\frac{\partial x}{\partial u}\frac{\partial \bar{x}}{\partial u}=0,\qquad \Sigma\frac{\partial x}{\partial u}\frac{\partial \bar{x}}{\partial v}+\Sigma\frac{\partial x}{\partial v}\frac{\partial \bar{x}}{\partial u}=0,\qquad \Sigma\frac{\partial x}{\partial v}\frac{\partial \bar{x}}{\partial v}=0.$$

Hence S and Σ correspond with orthogonality of linear elements and consequently the congruence under consideration is of the Ribaucour type,* and Σ is the director-surface.

Referring to (29), we have that another necessary condition that the asymptotic lines on S be parametric is

$$\frac{\partial}{\partial v}\left(\frac{\partial}{\partial u}\log\frac{ER}{F}+\left\{{12 \atop 2}\right\}'\right)=\frac{\partial}{\partial u}\left(\frac{\partial}{\partial v}\log\frac{GR}{F}+\left\{{12 \atop 1}\right\}'\right).$$

This equation and (31) are equivalent to the two

$$(35)\qquad \frac{\partial^2}{\partial u\,\partial v}\log\frac{G}{E}+\frac{\partial}{\partial u}\left\{{12 \atop 1}\right\}'-\frac{\partial}{\partial v}\left\{{12 \atop 2}\right\}'=0,\qquad \frac{\partial}{\partial u}\frac{G}{E}\left\{{11 \atop 2}\right\}'=\frac{\partial}{\partial v}\frac{E}{G}\left\{{22 \atop 1}\right\}'.$$

We shall show that these conditions are sufficient.

When equations (35) are satisfied, equation (31) is true and there exists a surface Σ, defined by (32). Since the parametric curves on Σ are the characteristic lines, we have †

$$(36)\qquad \frac{\bar{D}}{E}=\frac{\bar{D}''}{G},\qquad \bar{D}'=0,$$

and consequently the second of equations (35) is reducible to ‡

$$(37)\qquad \frac{\partial}{\partial u}\left\{{12 \atop 1}\right\}_1=\frac{\partial}{\partial v}\left\{{12 \atop 2}\right\}_1,$$

where the Christoffel symbols are formed with respect to the linear element of Σ. This is the condition that there exist a surface S corresponding to Σ with orthogonality of linear elements and such that its asymptotic lines are parametric. § If we take S for the middle surface of a congruence of Ribaucour whose director-surface is Σ, we have a congruence of the kind sought.

Since the developables of a congruence of Ribaucour correspond to the asymptotic lines on the director surface, the conditions (10) that the characteristic ruled surfaces be parametric are equivalent to equations (36). Hence we have

THEOREM 7. *The characteristic ruled surfaces of a congruence of Ribaucour cut the middle surface in the curves which correspond to the characteristic conjugate system on the director-surface.*

* E., p. 420.
† Loc. cit., p. 435.
‡ E., p. 201.
§ E., p. 380.

In order that these curves on the middle surface may be asymptotic lines, the condition (37) must be satisfied, and consequently we have

THEOREM 8. *Let Σ be a surface of positive curvature whose characteristic lines form a conjugate system with equal point invariants and let S be the unique surface corresponding to Σ with orthogonality of linear elements in such a way that its asymptotic lines correspond to the characteristic lines on Σ; then S is the middle surface and Σ the director-surface of a congruence of Ribaucour whose characteristic ruled surfaces cut S in its asymptotic lines; and these are the only congruences of this sort.*

§ 4. *The Case $E = G$.*

We shall establish the existence of such congruences, by showing that there exist upon the unit sphere systems of curves which are such that when they are parametric the conditions $E = G$ and (35) are satisfied. The latter conditions become in this case

$$\frac{\partial}{\partial u}\left\{{12 \atop 1}\right\}' = \frac{\partial}{\partial v}\left\{{12 \atop 2}\right\}', \qquad \frac{\partial}{\partial u}\left\{{11 \atop 2}\right\}' = \frac{\partial}{\partial v}\left\{{22 \atop 1}\right\}'. \tag{38}$$

If we put

$$E = G = \lambda, \qquad F = \lambda \cos \omega, \tag{39}$$

the first of these conditions may be written

$$\frac{\partial}{\partial u}\left(\frac{\dfrac{\partial \log \lambda}{\partial v} - \cos \omega \dfrac{\partial \log \lambda}{\partial u}}{\sin \omega}\right) = \frac{\partial}{\partial v}\left(\frac{\dfrac{\partial \log \lambda}{\partial u} - \cos \omega \dfrac{\partial \log \lambda}{\partial v}}{\sin \omega}\right),$$

and the second is reducible by means of the first to

$$\frac{\partial}{\partial u}\left(\frac{1}{\sin \omega}\frac{\partial \omega}{\partial u}\right) = \frac{\partial}{\partial v}\left(\frac{1}{\sin \omega}\frac{\partial \omega}{\partial v}\right),$$

or in other form

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \log \tan \frac{\omega}{2} = 0.$$

The general solution of this equation may be given the form

$$\tan^2 \frac{\omega}{2} = \frac{\psi(u - v)}{\phi(u + v)}, \tag{40}$$

where ϕ and ψ are arbitrary functions.

From the first of equations (38) it follows that there is a surface S_0 whose asymptotic lines have the given representation on the sphere. Since $E = G$, the equation of the lines of curvature on S_0 is $du^2 - dv^2 = 0$, so that if

$$u_1 = u + v, \qquad v_1 = u - v,$$

the curves $u_1 = \text{const.}$, $v_1 = \text{const.}$ are lines of curvature. From (40) it follows

that

$$(40') \qquad \cos\omega = \frac{\phi(u+v)-\psi(u-v)}{\phi(u+v)+\psi(u-v)} = \frac{U(u_1)-V(v_1)}{U(u_1)+V(v_1)},$$

and consequently the linear element of the spherical representation is

$$d\sigma^2 = \frac{\lambda}{U+V}(Udu_1^2 + Vdv_1^2).$$

Moreover, the linear element of S_0 is of the form *

$$ds_0^2 = \mu\left(\frac{du_1^2}{U} + \frac{dv_1^2}{V}\right),$$

where μ is a determinate function. Hence S_0 is an isothermic surface whose lines of curvature are represented on the sphere by an isothermal system. Conversely, it can be shown that when a surface of this kind is referred to its asymptotic lines conditions (38) and (40) are satisfied. †

To this group of surfaces belong the quadrics, cyclides of Dupin, surfaces of revolution, minimal surfaces, certain surfaces with plane lines of curvature in both systems, ‡ and a group of systems recently discussed by A. E. Young. §

Suppose that S_0 is such a surface referred to its asymptotic lines, and that S_1 is the unique surface whose characteristic lines have the same spherical representation as S_0. In consequence of (39) these lines on S_1 form an isothermal-conjugate system, and since the second of equations (38) is satisfied, this conjugate system has equal point invariants. || Hence there exists a surface S, corresponding to S_1 with orthogonality of linear elements, whose asymptotic lines correspond to the characteristic lines on S_1. From Theorem 7 it follows that the congruence of Ribaucour for which S is the middle surface and S_1 the director surface is such that the characteristic ruled surfaces cut S in its asymptotic lines, and thus we have a congruence of the kind sought.

It is evident that S_0 is an associate surface of S_1 and consequently determines a surface S' corresponding to S_1 with orthogonality of linear elements. When S_0 is referred to its asymptotic lines, the parametric curves on S' form a conjugate system. In view of Theorem 7 we have that the characteristic ruled surfaces of the congruence of Ribaucour, whose director-surface is S_1 and middle surface S_1', cut S' in a conjugate system.

In general, when the characteristic ruled surfaces of a congruence of Ribaucour

* E., p. 192.

† Cf. A. E. Young, *On a certain class of isothermic surfaces*, Transactions of the American Mathematical Society, vol. 10 (1909), pp. 79–94.

‡ Eisenhart, *Isothermal-conjugate systems of lines on surfaces*, American Journal of Mathematics, vol. 25 (1902), pp. 239–248.

§ l. c.

|| E., p. 380.

cut the middle surface in a conjugate system, the spherical representation of these surfaces is that of the asymptotic lines on that associate surface of the director-surface which is determined by the middle surface. Hence we have

THEOREM 9. *A necessary and sufficient condition that the characteristic ruled surfaces of a congruence of Ribaucour cut the middle surface in a conjugate system is that the spherical representation of these surfaces satisfy the conditions*

$$(41)\qquad \begin{gathered} \frac{\partial}{\partial u}\left\{ {12 \atop 1} \right\}' = \frac{\partial}{\partial v}\left\{ {12 \atop 2} \right\}', \\ \frac{\partial^2 \log}{\partial u \partial v}\frac{E}{G} + \frac{\partial}{\partial u}\left(\frac{G}{E}\left\{ {11 \atop 2} \right\}'\right) - \frac{\partial}{\partial v}\left(\frac{E}{G}\left\{ {22 \atop 1} \right\}'\right) = 0. \end{gathered}$$

The knowledge of one such system of curves on the sphere leads to the determination of another. This results from

THEOREM 10. *If the characteristic lines on a surface correspond to the asymptotic lines on an associate surface, the characteristic lines on the latter correspond to the asymptotic lines on the former.*

For, let S_1 and S_0 be two associate surfaces with the characteristic lines and asymptotic lines respectively parametric; then

$$(42)\qquad \frac{D_1}{E_1} = \frac{D_1''}{G_1}, \qquad D_1' = 0, \qquad D_0 = D_0'' = 0, \qquad D_0' = -\mu D_1'' = \sigma D_1,$$

where μ and σ are two functions such that the coördinates of the two surfaces are in the relations

$$\frac{\partial x_0}{\partial u} = -\mu\frac{\partial x_1}{\partial v}, \qquad \frac{\partial \bar{x}_0}{\partial v} = \sigma\frac{\partial x_1}{\partial u},$$

and similarly for the y's and z's.* From these we have

$$(43)\qquad E_0 = \mu^2 G_1, \qquad G_0 = \sigma^2 E_1,$$

the functions E_1, G_1 in (42) and (43) being the coefficients of the linear element of S_1, and E_0, G_0 the corresponding functions for S_0. The differential equation of the characteristic lines on S_0 is reducible to †

$$E_0 du^2 + G_0 dv^2 = 0,$$

which by means of (42) and (43) is equivalent to

$$D_1 du^2 + D_1'' dv^2 = 0;$$

consequently the theorem is proved.

* E., pp. 378, 380.

† E., p. 131.

Since asymptotic lines and characteristic lines are real only on surfaces of negative and positive curvature respectively, it follows that if a system satisfying (41) is real, the similar system obtained by means of Theorem 10 is imaginary, and *vice-versa*.

§ 5. *Isothermic Systems of Ruled Surfaces. The Pseudofocal Surfaces.*

Since the quadratic form

$$\Phi \equiv A du^2 + 2B du dv + C dv^2$$

is definite for a congruence of the elliptic type, there exist double systems of ruled surfaces which are such that, when a system of this kind is parametric, the coefficients of the form Φ satisfy the conditions

$$A = C, \qquad B = 0. \tag{44}$$

The determination of these double systems of ruled surfaces is the same analytical problem as that of isothermic orthogonal systems of curves on a surface.* For this reason we say that ruled surfaces satisfying the conditions (44) form an *isothermic system.*

On the assumption that such a system is parametric, we have from (6)

$$Ef' + Gf - F(e+g) = 0, \qquad Eg - Ge + F(f' - f) = 0. \tag{45}$$

Furthermore, if we take the middle surface of the congruence for the surface of reference, we have the condition (13). Combining the latter and (45), we find that

$$\frac{e}{F} = -\frac{f}{E} = \frac{f'}{G} = -\frac{g}{F} = \sigma, \tag{46}$$

where σ denotes the factor of proportionality.

When the values (46) are substituted in the first two of equations (22), we obtain

$$\gamma = -\frac{\partial \sigma}{\partial v} - \sigma S, \qquad \gamma_1 = \frac{\partial \sigma}{\partial u} + \sigma T, \tag{47}$$

where we have put

$$S = \left\{ {11 \atop 2} \right\}' + \left\{ {22 \atop 2} \right\}', \qquad T = \left\{ {22 \atop 1} \right\}' + \left\{ {11 \atop 1} \right\}'. \tag{48}$$

And the third of equations (22) is reducible to

$$\frac{\partial^2 \sigma}{\partial u^2} + \frac{\partial^2 \sigma}{\partial v^2} + T\frac{\partial \sigma}{\partial u} + S\frac{\partial \sigma}{\partial v} + \left(\frac{\partial T}{\partial u} + \frac{\partial S}{\partial v} + E + G\right)\sigma = 0. \tag{49}$$

* Cf., E., p. 93.

Moreover, from (21) we derive the equations of the middle surface in the form

$$
(50)\qquad
\begin{aligned}
\frac{\partial x}{\partial u} &= \sigma \frac{\partial X}{\partial v} - \left(\frac{\partial \sigma}{\partial v} + \sigma S\right) X, \\
\frac{\partial x}{\partial v} &= -\sigma \frac{\partial X}{\partial u} + \left(\frac{\partial \sigma}{\partial u} + \sigma T\right) X.
\end{aligned}
$$

Conversely, when the parametric lines on the sphere are any whatever, and σ is any solution of equation (49), equations (50) define the coördinates of the middle surface of a congruence referred to an isothermic system of ruled surfaces.*

We shall consider now the two surfaces S_1 and S_2, whose coördinates are given by

$$
(51)\qquad
\begin{aligned}
x_1 &= x - \sigma X, \qquad y_1 = y - \sigma Y, \qquad z_1 = z - \sigma Z, \\
x_2 &= x + \sigma X, \qquad y_2 = y + \sigma Y, \qquad z_2 = z + \sigma Z.
\end{aligned}
$$

In the first place we remark that S_1 and S_2, thus defined, are the same surfaces for all isothermic systems. In fact, if the values (46) be substituted in the second of (24), we find the following relation between σ and the abscissa of an imaginary focal point

$$
(52)\qquad \sigma = i\rho.
$$

Because of this result we call S_1 and S_2 the *pseudofocal surfaces* of the congruence.

§ 6. *Conjugate-Potential Surfaces. Congruences of Lilienthal.*

Of particular interest is the case when the tangent planes at corresponding points of S_1 and S_2 are parallel. In order to investigate this case, we differentiate equations (51), with the result

$$
(53)\qquad
\begin{aligned}
\frac{\partial x_1}{\partial u} &= -\sigma\left(\frac{\partial X}{\partial u} - \frac{\partial X}{\partial v}\right) - \left(\frac{\partial \sigma}{\partial u} + \frac{\partial \sigma}{\partial v} + \sigma S\right) X, \\
\frac{\partial x_1}{\partial v} &= -\sigma\left(\frac{\partial X}{\partial u} + \frac{\partial X}{\partial v}\right) + \left(\frac{\partial \sigma}{\partial u} - \frac{\partial \sigma}{\partial v} + \sigma T\right) X,
\end{aligned}
$$

$$
(54)\qquad
\begin{aligned}
\frac{\partial x_2}{\partial u} &= \sigma\left(\frac{\partial X}{\partial u} + \frac{\partial X}{\partial v}\right) + \left(\frac{\partial \sigma}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S\right) X, \\
\frac{\partial x_2}{\partial v} &= -\sigma\left(\frac{\partial X}{\partial u} - \frac{\partial X}{\partial v}\right) + \left(\frac{\partial \sigma}{\partial u} + \frac{\partial \sigma}{\partial v} + \sigma T\right) X.
\end{aligned}
$$

If X_1, Y_1, Z_1; X_2, Y_2, Z_2 denote the direction-cosines of the normals to S_1

*Cf. SANNIA, *Nuova esposizione della geometria infinitesimale delle congruenze rettilinee*, Annali di matematica, ser. 3, vol. 15 (1908), pp. 39, 40.

and S_2, we find from (53) and (54)

$$
(55)\quad
\begin{aligned}
X_1 &= \frac{\sigma^2}{\sqrt{E_1G_1 - F_1^2}}\left\{2XH + \left(Y\frac{\partial Z}{\partial u} - Z\frac{\partial Y}{\partial u}\right)\left(2\frac{\partial \log\sigma}{\partial u} + S + T\right)\right.\\
&\qquad\left. + \left(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\right)\left(2\frac{\partial \log\sigma}{\partial v} + S - T\right)\right\},\\
X_2 &= \frac{\sigma^2}{\sqrt{E_2G_2 - F_2^2}}\left\{2XH + \left(Y\frac{\partial Z}{\partial u} - Z\frac{\partial Y}{\partial u}\right)\left(-2\frac{\partial \log\sigma}{\partial u} + S - T\right)\right.\\
&\qquad\left. - \left(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\right)\left(2\frac{\partial \log\sigma}{\partial v} + S + T\right)\right\},
\end{aligned}
$$

where E_1, F_1, G_1; E_2, F_2, G_2 are the first fundamental coefficients of S_1 and S_2. From these expressions it follows that the necessary and sufficient condition that the tangent planes to S_1 and S_2 be parallel is that

$$
(56)\qquad E_1G_1 - F_1^2 = E_2G_2 - F_2^2
$$

and

$$
(57)\qquad \frac{\partial \log\sigma}{\partial u} = -\tfrac{1}{2}T, \qquad \frac{\partial \log\sigma}{\partial v} = -\tfrac{1}{2}S.
$$

When the values from (57) are substituted in (53) and (54), it is found that

$$
(58)\quad
\begin{aligned}
\frac{\partial x_1}{\partial u} &= \frac{\partial x_2}{\partial v} = -\sigma\left[\left(\frac{\partial X}{\partial u} - \frac{\partial X}{\partial v}\right) + \tfrac{1}{2}(S - T)X\right],\\
\frac{\partial x_1}{\partial v} &= -\frac{\partial x_2}{\partial u} = -\sigma\left[\left(\frac{\partial X}{\partial u} + \frac{\partial X}{\partial v}\right) - \tfrac{1}{2}(S + T)X\right].
\end{aligned}
$$

From these equations it follows that $x_1 + ix_2$, $y_1 + iy_2$, $z_1 + iz_2$ are analytic functions of $u + iv$. Hence we say that S_1 and S_2 are *conjugate-potential surfaces*, and the congruence consists of the joins of corresponding points of the pair.

Conversely, given any three analytic functions, consider the congruence of linesjoining corresponding points of the surfaces S_1 and S_2 so determined. The direction-cosines of the lines are of the form

$$
X = \frac{x_2 - x_1}{2\sigma}, \qquad Y = \frac{y_2 - y_1}{2\sigma}, \qquad Z = \frac{z_2 - z_1}{2\sigma},
$$

where 2σ denotes the distance between the points on S_1 and S_2. From these we obtain

$$
(59)\quad
\begin{aligned}
\frac{\partial X}{\partial u} &= -\frac{1}{2\sigma}\left(\frac{\partial x_1}{\partial v} + \frac{\partial x_1}{\partial u} + 2\frac{\partial \log\sigma}{\partial u}X\right),\\
\frac{\partial X}{\partial v} &= \frac{1}{2\sigma}\left(\frac{\partial x_1}{\partial u} - \frac{\partial x_1}{\partial v} - 2\frac{\partial \log\sigma}{\partial v}X\right).
\end{aligned}
$$

If x, y, z denote the coördinates of the mid-point of the join of corresponding

points of S_1 and S_2, equations (59) may be given the form

$$(60)\quad \frac{\partial X}{\partial u} = -\frac{1}{\sigma}\left(\frac{\partial x}{\partial v} + 2\frac{\partial \log \sigma}{\partial u} X\right), \qquad \frac{\partial X}{\partial v} = \frac{1}{\sigma}\left(\frac{\partial x}{\partial u} - 2\frac{\partial \log \sigma}{\partial v} X\right).$$

From these equations follow equations (46). Hence the surface S is the middle surface of the congruence, and the parametric ruled surfaces form an isothermic system. Congruences of this kind were considered by Lilienthal,* and so we shall refer to them as *congruences of Lilienthal.*

The preceding results may be stated thus:

THEOREM 11. *A necessary and sufficient condition that the pseudofocal surfaces of a congruence correspond with parallelism of tangent planes is that the congruence be of the Lilienthal type; in this case the pseudofocal surfaces are conjugate-potential.*

From (57) it follows that a necessary condition that the parametric lines on the sphere represent an isothermic system of ruled surfaces of a congruence of Lilienthal is

$$(61)\qquad \frac{\partial}{\partial v} T = \frac{\partial}{\partial u} S.$$

Furthermore, the function σ given by (57) must satisfy equation (49). This gives the further condition

$$(62)\qquad 2\left[\frac{\partial T}{\partial u} + \frac{\partial S}{\partial v} + 2(E+G)\right] = T^2 + S^2.$$

Conversely, from the general theory it follows that when conditions (61) and (62) are satisfied, there exists a congruence of Lilienthal which can be found by quadratures.

In consequence of the identity †

$$\frac{\partial}{\partial v}\left(\left\{{11 \atop 1}\right\}' + \left\{{12 \atop 2}\right\}'\right) = \frac{\partial}{\partial u}\left(\left\{{22 \atop 2}\right\}' + \left\{{12 \atop 1}\right\}'\right),$$

equation (61) is equivalent to

$$\frac{\partial}{\partial v}\left(\left\{{12 \atop 2}\right\}' - \left\{{22 \atop 1}\right\}'\right) = \frac{\partial}{\partial u}\left(\left\{{12 \atop 1}\right\}' - \left\{{11 \atop 2}\right\}'\right).$$

But this is the condition ‡ that the parametric lines on the sphere represent a real isothermal-conjugate system on a surface Σ whose coördinates ξ, η, ζ are given by the equations

$$(63)\quad \frac{\partial \xi}{\partial u} = \frac{t}{H^2}\left(-G\frac{\partial X}{\partial u} + F\frac{\partial X}{\partial v}\right), \qquad \frac{\partial \xi}{\partial v} = \frac{t}{H^2}\left(F\frac{\partial X}{\partial u} - E\frac{\partial X}{\partial v}\right),$$

* *Untersuchungen zur allgemeinen Theorie der krummen Oberflächen und geradlinigen Strahlensysteme* (Bonn, 1886), p. 80.

† E., p. 153.

‡ E., p. 202.

where t is given by

$$\frac{\partial \log t}{\partial u} = \left\{ {12 \atop 2} \right\}' - \left\{ {22 \atop 1} \right\}', \qquad \frac{\partial \log t}{\partial v} = \left\{ {12 \atop 1} \right\}' - \left\{ {11 \atop 2} \right\}'.$$

From (50) and (63) we find that

$$dx\,d\xi + dy\,d\eta + dz\,d\zeta = 0.$$

Hence:

THEOREM 12. *A congruence of Lilienthal is a congruence of Ribaucour.*

If we put

$$u = \bar{u} + \bar{v}, \qquad v = i(\bar{v} - \bar{u}),$$

the ruled surfaces $\bar{u} = \text{const.}$, $\bar{v} = \text{const.}$ are the developables. If $\bar{E}$, $\bar{F}$, $\bar{G}$ denote the fundamental coefficients of the sphere in this case, we find that the conditions (61) and (62) may be given the form

$$\frac{\partial}{\partial \bar{u}} \left\{ {\overline{12} \atop 1} \right\}' = \frac{\partial}{\partial \bar{v}} \left\{ {\overline{12} \atop 2} \right\}' = \left\{ {\overline{12} \atop 1} \right\}' \left\{ {\overline{12} \atop 2} \right\}' - \bar{F}, \tag{64}$$

where the symbols $\left\{ {\overline{rs} \atop t} \right\}$ are formed with respect to $\bar{E}d\bar{u}^2 + 2\bar{F}d\bar{u}\,d\bar{v} + \bar{G}d\bar{v}^2$.

When a real system of lines on the sphere satisfies (64), these lines represent the developables of a congruence whose focal surfaces are curves;* moreover, the semi-focal distance is given by quadratures.

Since the tangent planes to two conjugate-potential surfaces S_1 and S_2 at corresponding points are parallel, it is readily shown that the second fundamental coefficients of these surfaces satisfy the relations

$$D_1 = -D_1'' = D_2', \qquad D_1' = -D_2 = D_2'',$$

and consequently

$$D_1 D_2'' + D_1'' D_2 - 2D_1' D_2' = 0.$$

Hence we have

THEOREM 13. *Conjugate-potential surfaces are associate to each other.*

§ 7. *Congruences which consist of the Lines joining Corresponding Points on Associate Surfaces.*

We have seen that the abscissæ of the focal points are $i\sigma$, $-i\sigma$; hence, the above result follows also from the following known theorem †:

THEOREM 14. *In order that two surfaces S_1 and S_2 corresponding with parallelism of tangent planes be associate surfaces, it is necessary and sufficient that for the congruence formed by the joins of corresponding points M_1 and M_2 of these surfaces, the focal points and the points M_1 and M_2 form a harmonic range.*

* E., p. 412.

† E., p. 425.

We inquire whether a pair of associate surfaces is connected with every congruence after the manner described in the preceding theorem.

We assume that a given congruence possesses this property and that the two associate surfaces meet a line of the congruence in points whose distances from the middle point of the line are denoted by t_1 and t_2. By Theorem 14 we have $t_1 t_2 = \rho^2$, and by (52)

$$t_1 t_2 = -\sigma^2. \tag{65}$$

We must express the condition that the surfaces Σ_1 and Σ_2, defined by

$$\xi_1 = x + t_1 X, \qquad \eta_1 = y + t_1 Y, \qquad \zeta_1 = z + t_1 Z,$$
$$\xi_2 = x + t_2 X, \qquad \eta_2 = y + t_2 Y, \qquad \zeta_2 = z + t_2 Z,$$

correspond with parallelism of tangent planes.

By means of (50) we have

$$\begin{aligned} \frac{\partial \xi_1}{\partial u} &= t_1 \frac{\partial X}{\partial u} + \sigma \frac{\partial X}{\partial v} + \left(\frac{\partial t_1}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S \right) X, \\ \frac{\partial \xi_1}{\partial v} &= -\sigma \frac{\partial X}{\partial u} + t_1 \frac{\partial X}{\partial v} + \left(\frac{\partial t_1}{\partial v} + \frac{\partial \sigma}{\partial u} + \sigma T \right) X, \end{aligned} \tag{66}$$

and similar expressions in η_1 and ζ_1. From these we find

$$\begin{aligned} \sqrt{E_1 G_1 - F_1^2}\, X_1 = (t_1^2 + \sigma^2) HX &+ \left[t_1 \left(\frac{\partial t_1}{\partial v} + \frac{\partial \sigma}{\partial u} + \sigma T \right) \right. \\ &\left. + \sigma \left(\frac{\partial t_1}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S \right) \right] \left(Z \frac{\partial Y}{\partial u} - Y \frac{\partial Z}{\partial u} \right) \\ &+ \left[\sigma \left(\frac{\partial t_1}{\partial v} + \frac{\partial \sigma}{\partial u} + \sigma T \right) - t_1 \left(\frac{\partial t_1}{\partial u} - \frac{\partial \sigma}{\partial v} - \sigma S \right) \right] \left(Z \frac{\partial Y}{\partial v} - Y \frac{\partial Z}{\partial v} \right), \end{aligned}$$

where E_1, F_1, G_1 are the first fundamental coefficients of Σ_1 and X_1, Y_1, Z_1 are the direction courses of the normal to Σ_1.

The expressions for X_2, Y_2, Z_2 are similar to the above. The necessary and sufficient conditions that these respective quantities be equal are reducible to

$$t_1 \sqrt{E_2 G_2 - F_2^2} = -t_2 \sqrt{E_1 G_1 - F_1^2}, \tag{67}$$

$$\frac{\partial \theta}{\partial v} + 4\sigma \frac{\partial \sigma}{\partial u} + 2\sigma^2 T + S\theta = 0, \qquad \frac{\partial \theta}{\partial u} - 4\sigma \frac{\partial \sigma}{\partial v} - 2\sigma^2 S + T\theta = 0 \tag{68}$$

where, for the sake of brevity, we have put

$$\theta = t_1 \sigma - \frac{\sigma^3}{t_1}. \tag{69}$$

By means of (49) the condition of integrability of (68) is reducible to

$$
(70)\qquad \begin{aligned}
&\frac{1}{4}\left(\frac{\partial S}{\partial u}-\frac{\partial T}{\partial v}\right)\theta+\left(\frac{\partial \sigma}{\partial u}\right)^2+\left(\frac{\partial \sigma}{\partial v}\right)^2+\sigma\left(S\frac{\partial \sigma}{\partial v}+T\frac{\partial \sigma}{\partial u}\right)\\
&\qquad -\frac{\sigma^2}{2}\left[\frac{\partial T}{\partial u}+\frac{\partial S}{\partial v}+2(E+G)-(T^2+S^2)\right]=0.
\end{aligned}
$$

Hence in order that two associate surfaces Σ_1 and Σ_2 exist it is necessary that the function θ given by (70) satisfy equations (68). Moreover, it is readily shown that these conditions are sufficient.

When in particular $\theta = 0$, equations (68) reduce to (57) and then (70) is satisfied. This is the case of congruences of Lilienthal.

It may be shown that equation (61) is equivalent to the first of equations (64) and consequently is the condition that the congruence be of the Ribaucour type. From this and (70) it results that for the congruences of Ribaucour, for which σ is a solution of (49) and

$$
(71)\qquad \begin{aligned}
&\left(\frac{\partial \sigma}{\partial u}\right)^2+\left(\frac{\partial \sigma}{\partial v}\right)^2+\sigma\left(S\frac{\partial \sigma}{\partial v}+T\frac{\partial \sigma}{\partial u}\right)\\
&\qquad -\frac{\sigma^2}{2}\left[\frac{\partial T}{\partial u}+\frac{\partial S}{\partial v}+2(E+G)-(T^2+S^2)\right]=0,
\end{aligned}
$$

the function θ involves a parameter, and consequently there is an infinity of pairs of associate surfaces such as Σ_1 and Σ_2. Moreover, their determination requires quadratures only.

Since congruences of Lilienthal are of the Ribaucour type, the above result is applicable to these congruences; for equation (71) is satisfied by the function σ given by (57). Now equations (68) reduce to

$$
\frac{\partial \theta}{\partial v}-2\frac{\partial \log \sigma}{\partial v}\cdot\theta=0, \qquad \frac{\partial \theta}{\partial u}-2\frac{\partial \log \sigma}{\partial u}\theta=0,
$$

of which the general solution is

$$
\theta = c\sigma^2,
$$

c being an arbitrary constant. From this result, (69), and (65), we have

$$
t_1 = a\sigma, \qquad t_2 = -\frac{\sigma}{a},
$$

where a is an arbitrary constant. Hence we have

THEOREM 15. *With a congruence of Lilienthal there are associated an infinity of pairs of associate surfaces; two of these are conjugate-potential surfaces which cut a line of the congruence at points distant σ and $-\sigma$ from the middle point; and corresponding points of any other pair are at distances $a\sigma$, $-\sigma/a$, where a is a constant.*

PRINCETON, *September*, 1909.

CONJUGATE LINE CONGRUENCES CONTAINED IN A BUNDLE OF QUADRIC SURFACES*

BY

VIRGIL SNYDER

Introduction.

The cubic complex defined by the generators of both systems in a bundle of quadric surfaces has been investigated by MONTESANO,† and his methods employed by FANO‡ to determine a certain congruence of order three. A method, similar in part, had been previously found by STAHL,§ in connection with congruences of the second order whose lines can be arranged on a system of quadric surfaces.

The questions discussed in these papers suggest the more general one of conjugate congruences,‖ that is, those in which the two systems of generators of ∞^1 quadrics belonging to the bundle define two distinct congruences having the same focal surface. In the following paper I propose to derive such congruences and obtain a number of their properties. It will be seen that *the equations of such congruences of every order and of genus ranging from zero to a definite function of the order can be expressed rationally in terms of the eight points common to all the quadrics of the bundle.* The focal surface of each congruence remains invariant under a discontinuous birational transformation group of infinite order.

The preceding memoirs are all synthetic in their treatment; the present discussion is analytic, and the proofs of a few theorems already known will be given from a different point of view. Incidentally, extensive families of contact curves of a given quartic are obtained by algebraic methods.

* Presented to the Society, April 30, 1910.

† *Su di un complesso di rette di terzo grado*, Memorie di Bologna, ser. 5, vol. 3 (1893), pp. 549–577.

‡ *Nuove ricerche sulle congruenze di rette del 3° ordine*, Memorie di Torino, ser. 2, vol. 51 (1901), pp. 1–79.

§ *Ueber Strahlensysteme zweiter Ordnung*, Crelle's Journal, vol. 95 (1883), pp. 297–316.

‖ The idea of conjugate congruences has been developed from a different point of view by BALDUS: *Ueber Strahlensysteme, welche unendlich viele Regelflächen 2 Grades enthalten*, Erlangen dissertation, 1910. The only case to which his method and my own both apply is that of congruences of order two.

§ 1. *The involution I.*

1. Let $H_1(x) = 0$, $H_2 = 0$, $H_3 = 0$ be the equations of three quadric surfaces, and let $B_i (i = 1, \cdots, 8)$ be the eight points common to them. The bundle of quadrics

$$\Sigma = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 = 0 \tag{1}$$

defines a complex of order 3, the cone belonging to any point P being determined by the space quartic curve c_4 passing through P. This c_4 is the basis curve of the pencil of quadrics of (1) determined by P. The lines of the complex Σ can be put into (1, 1) correspondence with the points of space. Given any point $P \equiv (\xi_1, \xi_2, \xi_3, \xi_4)$, the line l joining P to B_1 will uniquely determine a quadric of Σ containing l. A generator m' of the other regulus passes through P. With P we may therefore associate the line m' of the complex. Conversely, any line m' of Σ will uniquely determine a quadric of the bundle. The quadric passes through B_1, through which passes a line l of the regulus to which m' does not belong. If l cuts m' in P, we may associate m' with the point P. When m' describes the whole regulus H determined by it, P will describe the line l; hence associated with every regulus is a line of the bundle B_1. The tangent plane to H at B_1 contains l and a line l' of the other regulus, so that there is associated with each quadric of Σ a pair of lines l, l' in the bundle B_1, each of which determines the other. These pairs determine a two-dimensional involution I.

The quadrics of Σ are in (1, 1) correspondence with the pairs of lines of the involution I in the bundle B_1.

2. Moreover, the quadrics of Σ are in (1, 1) correspondence with the points of the λ-plane. The pencil of quadrics in Σ passing through the point $P \equiv (\xi)$ corresponds to the straight line

$$\lambda_1 H_1(\xi) + \lambda_2 H_2(\xi) + \lambda_3 H_3(\xi) = 0.$$

The basis c_4 of the pencil of quadrics passes through B_1, hence the lines l, l' corresponding to quadrics of this pencil will describe a cubic cone K_3 passing through the other basis points B_i.

To the ∞^2 straight lines of the λ-plane correspond the ∞^2 cubic cones containing the seven fixed lines l_{1k} $(k = 2, \cdots, 8)$.

Any two lines in λ intersect in a point; two cones K_3 intersect in two lines which are conjugate in I; this pair of lines is the image of the given point.

The relations between (λ) and (H_2) are obtained by solving for $\lambda_1 : \lambda_2 : \lambda_3$ the equations of a quadric H and its tangent plane at B_1. Let $B_1 \equiv (\eta_1, \eta_2, \eta_3, \eta_4)$, and write $\eta_i x_4 - \eta_4 x_i = y_i$. The equation of the tangent plane becomes $\Sigma \lambda_i v_i = 0$, and H is of the form $\Sigma \lambda_i V_i + (\Sigma \lambda_i v_i) v_4 = 0$, wherein v_i, V_i are certain homogeneous linear and quadratic functions respectively in y_1, y_2, y_3, and $v_4 = 0$ defines a plane not passing through B_1. Solving for λ_i we have

$$(2) \qquad \lambda_1 : \lambda_2 : \lambda_3 = [v_2 V_3 - v_3 V_2) : (v_3 V_1 - v_1 V_3) : (v_1 V_2 - v_2 V_1).$$

3. Any congruence contained in Σ can be defined by a relation among the λ_i, or by a curve $f_n(\lambda)$ in the λ-plane. Corresponding to any line m of this congruence is a point P, the locus of which is the image cone of the given curve f_n. It will be of order $3n$ and have each line l_{1k} for an n-fold line. The order of the congruence will be $2n$ and the two systems of reguli will in general be contained in it.

§ 2. *Conjugate Congruences.*

4. It may happen that l, l' describe two different cones. For this purpose it is necessary that K_{3n}, image of an irreducible $f_n(\lambda)$, breaks up into two cones which are transformed into each other by the involution I. Such particular curves can be obtained by starting with the cone. Let l describe the cone $K_n(v)$ with vertex at B_1. From the relations

$$(3) \qquad \Sigma v_i \lambda_i = 0, \qquad \Sigma V_i \lambda_i = 0, \qquad K_n(v_1, v_2, v_3) = 0$$

the v_i should be eliminated. The resultant in λ is the equation of the required curve. If K_n passes through B_i, all the equations will be satisfied by its coördinates. If l_{ik} is the multiplicity of the line joining B_i to B_k, then Σl_{1k} must be subtracted from the order of the resultant which is $n + 2n$.

The image of a cone of order n passing l_{1k} times through B_k is a curve in the λ-plane of order $3n - \Sigma l_{1k}$. The lines of the cone and the points of the curve are in (1, 1) correspondence. When l describes K_n, l' will describe another cone, birationally equivalent to it, and conjugate in the involution I. If the values of λ_i from (2) be substituted in the equation of the curve, K_n will be a factor. The other factor will give the equation of the other cone.

But the relations between l, l' can be determined directly. In the net of cubic cones $\Sigma a_i K_3^{(i)}$ a pencil is determined by one more basis line r. The cones of this pencil now contain eight fixed basis lines, hence they contain a ninth r'. If r_i be the coördinates of r, the relations between r, r' may be expressed by the equations

$$\frac{K_3^{(1)}(r')}{K_3^{(1)}(r)} = \frac{K_3^{(2)}(r')}{K_3^{(2)}(r)} = \frac{K_3^{(3)}(r')}{K_3(r)}.$$

When r describes a plane, r' will describe a rational cone of order 8, having the seven lines l_{1k} for three-fold edges. The involution I represents a Cremona transformation of order 8. If l describes a plane pencil passing through B_i, K' is of order 5 and has each of the remaining lines l_{1k} for double edge. If the plane passes through B_i, B_k, K' is a quadric cone through the other five.**

*This interesting transformation was first discussed by GEISER: *Ueber zwei geometrische Probleme*, Crelle's Journal, vol. 67 (1867), pp. 78–95; it has been further considered by BERTINI: *Ricerche sulle trasformazioni univoche involutorie nel piano*, Annali di Mate-

From the method in which it is produced it is evident that the Jacobian J_6 of the net, a sextic cone having each line l_{1k} for double edge, is invariant in I in such a way that every line will go into itself.

Every line in which $K(l)$ intersects J_6 must lie on $K'(l')$ also.

Since every line of J_6 goes into itself when operated upon by I, the corresponding quadric of Σ is a cone; J_6 is the projection from B_1 of the locus of the vertices of the cones contained in the bundle (1). Moreover, since the lines which K, K' have in common apart from those on J_6 occur in pairs, the image in the λ-plane is a double point on the image $f(\lambda)$ and the quadric H belongs to both the congruences σ, the locus of l, and σ', the locus of l'.

5. The two congruences σ, σ' have the same focal surface ϕ. Its order is twice the class of $f(\lambda)$. The points B_k are all singular upon ϕ, of an order half the order of the surface. The other isolated singular points are of order two; they are the vertices of the quadric cones belonging to the congruence and contained in Σ. Their number is the number of lines common to K and J_6 apart from the lines l_{1k}. The focal surface also contains a cuspidal curve and a double curve, but the points of these are not singular points for either congruence.

§ 3. *Double λ-Plane.*

6. A cone K whose points are the images of the lines of σ is a singular cone of σ', and conversely. It was seen that a $(1, 1)$ correspondence exists between the points of the λ-plane and the quadrics of the bundle Σ. Now consider the λ-plane as double, or two-sheeted, the same point defining the l or the l' generation of the image quadric according as it is regarded as belonging to the first or second sheet. For values of λ_i corresponding to a cone of Σ the two systems l, l' coincide, hence at each such point of the λ-plane the two sheets must coincide. The values of λ_i that make the discriminant $\Delta_4(\lambda)$ of Σ vanish define the cones. This is a general non-singular quartic curve in the λ-plane. When a locus $f(\lambda)$ crosses $\Delta(\lambda)$ the two sheets cannot be distinguished from each other, hence a necessary condition that $f(\lambda)$ may define two conjugate congruences is that it shall touch $\Delta(\lambda)$ at every point which they have in common.

In case $f(\lambda)$ is rational, this is also sufficient, since rational curves are unipartite. In such cases λ_i can be expressed rationally in terms of a parameter t, which when substituted in Σ will define a family of quadrics of the form

$$\sum_{m=1}^{n} t^m H_m = 0.$$

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matica, ser. 2, vol. 8 (1877), pp. 244–282. It has also received minor mention by CAPORALI in the Rendiconti di Napoli (1879), by DOEHLEMANN in Schlömilch's Zeitschrift, vol. 36 (1891), and by STEINMETZ in the American Journal of Mathematics, vol. 14 (1891), pp. 39–66.

The necessary and sufficient condition that a relation of genus zero between the coefficients λ_i of the quadrics of a bundle may define two conjugate congruences is that the discriminant, expressed in terms of the parameter, shall be a square.

For curves of genus greater than zero the condition is not sufficient on account of the existence of multipartite curves; one part may touch $\Delta(\lambda)$ in one region and another in another. However, the elimination of v_i from (3) will always define a contact curve of the proper kind. All the contact curves of $\Delta_4(\lambda)$ which belong to an odd characteristic can be obtained in this way.*

7. The equation of the focal surface is obtained by expressing the condition that the line $\Sigma H_i \lambda_i = 0$ shall touch the curve $f(\lambda) = 0$. If in the result $\phi(H_1, H_2, H_3) = 0$ we think of H_i as point coördinates, every quadric of the congruence is represented by a tangent to the curve ϕ, and the quartic curve along which the quadric touches the focal surface is the image of the point of contact. The residual points of intersection of the tangent and the curve have other quartic curves for images, hence:

Every quadric of a congruence contained in a linear bundle of quadrics touches the focal surface along a quartic curve and cuts it in a residual curve composed of space quartics of the first kind, all passing through the eight basis points of the bundle. Each of these residual quartics is itself the curve of contact of some quadric of the congruence.

Moreover, corresponding to every point of inflexion on $f(\lambda)$ there is a cusp on $\phi(H)$ and to every double tangent on f a double point on ϕ.

The focal surface has a cuspidal curve consisting of as many quartics as there are points of inflexion on $f(\lambda)$, and a double curve consisting of as many quartics as there are bitangents of $f(\lambda)$.

§ 4. *Problem of Projectivity.*

8. The points of a singular cone K with vertex at B_1 uniquely define a congruence σ. The cone K' is found from K by the involution I, hence the conjugate congruence σ' is also defined. But each point B_i is the vertex of a singular cone, and any one of these would be sufficient to define the same congruence. These cones are thus all birationally equivalent to the given cone, and each must satisfy the relations analogous to (2) and (3). The double points on ϕ which are the images of lines of J_6 do not need to be considered, for they are vertices of quadric cones belonging to both σ and σ'.

Any two bundles B_i, B_k can be put into $(1, 1)$ correspondence. A line m_i of B_i determines a quadric H of Σ. Since H passes through B_k there is just one line m_k through B_k lying on H and belonging to the same regulus as m_i.

* For the literature of contact curves of $\Delta_4(\lambda)$ see the Encyklopädie der mathematischen Wissenschaften, III C5, §§ 60–62, 67, 74.

The quadric H passes through the other basis points $B_1, \cdots, B_8$, through each of which passes one line belonging to the other regulus, hence:

The six planes containing any line of the bundle B_i and any six basis points respectively are projective with the six planes through the corresponding lines of the remaining bundle and the same six basis points.

Since the eighth basis-point is uniquely fixed when the other seven are known, the problem of finding m_k when m_i is given is reduced to the problem of projectivity which, stated for the plane, is as follows: Given any five points $A_1, \cdots, A_5$, any other set of five points $A'_1, \cdots, A'_5$, and a point P, all in the same plane, to find a point P' such that the pencil $P'A'_i$ is projective with the pencil of five lines PA_i. This problem has been completely solved.*

The bundles B_i, B_k are equivalent under a Cremona transformation of order 5, having the remaining basis points for fundamental elements of the second order; to an arbitrary plane through B_i corresponds a rational quintic cone with vertex at B_k and having double edges passing through the other basis points. If the plane of B_i passes through B_1, the K_5 of B_1 is composite, consisting of a quadric cone through the other five edges and a cubic cone with a double edge passing through B_1 and passing once through each of the remaining basis points. If the pencil B_i passes through B_k, the K_3 of B_k passes through B_i. If the pencil B_i passes through B_1 and B_2, the proper image in B_k is the plane $B_k B_1 B_2$. If K_n of B_i has a double or cuspidal edge not passing through any other basis point, its image in B_k will also have a double or cuspidal edge not passing through any other basis point, and similarly for its image in the involution I. The equations of this transformation can be obtained readily by three quadric inversions and two collineations, hence the complete system of singular cones of both σ and σ' can be expressed analytically when any one such cone of either is given.

§ 5. *Infinite Birational Groups.*

9. The focal surface ϕ is invariant under the operation of interchanging the points of contact of every line of σ. This transformation T_1 is birational and is of order 2. The congruence σ' defines another birational transformation T_2 of order 2. Except for particular cases arising from relations among the basis point B_i, the operation $T_1 T_2$ is of infinite order.†

The equations of this transformation can be obtained directly by the same method as that previously employed in the case of rational conjugate congruences.‡

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* STURM: *Die Lehre von den geometrischen Verwandtschaften*, vol. 1 (1908), pp. 352–358. See also SCHOENFLIES: *Encyklopädie der math. Wissenschaften*, III AB 5, § 16 (p. 442).

† SNYDER: *Infinite discontinuous groups of birational transformations which leave certain surfaces invariant*, these Transactions, vol. 11 (1910), pp. 15–24.

‡ SNYDER: *Surfaces invariant under infinite discontinuous birational groups defined by line congruences*, American Journal of Mathematics, vol. 32 (1910), pp. 177–185.

The fundamental points are the eight basis points and the vertices of the quadric cones belonging to both congruences; the fundamental curves are the curves of contact of these cones with the focal surface.

Among the quadrics of the congruence there are a finite number which touch the focal surface along two quartic curves. These double quadrics may arise in two different ways, and have in consequence different relations between the two quartic curves of contact.

If $K(v)$ has a double edge not passing through a basis point, $f(\lambda)$ will have a double point and $\phi(H)$ a corresponding double tangent. Let the images of the points of contact be c_4, c_4'. A line of σ will cut c_4 in P_1, P_2 and cut c_4' in P_1', P_2'. Corresponding to one sheet of $K(v)$ the first pair of points are foci of the given line, and to the other sheet, the second pair. Similarly, P_1 is focus for the σ' line passing through it, for the same sheet of $K(v)$.

Now suppose $K(v)$ intersects $K'(v)$ in a pair of conjugate lines l, l' in I. These two lines belong to the same quadric H, define a double point on $f(\lambda)$, and a double tangent to $\phi(H)$. Every line of both reguli of H belongs to both congruences σ and σ'. Let c_4, c_4' be the quartics of contact, and P a point on c_4. The lines m, m' both touch ϕ at P. As lines of σ, m has its foci on c_4, but m' on c_4'; as lines of σ', m has its foci on c_4', and m' on c_4. Every rational congruence of order > 2 contained in Σ must have a double quadric of one or the other of these forms. For orders higher than 3 both forms may appear in the same congruence.

The details of a few illustrative cases will now be discussed.

§ 6. *Images of Contact Curves.*

10. Let K_1 be a plane pencil with vertex at B_1 and passing through B_2, B_3. The image in λ is a bitangent to $\Delta_4(\lambda)$, hence the congruence is of order 1. Since $B_k (k = 4, 5, 6, 7, 8)$ is also the vertex of a pencil passing through B_2, B_3 it follows that the line $B_2 B_3$ belongs to the σ' congruence, and all the lines of σ cut it. The pencil of quadrics in Σ which contains $B_2 B_3$ has for its remaining basis curve a space cubic passing through the other six basis points and cutting $B_2 B_3$ twice. The σ congruence therefore consists of the common secants of $B_2 B_3$ and the associated cubic.

The images of the 28 bitangents of $\Delta_4(\lambda)$ are the 28 lines $B_i B_k$ and the 28 associated cubics passing through the remaining basis points.

11. If K_1 passes through B_2 but not B_3, the λ image is a conic touching $\Delta_4(\lambda)$ in four points. The congruence is of order 2 and ϕ is of order 4. This is the well-known congruence of order 2, class 6, which has a singular plane. The plane K_1 can pass through any of the $8 \cdot 7/1 \cdot 2$ lines $B_i B_k$, and have ∞^1 positions in each case; each defines a contact conic of $\Delta_4(\lambda)$, hence:

The set of 28 series of ∞^1 contact conics of $\Delta_4(\lambda)$ are images of (2, 6) congruences having a singular plane.

12. A K_1 with vertex at B_1 but not containing any other basis points will cut J_6 in six lines. The image K_8' in I will cut K_1 in these six lines and also in one pair l, l' conjugate in I. The λ-image is a nodal cubic touching $\Delta_4(\lambda)$ in six points. The congruence is rational, of order 3, class 9, and has one double quadric of the second kind. The focal surface is of order 8, has eight fourfold points and six double points. Its cuspidal curve is composed of three quartics passing through the eight fourfold points. The singular cones at the other basis points are rational K_5 having the lines joining all the others except B_1 for double edges. The conjugate congruence has B_1 for the vertex of a rational K_8 having triple edges passing through all the other basis points, and a K_4 from each of the remaining basis points with a triple edge passing through B_1 and passing simply through the remaining six. The vertices of the six quadric cones all lie in the given K_1.

13. A K_2 through five basis points will define the (1, 3) congruence already discussed. In fact the singular cones through B_2 and B_3 are exactly of this type.

A K_2 through four basis points defines the well known (2, 6) congruence without a basis plane.

The conjugate congruence is of the same form, hence:

The set of 35 series of ∞^1 contact conics of $\Delta_4(\lambda)$ are images of (2, 6) congruences having no singular plane.

If K_2 has its vertex at B_1 and passes through B_2, B_3, B_4, it will cut J_6 in six lines apart from lines l_{ik}. The conjugate K_7' will have l_{12}, l_{13}, l_{14} for double edges, and the remaining l_{ik} for triple edges. It will therefore cut K_2 in one pair of lines l, l', conjugate in I. The image in the λ plane is a nodal cubic touching $\Delta_4(\lambda)$ in six points. The congruence is of order 3, class 9, has a double quadric of the second kind, and contains six quadric cones. The focal surface is of order 8 and has many properties in common with the preceding case, but the vertices of the quadric cones do not lie in a plane, and the configuration of the singular cones at the basis points is different.

The points B_2, B_3, B_4 are vertices of cones of order 6, K_6 having the lines to the remaining two for triple edges and double edges passing through the four points B_5, B_6, B_7, B_8. Each passes simply through B_1. The conjugate congruence has each of these three points for vertex of a cubic cone having a double edge, passing simply through those four basis points, but not through the two remaining ones of the triad. The given congruence has B_5, B_6, B_7, B_8 as vertices of singular cones of order four; each has a double edge through B_2, B_3, B_4, and passes simply through the remaining points of the tetrad, but does not pass through B_1. The conjugate congruence has each of the points B_5,

B_6, B_7, B_8 for vertex of a quintic cone having the line to B_1 for triple edge; the remaining points of the tetrad are on double edges, and each cone passes simply through the triad B_2, B_3, B_4. It is now easy to obtain congruences of orders 4, 5, 6 and having three, six, or ten double quadrics of the second kind as images of a quadric cone having two, one, or no lines l_{1k}. No rational congruence of order higher than six can have a singular cone of order less than three.

14. A K_3 with six simple lines l_{1k} will define a congruence of order 3 without a double quadric. If B_2 is the basis point not lying on $K_3(B_1)$, then B_2 is the vertex of a similar cone, passing through the same six basis points, which are vertices of K_5, each having lines joining the other five for double edges. The conjugate congruence has sextic cones at B_1 and B_2, each having the line B_1B_2 for triple edge and each having double edges through the other six basis points. The points $B_3, \cdots, B_8$ are vertices of K_4's having double edges passing through B_1 and B_2, and passing simply through the other basis points.

These two conjugate congruences are the only ones without double quadrics that can be mapped upon elliptic cones. Since there are 28 pairs B_iB_k and each cone contains three undefined constants, these congruences are imaged in the λ-plane by 28 series of ∞^3 contact cubics of $\Delta_4(\lambda)$. The remaining 36 series do not define conjugate congruences. If the original $K_3(B_1)$ has a double edge not containing any other basis point, the resulting rational congruences will have a double quadric of the first kind. *We have thus found six rational congruences of order 3 contained in* Σ; these results are believed to be new.*

15. Another interesting illustration is furnished by the $K_6(B_1)$ having double edges passing through all the other basis points. The singular cones at the other basis points are of the same kind, and the conjugate congruence has an exactly similar configuration. The image in λ is a non-singular quartic curve touching $\Delta_4(\lambda)$ in eight points. The congruence is therefore of order four. *The focal surface is of order 24, has eight twelve-fold points and eight double points; it has a cuspidal curve of order 96 composed of 24 quartics of the first kind passing through the eight singular points, and a nodal curve of order 112 composed of 28 quartics of the same bundle.* As before, K_6 may have one or more double edges, giving rise to quartic congruences having one or more double quadrics of the first kind.

The two congruences of order 2, of genus zero, the two conjugate congruences of order 3, of genus one, and the self-conjugate congruence of order 4, of genus three, are the only possible conjugate congruences contained in Σ which do not have a double quadric.

Congruences which can be mapped birationally upon cones of given genus can

* The elliptic congruences of order three were found by FANO (l. c.); he does not mention congruences having a double quadric.

be constructed, but the corresponding order is not less than the minimum order of a plane curve of that genus.

16. Among the $K_6(B_1)$ having seven double edges through the other basis points are two exceptions, J_6 and the image of a general c_2 in the λ-plane. The former corresponds to $\Delta_4(\lambda)$ itself and is defined as the congruence formed by the cones in Σ. From the preceding theorem we see that its focal surface is of order 24, has 24 cuspidal quartics and 28 double quartics, but from No. 10 the latter consists of 28 lines l_{ik} and the 28 associated cubics; moreover, instead of having eight other double points, it has as double point every point of the sextic which is projected from B_1 by J_6. The other exceptional type of K_6 is defined by quadratic relations among the cubics forming the net. Every such cubic is an adjoint of the sextic and every pencil of such cubics having one basis line of K_6 will, from the nature of its equation, have the other on K_6 also, that is, K_6 is hyperelliptic.*

The $c_2(\lambda)$ cuts $\Delta_4(\lambda)$ in eight points, but the vertices of the corresponding cones are not singular points on ϕ_4. If U_i be linear functions of the H_k, the equation has the form

$$U_1 U_3 - U_2^2 = 0.$$

When the equation of a quartic surface can be written as a quadratic function of three quadrics, the tangent cone of order 6 from each associated double point is either composite or hyperelliptic.

This theorem is true for surfaces of any order defined by functions of U_1, U_2, U_3.

§ 7. *Particular Positions of Basis Points.*

17. Let it be supposed that four basis points B_1, B_2, B_3, B_4 lie in a plane π_1, and the other four in another plane π_1'. When P describes a plane pencil with vertex at B_1, its image in the involution I will be a cone K_7' of order 7, with l_{12}, l_{13}, l_{14} for double edges, and the lines from B_1 to the other basis points for triple lines. As the basis points appear unsymmetrically a singular cone of order n and vertex at B_k will be designated by $K_n(k\,;\, l_h, m_p, \cdots)$, $l_h \cdots$ indicating that the line joining B_k to B_l is an h-fold line, m_p that $B_k B_m$ is a p-fold line, etc. Moreover, the involutions I will be different for different points. This difference will be indicated by subscripts. The result of operating with I_1 upon $K_1(1)$ will be indicated by the symbol

$$K_1(1)\, I_1\, K_7'(1\,;\, 2_2, 3_2, 4_2, 5_3, 6_3, 7_3, 8_3).$$

The line $\pi_1 \pi_1'$ is now a part of the locus of vertices of quadric cones contained in the bundle Σ. The residual curve is a hyperelliptic quintic whose projecting cone from B_1 may be expressed by the symbol $J_5(1\,;\, 2_1, 3_1, 4_1, 5_2, 6_2, 7_2, 8_2)$.

* SNYDER: *On a special algebraic curve having a net of minimum adjoint curves*, Bulletin American Mathematical Society, vol. 14 (1907), pp. 70–74.

The discriminant curve $\Delta_4(\lambda)$ has one double point, the coördinates of which define the planes π_1, π_1' in Σ.

The 16 double tangents of $\Delta_4(\lambda)$ are images of the lines $B_k B_h$ ($k = 1, 2, 3, 4$; $h = 5, 6, 7, 8$) and the six tangents from the node are each the image of a pair of lines, one in π_1, one in π_1'. The transformation T_{ik}, defining the cone at B_k when that at B_i is given, is not changed from that of the general case when B_i, B_k both lie in π_1 or π_1' but is of order 4 when they lie in different planes. Thus we may write

$$K_1(1)\, T_{15} K_4(5\,;\, 2_2, 3_2, 4_2, 6_1, 7_1, 8_1).$$

The focal surface of the congruence of cones in Σ is now of order 20, contains 18 cuspidal quartics of the first kind, six double planes, six pairs of intersecting double lines, 16 other double lines, 16 double space cubics, and a double quintic.

The planes through B_1 and any basis point in π_1' define the general $(2, 5)$ congruences. The line of intersection of this plane and π_1 does not pass through any other basis point, hence it is the image of a cone breaking up into π_1 and π_1'. The cone consists of two plane pencils, one in π_1, the other in π_1', the vertices of each being on the line $\pi_1 \pi_1'$. In the conjugate congruence the planes of these pencils are interchanged.

The image conic in the λ-plane touches $\Delta_4(\lambda)$ in three points and passes through the node.

The singular cone $K_1(1;\, 2_1)$ defines a particular form of the $(2, 6)$ congruence having one singular plane.

A general $K_1(1)$ defines a $(3, 8)$ congruence, in which one cone breaks up into π_1, π_1'. The focal surface has one more double point and two more double planes than in the general case. The congruence contains one double quadric of the second kind.

An interesting example is the $(3, 7)$ congruence defined by the singular cone $K_2(1;\, 5_1, 6_1, 7_1)$. The complete configuration of singular cones of the two conjugate congruences is as follows:

$$K_2(1;\, 5_1, 6_1, 7_1), \qquad K_4(2;\, 3_1, 4_1, 5_2, 6_2, 7_2, 8_1),$$
$$K_5'(1;\, 2_1, 3_1, 4_1, 5_2, 6_2, 7_2, 8_3), \qquad K_3'(2;\, 1_1, 5_1, 6_1, 7_1, 8_2),$$
$$K_4(5;\, 1_1, 2_2, 3_2, 4_2, 6_1, 7_1),$$
$$K_3'(5;\, 1_1, 2_1, 3_1, 4_1, 8_2).$$

Those at B_3, B_4 can be obtained from those at B_2 by interchanging 2 and 3, 2 and 4 respectively, and those at B_6, B_7, B_8 by making similar changes in those at B_5.

The congruences contain a double quadric of the second kind consisting of the two pencils in π_1, π_1'. The image in the λ-plane consists of a cubic having a double point at the node of $\Delta_4(\lambda)$ and touching the latter in four points.

The cone $K_2(1;\ 4_1, 5_1, 6_1)$ defines a $(3, 8)$ congruence having five cones, one pair of planes, and a double quadric of the second kind.

The cone $K_2(1;\ 3_1, 4_1, 5_1)$ defines a $(3, 9)$ congruence containing six cones of Σ and a double quadric of the second kind.

The cubic cone $K_3(1;\ 3_1, 4_1, 5_1, 6_1, 7_1, 8_1)$ defines a $(3, 8)$ congruence having 5 quadric cones of Σ. If K_3 is rational the congruence contains a double quadric of the first kind.

The cone $K_3(1;\ 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)$ defines a $(3, 9)$ congruence as in the general case.

The preceding eighteen forms are the only possible types of congruences in Σ of order less than 4.

The non-hyperelliptic cone of order 6 having a double edge passing through each remaining basis point defines a $(4, 12)$ congruence as in the general case. The conjugate congruence is of the same type.

18. Let the basis points lie on two pairs of planes:

$$\pi_1 \equiv B_1 B_2 B_3 B_4, \qquad \pi_1' \equiv B_5 B_6 B_7 B_8;$$
$$\pi_2 \equiv B_1 B_2 B_5 B_6, \qquad \pi_2' \equiv B_3 B_4 B_7 B_8.$$

The two lines $\pi_1\pi_1'$ and $\pi_2\pi_2'$ and a space quartic of the first kind constitute the locus of the vertices of cones in Σ. The latter is projected from B_1 by the cone $J_4(1;\ 3_1, 4_1, 5_1, 6_1, 7_2, 8_2)$. The involution I transforms a plane pencil into a cone of order 6 of the form

$$K_1(1)IK_6'(1;\ 2_1, 3_2, 4_2, 5_2, 6_2, 7_3, 8_3).$$

All the operations T_{ik} transform plane pencils into quartic cones except when ik is one of the combinations 12, 34, 56, or 78. Thus we have

$$K_1(1)T_{12}K_5(2;\ 3_2, 4_2, 5_2, 6_2, 7_2, 8_2); \quad K_1(1)T_{13}K_4(3;\ 2_2, 4_1, 5_2, 7_1, 8_1).$$

Of the congruences of the second order, $K_1(1;\ 2_1)$ defines a particular $(2, 6)$; $K_1(1;\ 3_1)$ defines a particular $(2, 5)$, and $K_1(1;\ 7_1)$ a general $(2, 4)$.

Of the congruences of order 3, $K_1(1)$ defines a $(3, 7)$ having four cones of Σ, two pairs of planes, and a double quadric of the second kind. The cone $K_2(1;\ 6_1, 7_1, 8_1)$ defines a $(3, 7)$ congruence having a pair of planes for a double quadric of the second kind. The image cubic curve in the λ-plane has a double point at one node of $\Delta_4(\lambda)$ and passes simply through the other; it touches $\Delta_4(\lambda)$ in three other points.

19. The basis points may lie in three pairs of planes:

$$\pi_1 \equiv B_1 B_2 B_3 B_4, \quad \pi_1' \equiv B_5 B_6 B_7 B_8, \quad \pi_2 \equiv B_1 B_2 B_5 B_6, \quad \pi_2' \equiv B_3 B_4 B_7 B_8,$$
$$\pi_3 \equiv B_1 B_3 B_6 B_8, \quad \pi_3' \equiv B_2 B_4 B_5 B_7.$$

The locus of the vertices of cones in Σ now consists of the three lines $\pi_1\pi_1'$, $\pi_2\pi_2'$, $\pi_3\pi_3'$, and a space cubic, the latter being projected from B_1 by the cone $J_3(1;\ 4_1, 5_1, 7_2, 8_2)$. The involution I transforms a plane pencil into a cone of order 5 of the form $K_1(1)IK_5'(1;\ 2_1, 3_1, 4_2, 5_2, 6_1, 7_3, 8_2)$. The transformations T_{ik} can be most easily determined from the details of the $(3, 6)$ congruence defined by $K_1(1)$. The singular cones of both this congruence and its conjugate at B_1 are given above; the others are as follows:

$$K_4(2;\ 3_2, 4_1, 5_1, 6_2, 7_1, 8_2), \qquad K_4(3;\ 2_2, 4_1, 5_2, 6_2, 7_1, 8_1),$$
$$K_2'(2;\ 1_1, 7_1, 8_1), \qquad K_2'(3;\ 1_1, 7_1, 8_1),$$
$$K_3(4;\ 2_1, 3_1, 5_1, 6_2, 8_1), \qquad K_4(6;\ 2_2, 3_2, 4_2, 5_1, 7_1, 8_1),$$
$$K_3'(4;\ 1_2, 5_1, 6_1, 7_1, 8_1), \qquad K_2'(6;\ 1_1, 4_1, 7_1),$$
$$K_3(5;\ 2_1, 3_2, 4_1, 5_1, 8_1), \qquad K_3(8;\ 2_2, 3_1, 4_1, 5_1, 6_1),$$
$$K_3'(5;\ 1_2, 3_1, 4_1, 7_1, 8_1), \qquad K_3'(8;\ 1_2, 2_1, 4_1, 5_1, 7_1),$$
$$K_2(7;\ 2_1, 3_1, 5_1),$$
$$K_4'(7;\ 1_3, 2_1, 3_1, 4_1, 5_1, 6_1, 8_1).$$

The image of the congruence of cones is a trinodal quartic in the λ plane. Its four proper bitangents are images of the congruences cutting the lines l_{17}, l_{28}, l_{35}, l_{45} and their associated space cubics.

Of the congruences of order 2, $K_1(1;\ 7_1)$ defines the general $(2, 3)$ case. Its image in the λ-plane is a conic passing through all three nodes of $\Delta_4(\lambda)$ and touching it once. The cone $K_1(1;\ 8_1)$ defines a particular $(2, 4)$ congruence, and $K_1(1;\ 2_1)$ defines a particular $(2, 5)$. A $(2, 6)$ congruence cannot be contained in Σ in this case.

Among the congruences of order 3, that of lowest class is the $(3, 5)$ defined by $K_2(1;\ 5_1, 7_1, 8_1)$. It contains four of the basis planes simply and the remaining pair doubly. The section of the focal surface made by the latter consists of two conic sections, each counted twice. The image in the λ-plane is a cubic having a double-point at one of the double-points of $\Delta_4(\lambda)$, passing simply through each of the remaining nodes, and touching $\Delta_4(\lambda)$ in two other points.

As in the preceding cases, the non-hyperelliptic $K_6(1, 2_2, \cdots, 8_2)$ defines a $(4, 12)$ congruence with a singular cone of order 6 at each basis point, and containing 8 quadric cones of Σ.

20. The eight basis points may be arranged on four pairs of planes. In addition to the preceding pairs of planes π_i, π_i' we may write

$$\pi_4 \equiv B_1B_4B_5B_8, \qquad \pi_4' \equiv B_2B_3B_6B_7.$$

The locus of the vertices of cones now consists of the four lines $\pi_1\pi_1'$, etc., and

of their two transversals. The latter project from B_1 into $J_2(1;\ 7_2)$. The curve $\Delta_4(\lambda)$ consists of two conics passing through the images of the four lines $\pi_1\pi_1'$, etc. In the involution I,

$$K_1(1)IK_4'(1;\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_3,\ 8_1),$$

that is, in the Cremona transformation defined by the involution I, the image of a plane passing through B_1 is in this case a quartic cone having its vertex at B_1, containing B_1B_7 as triple edge and passing simply through the remaining basis points.

The only congruences of the second order that can exist in this case are a particular $(2,\ 4)$ defined by $K_1(1;\ 2_1)$ and the general $(2,\ 2)$ defined by $K_1(1;\ 7_1)$. In the latter all the basis-points are vertices of plane pencils, and the conjugate congruence has the same configuration of singular planes. The image in the λ-plane consists of a conic belonging to the pencil defined by the factors of $\Delta_4(\lambda)$. The focal surface is the general Kummer surface.

Of congruences of the third order, $K_1(1)$ defines a $(3,\ 5)$ having a double quadric of the second kind; the elliptic $K_3(1;\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_1)$ defines a $(3,\ 7)$ without a double quadric, and $K_3(1;\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 8_1)$ defines a particular form of $(3,\ 9)$. When the cone is rational the corresponding congruence acquires a double quadric of the first kind.

As in the general case, the non-hyperelliptic $K_6(1;\ 2_2,\ \cdots,\ 8_2)$ defines a $(4, 12)$ congruence without any singular planes.

21. The basis points may be arranged on five pairs of planes, the four preceding, and

$$\pi_5 \equiv B_1,\ B_2,\ B_7,\ B_8, \qquad \pi_5' \equiv B_3,\ B_4,\ B_5,\ B_6.$$

The locus of the vertices of cones consists of the five lines $\pi_1\pi_1'$ etc., and of one other line, the latter projecting from B_1 into $J_1(1:7_1)$. The curve $\Delta_4(\lambda)$ consists of a conic and a pair of straight lines, each of the latter defining a pencil of quadric cones having common vertices. In the involution I, the image of a plane through B_1 is a cubic cone having B_1B_7 for double edge and passing simply through B_3, B_4, B_5, B_6. The symbol becomes

$$K_1(1)IK_3'(1:\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_2).$$

There are three particular congruences of the second order, a $(2,\ 4)$ defined by $K_1(1:2_1)$, a $(2,\ 3)$ by $K_1(1:3_1)$, and a $(2,\ 2)$ by $K_1(1:7)$.

Besides the rational $(3,\ 4)$ congruences defined by $K_1(1)$, the elliptic $(3,\ 6)$ defined by $K_3(1:\ 2_1,\ 3_1,\ 4_1,\ 5_1,\ 6_1,\ 7_1)$ is of interest. For the first time the conjugate congruence of that defined by $K_6(1;\ 2_2,\ \cdots,\ 8_2)$ is not directly expressed by substitution of the values of the coördinates. The lines l_{11}, l_{18} are already three fold in the extraneous factors that divide out of $K_8'(1:\ 2_3,\ \cdots,\ 8_3)$, hence

are neutral. But by substituting the coördinates of these lines in K_6, both appear as double edges.

22. The basis points may be arranged on six pairs of planes, the five preceding and also

$$\pi_6 \equiv B_1 B_4 B_6 B_7, \qquad \pi_6' \equiv B_2 B_3 B_5 B_8.$$

The locus of the vertices of cones consists of the six lines $\pi_i \pi_i'$, four of which meet in a point. The cones having this point for vertex contain four common edges each containing two basis points, hence form a pencil belonging to Σ. The curve $\Delta_4(\lambda)$ consists of the sides of a quadrilateral. The involution I is defined by the transformation

$$K_1(1) I K_2'(1 : 3_1, 5_1, 7_1).$$

The plane $K_1(1)$ defines a $(3, 3)$ congruence having a double quadric of the first kind. It contains no cones, but has six pairs of planes. The conjugate congruence is of the same type. Congruences of order 3 and class 2 are impossible when the basis points are distinct. Particular $(2, 2)$ and $(2, 3)$ forms as well as $(3, k)$, $k = 3, \cdots, 6$, can be obtained. The involution reduces to ordinary quadratic inversion. As in the preceding cases $K_6(1 ; 2_2, \cdots, 8_2)$ defines a congruence of type $(4, 12)$, having a conjugate of the same kind.

When the points are arranged on six pairs of planes, they may be assumed to be $(\pm 1, \pm 1, \pm 1, 1)$. The equation of the bundle of quadrics is of the form

$$\lambda_1(x_1^2 - x_4^2) + \lambda_2(x_2^2 - x_4^2) + \lambda_3(x_3^2 - x_4^2) = 0,$$

and $\Delta_4(\lambda)$ becomes

$$\lambda_1 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + \lambda_3) = 0.$$

The equation of a line through $B_1 \equiv (1, 1, 1, 1)$ are $x_i = 1 + \rho \xi_i$, from which, if it belongs to Σ_1,

$$\lambda_1(\xi_1 - \xi_4) + \lambda_2(\xi_2 - \xi_4) + \lambda_3(\xi_3 - \xi_4) = 0,$$

$$\lambda_1(\xi_1^2 - \xi_4^2) + \lambda_2(\xi_2^2 - \xi_4^2) + \lambda_3(\xi_3^2 - \xi_4^2) = 0.$$

A congruence of the second order is defined by $K_1(1;\ 2_1)$, wherein $B_2 \equiv (1, -1, 1, 1)$. This cone requires relations of the form

$$A(\xi_1 - \xi_4) = B(\xi_2 + \xi_4) = C(\xi_3 - \xi_4)$$

among the ξ_i. The ξ-eliminant of these equations is the equation of the λ-curve, image of the congruence; it is the conic

$$(c\lambda_1 + A\lambda_3)^2 + \lambda_2(c^2\lambda_1 + A^2\lambda_3) = 0,$$

which passes through the vertices $(0, 1, 0)$, $(0, -1, 1)$, $(1, -1, 0)$ of the quadrilateral and touches the line $\lambda_2 = 0$ at $(A, 0, -c)$. The $(2, 3)$ congru-

ence contains one cone of Σ besides four singular cones whose vertices are at the basis points, and three pairs of planes besides the four singular pencils at four basis points. The conjugate is of the same form.

23. Instead of a finite number of basis points, the bundle Σ may have a basis curve. There are four general forms.

(1) When H_1, H_2, H_3 can be written in the form

$$u_1 v_2 - u_2 v_1 = 0, \quad u_1 w_2 - u_2 w_1 = 0, \quad v_1 w_2 - v_1 w_1 = 0,$$

wherein u_i, v_i, w_i are linear in the variables x_i, the bundle has a space cubic for basis curve.

(2) The equation

$$x_4(\lambda_1 u_1 + \lambda_2 u_2) + \lambda_3(x_1 x_2 - x_3^2) = 0$$

defines a bundle of quadrics having the conics $x_4 = 0$, $x_1 x_2 - x_3^2 = 0$ for basis curve and the two basis points $u_1 = 0$, $u_2 = 0$, $x_1 x_2 - x_3^2 = 0$.

(3) The equation

$$\alpha x_1 x_4 + \beta x_1 x_3 + \gamma x_2 x_3 + \delta x_2 x_4 = 0,$$

in which α, β, γ, δ are linear homogeneous functions of λ_1, λ_2, λ_3, defines a bundle having the two basis lines $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, $x_4 = 0$.

(4) The bundle defined by

$$x_1 u_4 - x_4 u_1 = 0, \qquad x_2 u_4 - x_4 u_2 = 0, \qquad x_3 u_4 - x_4 u_3 = 0$$

has the line $x_4 = 0$, $u_4 = 0$ for basis curve. It has four basis points not lying on the line.

Any two quadrics contained in one of these forms (1), $\cdots$, (4) will intersect in the basis curve and a residual curve of order 1, 2, 2, or 3. In the first and third cases the residual intersection consists of one or of two generators. In the fourth case only will the residual intersection be a curve which is cut by every generator of one system in more than one point. On the focal surface of any congruence contained within the bundle the points of contact with a line of the congruence lie upon the complete intersection of two quadrics of the bundle. In the first two cases the lines are not proper bitangents; in the third the focal surface is a ruled surface contained within a linear congruence, while in the fourth the points of contact of lines of one congruence lie upon a bundle of space cubics of which they are bisecants. The operation of interchanging these points of contact is a birational transformation of order two. In case (3) it is simply an axial involution, hence case (4) will be the only one to be considered as defining a non-linear transformation.

24. The two systems of generators are rationally separable, hence the cubic complex is composite. One system may be written in the form

$$\frac{\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3}{u_4} = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3}{x_4} = \sigma,$$

and the other becomes

$$\frac{\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3}{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3} = \frac{u_4}{x_4} = \sigma';$$

thus it is seen that Σ consists of a quadratic and of a special linear complex, the axis of the latter being the basis line $u_4 = 0$, $x_4 = 0$.

The four distinct basis-points are the vertices of the invariant tetrahedron of the collineation defined by

$$x_i = \rho u_i.$$

A congruence of order n is defined by a relation of the form $f_n(\lambda_1, \lambda_2, \lambda_3) = 0$.

As before, if ν is the class of f_n, the order of the focal surface will be 2ν; the line $u_4 = 0$, $x_4 = 0$ is an ν-fold line upon the surface, and each of the distinct basis points is also ν-fold upon it. Upon the focal surface there is moreover a cuspidal curve composed of as many space cubics as $f_n(\lambda)$ has points of inflexion.

One or more of the quadric cones contained in σ may break up into a pair of planes; one plane must contain the basis line and pass through one basis point, and the other contains the three remaining basis points, hence the maximum number of such degenerate cones is four.

The equation

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = \sigma u_4$$

defines a pencil of planes through every tangent line to the curve $f_n(\lambda)$ in $u_4 = 0$. The equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = \sigma x_4$$

defines a similar pencil through the tangents to the curve $f_n(\lambda)$ in $x_4 = 0$. From the equations $x_i = \rho u_i$ these two systems are projective and the congruence is defined as the locus of intersection of corresponding planes. The complex σ is thus a tetrahedral complex.

CORNELL UNIVERSITY,
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ON THE FUNDAMENTAL NUMBER OF THE ALGEBRAIC NUMBER-FIELD $k(\sqrt[p]{m})$

BY

JACOB WESTLUND

Introduction.

The object of the present paper is the determination of an integral basis and the fundamental number of the algebraic number-field $k(\sqrt[p]{m})$ generated by the real pth root of m, where m is a positive integer greater than unity which is not divisible by the pth power of an integer, and where p is any odd prime. The case $p = 3$ has already been discussed by DEDEKIND.* The conjugate values of $\sqrt[p]{m}$ being $\sqrt[p]{m}, \rho\sqrt[p]{m}, \cdots, \rho^{p-1}\sqrt[p]{m}$, where $\rho = e^{2\pi i/p}$, the number-fields $k(\rho\sqrt[p]{m}), \cdots, k(\rho^{p-1}\sqrt[p]{m})$ are all different from $k(\sqrt[p]{m})$.

In order to obtain all possible number-fields of this type we let m run through all positive integers which are not divisible by the pth power of a prime. But the fields generated in this way are not all distinct. For any positive integer m which is not divisible by the pth power of a prime may be expressed in one way only in the form

$$m = a_1 a_2^2 a_3^2 \cdots a_{p-1}^{p-1}$$

where $a_1 a_2 \cdots a_{p-1}$ is not divisible by the square of a prime. If we then set

$$\alpha_i = \sqrt[p]{a_1^{i_1} a_2^{i_2} a_3^{i_3} \cdots a_{p-1}^{i_{p-1}}}$$

where $i_s \equiv si \pmod{p}$ and $0 < i_s < p$ for $s = 1, 2, 3, \cdots, p-1$, it is evident that $\alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ are algebraic integers in $k(\alpha_1)$, and hence $k(\alpha_1), k(\alpha_2), \cdots, k(\alpha_{p-1})$ are identical, while $k(\alpha_1)$ is a primitive field.

1. *Rational basis.*

As a rational basis of $k(\alpha_1)$ we may take either

$$1, \alpha_1, \alpha_1^2, \cdots, \alpha_1^{p-1}$$

or

$$1, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}.$$

**Über die Anzahl der Idealklassen in reinen kubischen Zahlkörpern*, Journal für die reine und angewandte Mathematik, vol. 121 (1899).

Denote the discriminants of these bases by D_1 and D_2, respectively. We have

$$D_1 = \begin{vmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{p-1} \\ 1 & \rho\alpha_1 & \cdots & \rho^{p-1}\alpha_1^{p-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \rho^{p-1}\alpha_1 & \cdots & \rho^{(p-1)^2}\alpha_1^{p-1} \end{vmatrix}^2 = m^{p-1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \rho & \cdots & \rho^{p-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \rho^{p-1} & \cdots & \rho^{(p-1)^2} \end{vmatrix}^2$$

Hence*

$$D_1 = (-1)^{\frac{1}{2}(p-1)} p^p m^{p-1}.$$

In a similar way we obtain

$$D_2 = (-1)^{\frac{1}{2}(p-1)} p^p (a_1 a_2 \cdots a_{p-1})^{p-} .$$

If Δ be the fundamental number of $k(\alpha_1)$, we must have $D_2 = n^2\Delta$ where n is a rational integer. Hence

$$\Delta = (-1)^{\frac{1}{2}(p-1)} p \left[\frac{(pa_1 a_2 \cdots a_{p-1})^{\frac{1}{2}(p-1)}}{n} \right]^2 = (-1)^{\frac{1}{2}(p-1)} pd^2$$

where d is a rational integer, and this shows that Δ contains the factor p.

2. *Ideal Prime Factors of p and m.*

Let q be a prime factor of m and Q an ideal prime factor of q. Then since $\alpha_1^p = q^i r$, where r is prime to q and $0 < i < p$, it follows that α_1 is divisible by Q. Suppose that Q^s is the highest power of Q contained in q. Then α_1^p must be divisible by Q^{si} and $si \equiv 0 \pmod{p}$. Hence $s = p$ and $(q) = Q^p$, i. e., *every prime factor of m is equal to the pth power of a prime ideal of the first degree.*

Let us next consider the prime p. If p is a factor of m it comes under the case already considered. Suppose then that p is not contained in m. Since p is a factor of the fundamental number, it is divisible by the square of a prime ideal P. Now consider the integer $\mu = \alpha_1 - b$, where $b = a_1 a_2^2 \cdots a_{p-2}^{p-2}$. We have

$$(\mu + b)^p - ba_{p-1}^{p-1} = 0$$

or, if we set $d = b^{p-1} - a_{p-1}^{p-1}$,

$$\mu^p + pb\mu^{p-1} + \cdots + pb^{p-1}\mu + bd = 0.$$

Since $d \equiv 0 \pmod{p}$ it follows that μ^p is divisible by p and μ by P and hence d is divisible by P^3. Two cases arise according as d is divisible by p^2 or not.

I. *d not divisible by p^2.* In this case p must be divisible by P^3. Hence, if $p > 3$, d must be divisible by P^4 and therefore p divisible by P^4. Reasoning

* PASCAL, *Determinanten*, p. 139.

in this way we find that p must be divisible by P^p. Hence $(p) = P^p$, i. e., *if p is prime to m and $d = b^{p-1} - a_{p-1}^{p-1}$ not divisible by p^2, then p is equal to the p-th power of a prime ideal of the first degree.*

II. *d divisible by p^2.* Let $p^s (s \geqq 2)$ be the highest power of p contained in d and P^r the highest power of P contained in μ. The equation satisfied by μ may be written

$$\mu(\mu^{p-1} + p\beta) + bd = 0,$$

where β is prime to P. If r were greater than unity, μ^{p-1} would be divisible by a higher power of P than P^p, and since p cannot contain a higher power of P than P^p, it follows from the equation above that μ would be divisible by p. But if μ were divisible by p, its conjugates would be divisible by p, but this is impossible, since the coefficient of μ in the equation above contains only the first power of p. Hence $r = 1$. It is then easily seen that p must be divisible by P^{p-1} and by no higher power of p. *Hence if p is prime to m, and $b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2, we have $(p) = P^{p-1}Q$, where P and Q are different prime ideals of the first degree.*

3. *Integral basis.*

Any integer ω in $k(\alpha_1)$ may be expressed in the form

$$\omega = \frac{x_0 + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}}{D_2},$$

where $x_0, x_1, \cdots, x_{p-1}$ are rational integers. Let q be a prime factor of a_s and let $(q) = Q^p$. Then the highest power of Q contained in α_i is Q^{i_s}, where $i_s \equiv si \pmod p$ and $0 < i_s < r$. Hence x_0 must be divisible by Q and hence by q. Denote by $\alpha_{r_1}, \alpha_{r_2}, \cdots, \alpha_{r_{p-1}}$ the numbers $\alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ arranged according to increasing powers of Q. It then follows that x_{r_1} must be divisible by Q and hence by q. In the same way we find that $x_{r_2}, \cdots, x_{r_{p-1}}$ are divisible by q. It is then easily seen that ω may finally be written in the form

$$\omega = \frac{x_0 + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}}{p^p},$$

where $x_0, x_1, \cdots, x_{p-1}$ are rational integers.

If p is a factor of m we proceed as above and find that

$$\omega = y_0 + y_1\alpha_1 + \cdots + y_{p-1}\alpha_{p-1},$$

where $y_0, y_1, \cdots, y_{p-1}$ are rational integers.

If p is prime to m, two cases arise, according as $d = b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2 or not.

I. *d not divisible by p^2.* Introducing the algebraic integer $\mu = \alpha_1 - b$ mentioned above and making use of the fact that $\alpha_i = \alpha_1^i / c_i$, where c_i is a rational

integer prime to p, we obtain

$$c\omega = \frac{y_0 + y_1\mu + \cdots + y_{p-1}\mu^{p-1}}{p^p}.$$

In this case we have $(p) = P^p$ and, as is easily seen, μ is divisible by P but not by P^2. Reasoning in exactly the same way as above we find that $y_0, y_1, \cdots, y_{p-1}$ are all divisible by p. Hence we finally get

$$c\omega = z_0 + z_1\mu + \cdots + z_{p-1}\mu^{p-1},$$

where $z_0, z_1, \cdots, z_{p-1}$ are rational integers. But since all the prime factors of c are contained in m, it follows that ω may be written in the form

$$\omega = x_0 + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1},$$

where $x_0, x_1, \cdots, x_{p-1}$ are rational integers. We then have the following result:

If $b^{p-1} - a_{p-1}^{p-1}$ is not divisible by p^2, the p numbers $1, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ form an integral basis of $k(\alpha_1)$ and $\Delta = D_2 = (-1)^{p-1/2}p^p(a_1a_2\cdots a_{p-1})^{p-1}$.

II. *d divisible by p^2.* In this case we know that $(p) = P^{p-1}Q$. We also know that μ^p is divisible by p, and hence μ is divisible by PQ and μ^{p-1} divisible by pQ^{p-2}. But

$$\begin{aligned}\mu^{p-1} &= (\alpha_1 - b)^{p-1} = \alpha_1^{p-1} - (p-1)\alpha_1^{p-2}b \cdots + b^{p-1}\\ &= [\alpha_1^{p-1} + \alpha_1^{p-2}b + \cdots \alpha_1 b^{p-2} + 1]\\ &\qquad - \left[p\alpha_1^{p-2}b - \left\{\frac{(p-1)(p-2)}{2!} - 1\right\}\alpha_1^{p-3}b^2 + \cdots - b^{p-1} + 1\right]\end{aligned}$$

and since $b^{p-1} \equiv 1 \pmod{p}$, it follows that

$$\gamma = \frac{\alpha_1^{p-1} + \alpha_1^{p-2}b + \cdots + \alpha_1 b^{p-2} + 1}{p}$$

is an algebraic integer. We shall now prove that the p numbers

$$\gamma, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}$$

form an integral basis of $k(\alpha_1)$. It is evident that these numbers form a rational basis. Denoting the discriminant of this basis by D_3 we get the following value

$$D_3 = (-1)^{p-1/2}p^{p-2}(a_1a_2\cdots a_{p-1})^{p-1}.$$

Now any algebraic integer ω may be written in the form

$$\omega = \frac{x_0\gamma + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}}{D_3}$$

where $x_0, x_1, \cdots, x_{p-1}$ are rational integers. It is easily seen that x_0 must be

divisible by D_3. For, denoting by $\omega, \omega', \cdots, \omega^{(p-1)}$ the conjugate values of ω, we have

$$\omega + \omega' + \cdots + \omega^{(p-1)} = \frac{x_0}{D_3}.$$

Hence x_0 is divisible by D_3. Let us then consider the algebraic integer

$$\omega_1 = \frac{x_1 \alpha_1 + \cdots + x_{p-1} \alpha_{p-1}}{D_3}.$$

If q be a prime factor of m we infer in the same way as above that $x_1, \cdots, x_{p-1}$ are divisible by q, and hence that ω_1 may be written in the form

$$\omega_1 = \frac{y_1 \alpha_1 + \cdots + y_{p-1} \alpha_{p-1}}{p^{p-2}}.$$

Replacing α_1 by $\mu + b$ we get

$$p^{p-2} c\omega_1 = z_0 + z_1 \mu + \cdots + z_{p-1} \mu^{p-1}$$

where $z_0, z_1, \cdots, z_{p-1}$ are rational integers and c is prime to p. By a simple argument it can then be shown that $z_0, z_1, \cdots, z_{p-1}$ and hence also $y_1, y_2, \cdots, y_{p-1}$ must be divisible by p and that ω may finally be written in the form

$$\omega = x_0 \gamma + x_1 \alpha_1 + \cdots + x_{p-1} \alpha_{p-1}.$$

Hence we have the following result: *If $b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2, the p numbers $\gamma, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ form an integral basis of $k(\alpha_1)$ and*

$$\Delta = D_3 = (-1)^{p-1/2} p^{p-2} (a_1 a_1 \cdots a_{p-1})^{p-1}.$$

PURDUE UNIVERSITY.

VOLTERRA'S INTEGRAL EQUATION OF THE SECOND KIND, WITH DISCONTINUOUS KERNEL*

BY

GRIFFITH C. EVANS

The integral equation of the second kind, of VOLTERRA, is written

$$u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi) d\xi. \tag{1}$$

In this equation the function $K(x, \xi)$ is called the kernel; the desired function is $u(x)$. When the functions $K(x, \xi)$ and $\phi(x)$ are continuous there is no difficulty in finding a continuous $u(x)$ that shall satisfy equation (1). This general case of the equation (1), under the conditions solely that $\phi(x)$ be continuous when $a \leqq x \leqq b$ and that $K(x, \xi)$ be continuous in the triangular region $a \leqq \xi \leqq x \leqq b$, was first investigated by VOLTERRA,† who showed that there is one and only one continuous solution in the interval $a \leqq x \leqq b$. His method applies without essential change to equations whose kernels are finite in the region $a \leqq \xi \leqq x \leqq b$ and have discontinuities, provided the discontinuities are regularly ‡ distributed.

Let us consider, however, a certain equation to which we are led by a hydrostatical problem. Suppose we are given a tube lying in a vertical plane along a curve of arbitrary shape, $s = u(x)$, where s is the distance along the curve and x the altitude. Let us fill this tube with a liquid of variable linear density ν, and then regulate its height x in the tube by allowing various amounts to flow through the bottom. Let us then regard ν as an analytic function of the depth in the liquid, i. e. $\nu = \nu(x - \xi)$. The average linear density is given by the formula

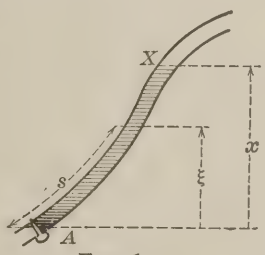

FIG. 1.

$$h(x) = \frac{\int_A^X \nu(x - \xi) ds}{\int_A^X ds} = \frac{\int_0^x \nu(x - \xi) u'(\xi) d\xi}{\int_0^x u'(\xi) d\xi},$$

* Presented to the Society, September 13, 1909.

† V. VOLTERRA, Atti dell' Accademia dei Lincei, ser. 5, vol. 5 (1896), pp. 177–184.

‡ M. BÔCHER: *An Introduction to the Study of Integral Equations;* Cambridge Tracts in Mathematics and Mathematical Physics, No. 10, 1909.

so that, when the curve is given, the average density is determined explicitly. Let us propose to ourselves the inverse problem: given the average linear density, to find the shape of the curve.

In the first place, if $u'(\xi)$ is continuous $h(x)$ is continuous, and if we let

$$\nu(x-\xi) = \alpha + \alpha_1(x-\xi) + \alpha_2(x-\xi)^2 + \cdots$$
$$= \alpha + G(x-\xi),$$

where $G(0)=0$, then $h(x)=\alpha+g(x)$, where $g(x)$ is some function that vanishes at least to the first order when $x=0$. We shall look only for solutions $u(x)$ that have continuous derivatives. Hence we may write the equation to determine $u'(x)$ as

$$g(x)\int_0^x u'(\xi)d\xi = \int_0^x G(x-\xi)u'(\xi)d\xi,$$

or

$$0 = \int_0^x [G(x-\xi)-g(x)]v(\xi)d\xi,$$

where $v(\xi)=u'(\xi)$.

This is an equation of which the kernel vanishes when $x=\xi=0$. If we differentiate once and divide by $g(x)$, we get the equation

$$v(x) = \int_0^x \frac{G(x-\xi)-g'(x)}{g(x)} v(\xi)d\xi,$$

which is a homogeneous equation of the form (1), with *discontinuous* kernel.*

The corresponding extension in the general equation (1) of the second kind is found by differentiation of the general integral equation of the first kind

$$\phi(x)-\phi(a) = \int_a^x K(x,\xi)u(\xi)d\xi. \tag{2}$$

It was originally an equation of this kind,

$$\psi(a) = \int_{x=0}^{x=a} \frac{ds}{(a-x)^n}, \qquad 0<n<1,$$

by which ABEL in 1823 † introduced the subject of integral equations; an equation of the first kind with discontinuous kernel. The general equation (2) where the kernel is continuous is itself not easy to treat. Yet in his first paper VOLTERRA shows that if $\phi(y)$ and $\phi'(y)$ remain continuous when $a \leqq y \leqq b$, and if $K(x,y)$ and $\partial K/\partial x = K_1(x,y)$ remain continuous in the triangular

* This equation will be treated in a later paper.

† VOLTERRA: Annali di Matematica, ser. 2, vol. 25 (1897), p. 140.

region $a \leqq y \leqq x \leqq b$, while $K(x, x) \neq 0$, $a \leqq x \leqq b$, then there is one and only one solution of (2) continuous in the interval $a \leqq x \leqq b$.*

There are two manifest ways of enlarging the kernel of (2) beyond VOLTERRA's first restrictions. One way is to generalize the equation of ABEL to the form

$$\phi(x) - \phi(a) = \int_a^x \frac{G(x, \xi)}{(x - \xi)^\lambda} u(\xi) d\xi, \qquad \lambda < 1,$$

where $G(x, \xi)$ and $\partial G/\partial x$ are continuous in the triangular region and $G(x, x) \neq 0$; this generalization VOLTERRA carries out in the second of his papers † appearing in the Atti di Torino, and shows that there is one and only one continuous solution. A second way is to allow the kernel of (2) to vanish when $x = \xi = a$. This generalization, with the restriction that $K(x, \xi)$ shall not vanish to higher order when ξ and x are equal and both approach a, than when ξ and x approach a independently, is considered by VOLTERRA in his third and fourth Turin papers. ‡ And here it turns out that there is in general more than one solution continuous in the neighborhood of a.

The same problem is considered by E. HOLMGREN § in 1900, and the special case is treated in which the kernel is in the form

$$K(x, y) = a_0 y + a_1 x + K'(x, y)$$

and $a_0 + a_1$ is not necessarily unequal to zero. The present state of the question is best set forth in a recent paper of LALESCO. ‖ He shows among other things that the equation

$$\int_0^x f(x, s) u(s) ds = F(x), \tag{3}$$

in which

$$f(x, s) = A_n x^n + A_{n-1} x^{n-1} s + \cdots + A_0 s^n + \psi(x, s),$$

— here $\psi(x, s)$ contains only terms whose total degree in x and s is greater than n, and $A_0 + A_1 + \cdots + A_n \neq 0$, — has a solution $u(x)$, finite at $x = 0$ and depending linearly on k parameters, provided that $F(x)$ vanishes to a sufficiently high order and that k distinct solutions of the equation

* Atti della Reale Accademia delle Scienze di Torino, vol. 31, Jan., 1896, pp. 311–323.

† Atti della Reale Accademia delle Scienze di Torino, vol. 31, Jan., 1896, pp. 400–408.

‡ Atti della Reale Accademia delle Scienze di Torino, vol. 31, March, 1896, pp. 557–567 and April, 1896, pp. 693–708.

§ E. HOLMGREN: Sur une théorème de M. Volterra sur l'inversion des intégrales définies; Atti della Reale Acc. delle Scienze di Torino, vol. 35 (1900), p. 570.

‖ T. LALESCO: Sur l'équation de Volterra; Journal de Mathématiques, ser. 6, vol. 4 (1908) p. 125.

$$\frac{A_0}{r-1}+\frac{A_1}{r-2}+\cdots+\frac{A_n}{r-n-1}=0\,*$$

have their real parts negative. LALESCO considers also the special case in which

$$f(x, s)=A_1x+A_0s+\alpha x^2+\beta xy+\gamma y^2+\psi_3(x, s),$$

where $A_1+A_0=0$ and $\alpha+\beta+\gamma\neq 0$, and shows, as was previously announced by E. HOLMGREN (see page 395, note), that if

$$\frac{A_0}{\alpha+\beta+\gamma}>0$$

there will be a single real solution, while if

$$\frac{A_0}{\alpha+\beta+\gamma}<0$$

there will be an infinity of solutions depending on a single parameter.

LALESCO's method of procedure, to which the method of our later paper is analogous, is to differentiate the equation (3) in the general case $n+1$ times, and then to base on the resulting Fuchsian equations an approximation method by which the given equation may be solved. It is essential in order to reach a Fuchsian equation that $A_0+A_1+\cdots+A_n\neq 0$.

The equation to which (3) is reduced by differentiating more than once is a special case of a type of equations proposed by BURGATTI,† and investigated in a short note by FUBINI,‡ i. e., equations of the form

$$(4)\qquad \int_0^x\left[f_0(x, s)u(s)+f_1(x, s)\frac{du}{ds}+\cdots+f_n(x, s)\frac{d^nu}{ds^n}\right]ds=F(x).$$

The connection between equations of the first kind with kernel vanishing at the initial point of the interval, and equations of the second kind with kernels discontinuous at the initial point of the interval is seen, as we have said, by differentiating once. Equation (3) thus becomes

$$f(x, x)u(x)+\int_0^x\frac{\partial f(x, s)}{\partial x}u(s)\,ds=F'(x)$$

or

$$u(x)=\frac{F'(x)}{f(x, x)}-\int_0^x\frac{1}{f(x, x)}\frac{\partial f(x, s)}{\partial x}u(s)\,ds,$$

* This equation was originally given by VOLTERRA, and the fact shown that there is but one finite solution if the real parts of all the roots are positive, while otherwise the solution is not determined uniquely.

† BURGATTI: Rendiconti dell' Accademia dei Lincei, 2. semestre 1903, p. 443 and p. 596.

‡ FUBINI: Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche di Napoli, ser. 3, vol. 10 (1904), pp. 61–64.

which on account of the restriction $A_0 + A_1 + \cdots + A_n \neq 0$ is an equation the kernel of which becomes infinite at $x = \xi = 0$ to not higher than the first order. The special equation on page 396 corresponds to a case where the kernel of an equation of the second kind becomes infinite at $x = \xi = 0$ to the second order.

In the present paper we consider equations whose kernels are absolutely integrable, and equations related to them by change of dependent variable. Later we shall treat equations whose kernels, though not of so general a type, are not restricted in their order of becoming infinite, and endeavor to define as to continuity the character of functions that can possibly satisfy the equations under investigation.

Throughout the course of the work it will be convenient to be able to refer to the following two conditions, (A) and (B).

Condition (A). A real function of the two variables x, ξ is to be continuous in the triangle $T: a \leqq \xi \leqq x \leqq b$, $b > a \geqq 0$ except on a finite number of curves, each composed of a finite number of continuous pieces with continuously turning tangents. Any vertical portion is to be considered a separate piece, and of such pieces there are to be merely a finite number, $x = \beta_1$, $x = \beta_2$, $\cdots$, $x = \beta_r$. On the other portions of the system of curves there are to be only a finite number of vertical tangents. Hence by any line $x = x_0$, $x_0 \neq \beta_1, \cdots, \beta_r$, the system of curves will be cut in only a finite number of points.

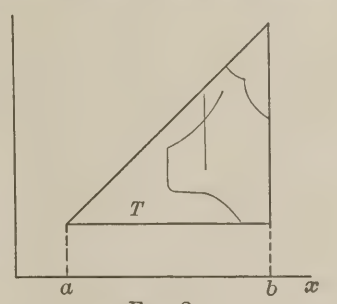

FIG. 2.

Condition (B). In the region $t: a \leqq x \leqq b$, a real function of the single variable x is to be continuous except at a finite number of points, $\gamma_1, \cdots, \gamma_s$, and is to remain finite.

§ 1. *A General Theorem.*

The existence of the unique continuous solution of (1) when $\phi(x)$ and the kernel are continuous is established by the development of $u(x)$ into a series by substitution of the equation into itself.* If we assume for the moment that the equation (1) has a continuous solution, $u(x)$, then the following equations will hold:

$$u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi) d\xi,$$

$$u(x) = \phi(x) + \int_a^x K(x, \xi)\phi(\xi) d\xi + \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') u(\xi') d\xi' d\xi,$$

* Method of Successive Substitutions, LIOUVILLE: Journal de Mathématiques, vol. 2 (1837), pp. 16–36 (applied to a particular equation).

etc. As n becomes infinite, the general expression becomes an infinite series. This series is shown to be uniformly convergent, and a solution of the given integral equation (1). The same method applies when $K(x, \xi)$ is not so much restricted.*

Let us define besides the linear region t and the two dimensional region T, a linear region t_δ formed from t by removing the small portions $\alpha_i - \delta < x < \alpha_i + \delta$ $(i = 1, 2, \cdots, l)$; and a two dimensional region T_δ formed from T by removing the small strips $\alpha_i - \delta < x < \alpha_i + \delta$ $(i = 1, 2, \cdots, l)$, where the δ is an arbitrarily small magnitude, and the α's are finite in number.

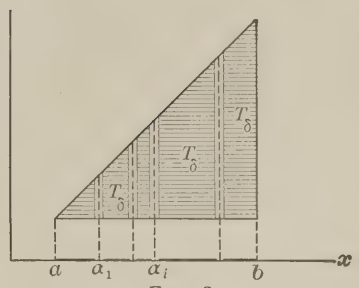

FIG. 3.

THEOREM. *Consider the integral equation* (1). *Let $K(x, \xi)$ satisfy Condition* (A), *and $\phi(x)$ Condition* (B). *Furthermore let us assume the following conditions:*

(a) $\int_a^x |K(x, \xi)| d\xi$ *converges for all values of x in the interval $a \leqq x \leqq b$ except for a finite number of such values,*† $x = \lambda_1, \lambda_2, \cdots, \lambda_t$, *and the function thus defined remains finite, less than M;*

(b) *Let ϵ and δ be two arbitrary positive quantities. Then there is a positive quantity η and a region T_δ such that*

$$\int_y^{y+\eta} |K(x, \xi)| d\xi < \epsilon, \qquad (x, y) \text{ and } (x, y+\eta) \text{ in } T_\delta.$$

The points $\alpha_1, \cdots, \alpha_l$ that appear in the definition of T_δ are independent of ϵ and δ and include the points β of (A) *and λ of* (a);

(c) *t can be divided into k parts, bounded by the points $a = a_0, a_1, \cdots, b = a_k$, such that*

$$\int_{a_i}^x |K(x, \xi)| d\xi \leqq H < 1 \qquad a_i \leqq x \leqq a_{i+1}\,;\ x \neq a_1, \cdots, a_l.$$

Under these conditions there is one and only one solution $u(x)$ of the integral equation (1), *continuous in t except for a finite number of points, and these points will be among the points $\gamma_1, \cdots, \gamma_s$, of* (B), *and $\alpha_1, \cdots, \alpha_l$, of* (b).

In the proof of this theorem (a) and (c) are used in showing the convergence of the expansion of the solution, and (b) in developing what continuity exists.

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* The principal theorem of this part is a generalization of theorems submitted by Dr. W. A. HURWITZ, Dr. C. N. MOORE and myself as answers to a problem given out by Professor MAXIME BÔCHER in his course on integral equations, Harvard University, 1907–08.

† An example of a point λ not a point β is given by the point $x = 1$, where $K(x, \xi) = 1$ in the horizontally shaded region of the adjoined figure, and $K(x, \xi) = 1/(x-1)$ in the vertically shaded region.

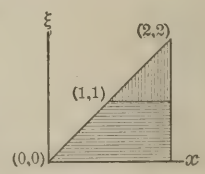

FIG. 4.

§ 2. *Convergence of the Expansion.*

We shall show that the series

$$\bar{u}(x) = \int_a^x K(x, \xi)\phi(\xi)d\xi + \int_a^x K(x, \xi)\int_a^\xi K(\xi, \xi')\phi(\xi')d\xi' d\xi + \cdots$$

converges absolutely and uniformly in the region t_δ and represents in t a function finite and continuous except for a finite number of points.

Every term of this series is continuous in the region t_δ. For if $r(x)$ is any function continuous in t except for a finite number of points and finite, in absolute value less than N, the integral

$$F(x) = \int_a^x K(x, \xi)r(\xi)d\xi$$

is continuous in the region t_δ and finite in t.

To prove this take x_0, any point in t_δ, and $x_0 + \Delta x$, a point near it and in t_δ, Fig. 5, and consider

$$\left|\int_a^{x_0+\Delta x} K(x_0 + \Delta x, \xi)r(\xi)d\xi - \int_a^{x_0} K(x_0, \xi)r(\xi)d\xi\right| = |\Delta F|$$

$$\leqq \left|\int_{x_0}^{x_0+\Delta x} K(x_0+\Delta x, \xi)r(\xi)d\xi\right| + \left|\int_a^{x_0} \{K(x_0+\Delta x, \xi) - K(x_0, \xi)\}r(\xi)d\xi\right|$$

$$\leqq N\int_{x_0}^{x_0+\Delta x} |K(x_0+\Delta x, \xi)|\,|d\xi| + N\int_a^{x_0} |K(x_0+\Delta x, \xi) - K(x_0, \xi)|\,|d\xi|.$$

Let us assign the value ρ, arbitrarily small, in advance. On each side of every one of the p points $\xi_1, \xi_2, \cdots, \xi_p$ in which the line $x = x_0$ is cut by the system of curves of discontinuity of $K(x, \xi)$ lay off a distance $\frac{1}{2}\eta$ such that

$$\int_{\xi_i-\frac{1}{2}\eta}^{\xi_i+\frac{1}{2}\eta} |K(x, \xi)|\,|d\xi| < \epsilon$$

in accordance with the condition (b) of the theorem. Take

$$\epsilon = \left(\frac{\rho}{p+1}\right)\left(\frac{1}{2N}\right).$$

Take $\Delta x < \eta$, and small enough so that none of the curves of the system are cut by any of the lines $\xi = \xi_i \pm \eta/2$ in the interval $x_0 \leqq x \leqq x_0 + \Delta x$. We shall then have

$$|\Delta F| \leqq \frac{\rho}{2} + N\left[\Sigma\int_{A_i}^{B_i} |K(x_0 + \Delta x, \xi) - K(x_0\xi)|\,|d\xi|\right],$$

where the integration is now extended over the sections of the line $x = x_0$

remaining after the p pieces have been removed. And since $K(x, \xi)$ is continuous in these closed regions (the shaded regions in the diagram, Fig. 5), Δx may be chosen small enough so that

$$|K(x_0 + \Delta x, \xi) - K(x, \xi)| \leqq \frac{\rho}{(x_0 - a)} \frac{1}{2N}.$$

Accordingly we have, finally,

$$|\Delta F| \leqq \frac{\rho}{2} + \frac{\rho}{2(x-a)} \Sigma \{B_i - A_i\} \leqq \frac{\rho}{2} + \frac{\rho}{2(x-a)}(x-a) \leqq \rho;$$

and F is continuous in t_δ, i. e. continuous in t except for the points $\alpha_1, \cdots, \alpha_l$, since δ is as small as we please to take it. Moreover, F is finite in t since $|F| \leqq NM$.

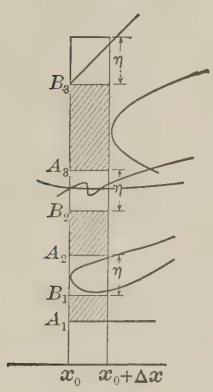

FIG. 5.

The same proof applies to the function

$$R(x) = \int_a^R K(x, \xi) r(\xi) d\xi, \qquad a \leqq R \leqq x,$$

showing that it is continuous except for the points α and finite, in t.

The function

$$\int_a^x K(x, \xi) r(\xi) d\xi$$

itself satisfies the conditions laid on $r(x)$. For it is continuous except for the points α and finite. Consequently, as is seen by mathematical induction, every term, and every integral appearing in every term, of the series is an $r(x)$, with points of discontinuity included in the points $\alpha_1, \cdots, \alpha_l$.

We now show the absolute and uniform convergence of the series

$$(3) \quad \bar{u}(x) = \int_a^x K(x, \xi)\phi(\xi) d\xi + \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi')\phi(\xi') d\xi' d\xi + \cdots$$

in the region t_δ. If we write it as

$$\sum_{n=0}^{\infty} u_{n+1}(x)$$

its general term is

$$u_{n+1}(x) = \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') \int_a^{\xi'} \cdots \int_a^{\xi^{(n-1)}} K(\xi^{(n-1)}, \xi^{(n)}) \phi(\xi^{(n)}) d\xi^{(n)} \cdots d\xi' d\xi.$$

We have immediately

$$(4) \quad |u_{n+1}(x)| \leqq N \int_a^x |K(x, \xi)| \int_a^\xi |K(\xi, \xi')| \int_a^{\xi'} \cdots \int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)}, \xi^{(n)})| d\xi^{(n)} \cdots d\xi' d\xi,$$

and in that part of the region t_δ for which x lies in the first of the intervals mentioned in (c), i. e., $a \leqq x \leqq a_1$, it follows that

$$|u_{n+1}(x)| \leqq NH^{n+1},$$

independently of δ. We may write this, for convenience, as

$$|u_{n+1}(x)| \leqq N\{(n+1)M\}^{j-1}H^{n+2-j}, \qquad j=1. \tag{5}$$

We can establish corresponding inequalities of the type (5) for any interval a_{i-1} to a_i, by a process of mathematical induction. For if we assume the formula

$$\begin{aligned} |u_{n+1}(x)| &\leqq N\int_a^x |K(x,\xi)|\int_a^\xi |K(\xi,\xi')|\int_a^{\xi'}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|d\xi^{(n)}\cdots d\xi' d\xi \\ &\leqq N\{(n+1)M\}^{j-1}H^{n+2-j} \end{aligned} \tag{6}$$

to hold independently of the value of δ for the first $i-1$ intervals of t, i. e., when $j = 1, 2, \cdots, i-1$, respectively, we can show that it holds for the ith interval, $j = i$, as well. It is obvious, since $M \geqq H$ and $|u_{n+1}(x)| \leqq NM^{n+1}$, that the formula holds in the ith interval when $n \leqq i-2$. Also, since $M \geqq H$,

$$N\{(n+1)M\}^{j_1-1}H^{n+2-j_1} \leqq N\{n+1)M\}^{j_2-1}H^{n+2-j_2}$$

if $j_2 > j_1$; hence the formula

$$\begin{aligned} |u_{n+1}(x)| &\leqq N\int_a^x |K(x,\xi)|\int_a^\xi |K(\xi,\xi')|\int^{\xi'}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|d\xi^{(n)}\cdots d\xi' d\xi \\ &\leqq N\{(n+1)M\}^{i-2}H^{n+3-i} \qquad (j=i-1), \end{aligned} \tag{7}$$

holds if x is in any of the first $i-1$ intervals.

Now, by rearrangement of the integral in (4) we have

$$|u_{n+1}(x)| \leqq$$

$$\begin{aligned} &\int_a^{a_{i-1}} |K(x,\xi)|\left\{N\int_a^\xi |K(\xi,\xi')|\int_a^{\xi'}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|\,d\xi^{(n)}\cdots d\xi'\right\}d\xi \\ &+\int_{a_{i-1}}^x |K(x,\xi)|\left[\int_a^{a_{i-1}} |K(\xi,\xi')|\left\{N\int_a^{\xi'} |K(\xi',\xi'')|\int_a^{\xi''}\right.\right. \\ &\left.\left.\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|\,d\xi^{(n)}\cdots d\xi''\right\}d\xi'\right]d\xi+\int_{a_{i-1}}^x |K(x,\xi)|\int_{a_{i-1}}^\xi |K(\xi,\xi')| \\ &\times\left[\int_a^{a_{i-1}} |K(\xi',\xi'')|\left\{N\int_a^{\xi''}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|d\xi^{(n)}\cdots d\xi'''\right\}d\xi''\right]d\xi' d\xi \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &+N\int_{a_{i-1}}^x |K(x,\xi)|\int_{a_{i-1}}^\xi |K(\xi,\xi')|\int_{a_{i-1}}^{\xi'}\cdots\left[\int_{a_{i-1}}^{\xi^{(n+1-i)}} |K(\xi^{(n+1-i)},\xi^{(n+2-i)})|\right. \\ &\left.\times\int_a^{\xi^{(n+2-1)}}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|\,d\xi^{(n)}\cdots d\xi^{(n+2-i)}\right]d\xi^{(n+1-i)}\cdots d\xi. \end{aligned}$$

These $n+3-i$ terms we have already a means for treating. For, as we have shown on page 401, we can apply to the integrals within the { } the formula (7). For the single remaining integral inside the square bracket, in the first $n+2-i$ terms, we may write

$$\int_a^{a_{i-1}} |K(\xi^{(r)}, \xi^{(r+1)})| d\xi^{(r+1)} \leqq M.$$

The terms outside both brackets we may write as less than or equal to a power of H. The last term we have directly $\leqq NH^{n+2-i}M^{i-1}$. We may write, then, for the ith interval of t_δ

$$|u_{n+1}(x)| \leqq$$

$$\begin{aligned}
& MN\{(n-1+1)M\}^{i-2}H^{n-1+3-i} && (\text{or } Nn^{i-2}M^{i-1}H^{n+2-i}) \\
+\ & HMN\{(n-2+1)M\}^{i-2}H^{n-2+3-i} && (\text{or } N(n-1)^{i-2}M^{i-1}H^{n+2-i}) \\
+\ & H^2MN\{(n-3+1)M\}^{i-2}H^{n-3+3-i} && (\text{or } N(n-2)^{i-2}M^{i-1}H^{n+2-i}) \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
+\ & H^{n+1-i}MN\{(i-1)M\}^{i-2}H && (\text{or } N(i-1)^{i-2}M^{i-1}H^{n+2-i}) \\
+\ & NM^{i-1}H^{n+2-i}.
\end{aligned}$$

Hence we have

$$|u_{n+1}(x)| \leqq NM^{i-1}H^{n+2-i}[n^{i-2}+(n-1)^{i-2}+(n-2)^{i-2}+\cdots+(i-1)^{i-2}+1],$$

where there are in the bracket $n+3-i$ terms. Hence, since $i \geqq 2$,

$$|u_{n+1}(x)| \leqq NM^{i-1}H^{n+2-1}(n+1)n^{i-2} \leqq NM^{i-1}H^{n+2-i}(n+1)(n+1)^{i-2},$$

and finally,

$$|u_{n+1}(x)| \leqq N\{(n+1)M\}^{i-1}H^{n+2-1},$$

the desired result, which holds now not only for the ith interval of t_δ, but also for any point x in the first i intervals. Hence we have for any point x of the k intervals of t_δ,

$$|u_{n+1}(x)| \leqq N\{(n+1)M\}^{k-1}H^{n+2-k},$$

a value independent of the magnitude of δ. The series of constants, composed of terms of the type of the right hand side, is convergent by the simplest test. Suppose its value is Σ; then

$$|u(x)| \leqq \Sigma,$$

where Σ is independent of δ. Since the series is a uniformly convergent series of continuous functions in t_δ, it represents in t_δ a continuous function, regardless of the value of δ, $(\delta > 0)$.

§ 3. *Existence of a Solution.*

If we define

$$U(x) = \phi(x) + \bar{u}(x) \tag{8}$$

we know that $U(x)$ is a function continuous in t except for a finite number of points, which are among the points $\gamma_1, \cdots, \gamma_s, \alpha_1, \cdots, \alpha_l$; and finite $\leqq \Sigma + N$. It is now easy to show that $U(x)$ is a solution of the integral equation (1).

If we write for this purpose

$$U(x) = S_n(x) + R_n(x),$$

where $S_n(x)$ is the sum of the first n terms, the three functions $U(x)$, $S_n(x)$, $R_n(x)$ are of the type $r(x)$, § 2, and

$$\int_a^x K(x, \xi)U(\xi)d\xi = \int_a^x K(x, \xi)S_n(\xi)d\xi + \int_a^x K(x, \xi)R_n(\xi)d\xi$$

or

$$\int_a^x K(x, \xi)U(\xi)d\xi - \int_a^x K(x, \xi)S_n(\xi)d\xi = \int_a^x K(x, \xi)R_n(\xi)d\xi.$$

Hence

$$\lim_{n=\infty}\left(\int_a^x K(x, \xi)U(\xi)d\xi - \int_a^x K(x, \xi)S_n(\xi)d\xi\right) = 0$$

for

$$\left|\int_a^x K(x, \xi)R_n(\xi)d\xi\right| \leqq \int_a^x |K(x, \xi)||R_n(\xi)|d\xi;$$

and since $|R_n(x)|$ can be made less than an arbitrarily assigned ϵ by a suitable choice of n, we have

$$\left|\int_a^x K(x, \xi)R_n(\xi)d\xi\right| \leqq \epsilon \int_a^x |K(x, \xi)|d\xi,$$

which in t_δ is $\leqq \epsilon M$ and can therefore be made as small as we please by taking n large enough.

Hence

$$\int_a^x K(x, \xi)U(\xi)d\xi = \lim_{n=\infty} \int_a^x K(x, \xi)S_n(\xi)d\xi$$

and we can integrate the series term by term. We see in this way that $U(x)$ and

$$\phi(x) + \int_a^x K(x, \xi)U(\xi)d\xi$$

are identical, and that consequently $U(x)$ is a solution of the integral equation (1).

§ 4. *Uniqueness of the Solution.*

If $u(x)$ is any solution whatever, finite, and continuous in t except for a finite number of points,

$$v(x) = u(x) - U(x)$$

will be a function of the same kind, and since

$$u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi) d\xi$$

and

$$U(x) = \phi(x) + \int_a^x K(x, \xi) U(\xi) a\xi$$

$v(x)$ will be a solution of the homogeneous integral equation

$$v(x) = \int_a^x K(x, \xi) v(\xi) d\xi.$$

Consequently

$$v(x) = \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') \int_a^{\xi'} \cdots \int_a^{\xi^{(n-1)}} K(\xi^{(n-1)} \xi^{(n)}) d\xi^{(n)} \cdots d\xi, \qquad n = 1, 2, \cdots,$$

and

$$|v(x)| \leqq N \int_a^x |K(x, \xi)| \int_a^\xi |K(\xi, \xi')| \int_a^{\xi'} \cdots \int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)}, \xi^{(n)})| d\xi^{(n)} \cdots d\xi.$$

But by the analysis of § 2, this last inequality leads to

$$|v(x)| \leqq N\{(n+1)M\}^{k-1} H^{n+2-k}, \qquad x \text{ in } t_\delta,$$

and since $H < 1$ and n is arbitrary, the right hand side can be made as small as we please. Hence for any point of t_δ

$$v(x) = 0$$

and $U(x)$ is the only solution of the given type. Thus the theorem of § 1 is proved.

A few examples will show its scope. A kernel $K(x, \xi)$ that satisfies the given conditions is $1/2(x-a)$. Here

$$\int_a^x |K(x, \xi)| d\xi = \tfrac{1}{2},$$

and we know that there is one and only one finite solution of

$$u(x) = \phi(x) + \int_a^x \frac{u(\xi)}{2(x-a)} d\xi$$

through any finite range, a to b, whatever. A more complicated example of the same kind is

$$K(x, \xi) = \begin{cases} \dfrac{1}{5(x-a)}, & \cdots\cdots\cdots, \quad a \leqq x \leqq x_0; \\ \dfrac{1}{5(x_0-a)}, & \cdots, \quad a \leqq \xi \leqq x_0 \\ \dfrac{1}{5(x_0-a)} + \dfrac{1}{2(x-x_0)}, & x_0 < \xi \leqq x \end{cases} \Bigg\} x_0 < x \leqq b.$$

Other examples, of theoretical importance, will appear later. An example where the theorem does not hold we get by putting $K(x, \xi) = 2/(x-a)$. This fails to satisfy (c), for

$$\int_a^x |K(x, \xi)| d\xi = \infty$$

and we can choose no first interval small enough so that

$$\int_a^x |K(x, \xi)| d\xi \leqq H < 1$$

within it. As a matter of fact, in this case there is more than one finite solution. A second case, where (c) is satisfied but where (a) fails to be, is obtained by choosing

$$K(x, \xi) = \begin{cases} \dfrac{1}{5(x-a)}, & a \leqq x \leqq x_0; \\ \dfrac{1}{5(\xi-a)}, & x_0 < x \leqq b. \end{cases}$$

§ 5. *Related Theorems.*

There are other theorems more or less closely related to the one that we have developed, as extensions or as special cases. Practically the same proof holds when we replace (A), (B), (a), (b), (c), by:

(A') $K(x, \xi)$ is continuous in general* in T except in the neighborhood of a system of curves of the kind specified in Condition (A).

(B') $\phi(x)$ is continuous in general in t except in the neighborhood of a finite number of points, and is finite.

* By "$K(x, \xi)$ is continuous in general except in the neighborhood of a system of curves of the kind specified in Condition (A)," we mean that there exists a finite number of curves of the kind specified in Condition (A) such that if every one is enclosed by a strip of width 2ρ, i. e., ρ on each side, where ρ is arbitrarily small, and a region T_ρ formed from T by taking away these strips, then $K(x, \xi)$ will be continuous throughout T_ρ except for a finite number of curves of the kind specified in Condition (A). An analogous meaning is attached to "$\phi(x)$ continuous in general in t except in the neighborhood of a finite number of points." Thus $1/\sin 1/x$ is "continuous in general except in the neighborhood" of the point $x = 0$, Fig. 6.

(a') The integral

$$\int_a^x |K(x, \xi)| d\xi$$

converges in t except in the neighborhood of a finite number of points $\lambda_1, \cdots, \lambda_t$, and remains finite.

(b') There is a finite number of points $\alpha_1, \cdots, \alpha_l$ such that when the region t_δ is constructed there remain in it merely a finite number of points $\alpha_1', \cdots, \alpha_{l'}'$, l' not necessarily independent of δ, such that when ϵ and δ', $(0 < \delta' \leqq \delta)$, are chosen at pleasure and $t_{\delta'}$ constructed by removing from t_δ regions of width $2\delta'$ containing respectively the points α_1, there is a length η for which

$$\int_y^{y+\eta} |K(x, \xi)| d\xi < \epsilon \quad \begin{cases} x \text{ in } t_{\delta'} \\ (x, y) \text{ and } (x, y+\eta) \text{ in } T. \end{cases}$$

(c') t can be divided into k parts, bounded by points $a = a_0, a_1, \cdots, a_{k-1}, b = a_k$ such that

$$\int_{a_i}^x |K(x, \xi)| d\xi \leqq H < 1 \quad \begin{cases} a_i \leqq x \leqq a_{i+1} \\ H \text{ independent of } \delta, \delta'. \end{cases}$$

The conclusion in this case is that there is one and only one solution of the equation continuous in t except in the neighborhood of a finite number of points.

An obvious special case of the theorem of § 1 is where the kernel is continuous throughout T. Another is the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{(x-\xi)^\lambda (x-a)^\mu (\xi - a)^\nu} u(\xi) d\xi,$$

where $K(x, \xi)$ is continuous throughout T, and $\lambda + \mu + \nu < 1$.

The condition (A) can be replaced by conditions on the integral of the kernel; for instance (A) and (b) can together be replaced by the condition which follows:

The integral

$$\int_a^x K(x, \xi) r(\xi) d\xi$$

where $r(x)$ is finite in t and continuous except for a finite number of points, shall converge except at most for a finite number of values of x, and the function of x thus defined shall remain finite; furthermore, it shall be continuous except at most for a finite number of values of x, denoted by $\alpha_1, \cdots, \alpha_l$, which are independent of the choice of $r(x)$.

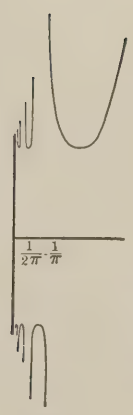

FIG. 6.

A special case of this theorem, elegant on account of the simplicity of the

conditions imposed, has been treated by Dr. W. A. HURWITZ.* The hypotheses for this case were:

(B'') $\phi(x)$ continuous in t;

(a'') $\int_a^t |K(x, \xi)| d\xi$ converges in t;

(b'') $\int_a^x |K(x, \xi)| d\xi$ represents a continuous function in t;

(c'') $|K(x_1, \xi)| \geqq |K(x_2, \xi)|$ when $x_1 > x_2$.

Here (a'') implies (a) of § 1, and (b'') and (c'') imply (b) and (c). For since

$$|K(x_1, \xi)| \geqq |K(x_2, \xi)|, \quad x_1 > x_2,$$

$$\int_{a_i}^x |K(x, \xi)| d\xi \leqq \int_{a_i}^x |K(b, \xi)| d\xi$$

and

$$\int_y^{y+\eta} |K(x, \xi)| d\xi \leqq \int_y^{y+\eta} |K(b, \xi)| d\xi,$$

and it follows from the convergence of

$$\int_a^x |K(b, \xi)| d\xi$$

that both (b) and (c) must hold.

§ 6. *Application to a Non-Integrable Kernel.*

An interesting application of the general theorem is to integral equations where the kernels have such singularities that they are no longer integrable. Let us consider the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi) d\xi, \tag{9}$$

where $K(x, \xi)$ satisfies Condition (A) and is finite, where $\phi(x)$ satisfies Condition (B) and vanishes at $x = a$ in a way to be explained below, and where $f(\xi)$ is continuous in t and unequal to zero except at the point a, where it may vanish in any way. We assume for convenience that

$$\lim_{y=a} \int_y^x \frac{d\xi}{f(\xi)}$$

---

*As a problem in Professor BÔCHER's course in integral equations: Harvard University, 1907–08. Dr. HURWITZ also suggests the theorem: If

(1) $K(x, \xi)$ increases with x for any value of ξ;

(2) $\int_a^x |K(x, \xi)| d\xi$ converges and represents a continuous function

(3) $\phi(x)$ is finite and integrable;

then the equation has one and only one finite and integrable solution.

does not exist; otherwise the case comes under the theorem of § 1. Without loss of generality we may take

$$|K(x, \xi)| \leqq H < 1,$$

putting whatever multiplicative constant is necessary into the denominator with $f(\xi)$.

If we transform this equation by means of the substitution

$$u(x) = r(x)v(x) \tag{10}$$

we obtain

$$r(x)v(x) = \phi(x) + \int_a^x \frac{K(x, \xi)r(\xi)}{f(\xi)} v(\xi)d\xi,$$

so that

$$v(x) = \frac{u(x)}{r(x)}$$

will be a solution of the equation

$$v(x) = \frac{\phi(x)}{r(x)} + \int_a^x K(x, \xi)\frac{r(\xi)}{f(\xi)r(x)} v(\xi)d\xi; \tag{11}$$

and conversely, if $v(x)$ is a solution of this last equation,

$$u(x) = r(x)v(x)$$

will be the solution of the first.

Now let us see if we can so choose $r(x)$ that the kernel in equation (11) shall satisfy the conditions (a), (b), (c) of the fundamental theorem. That is, $r(\xi)$ will have to be such a function that it will not only erase the singularity due to $f(\xi)$, but will also have enough vanishing left so that by means of the integration it will suffice also for the singularities due to $r(x)$. Obviously, no function with a vanishing point of finite order at a, except in the single case where $f(\xi)$ vanishes to the first order (a case that is taken care of also by the succeeding formulas), will suffice, for the order of vanishing of its integral is greater only by one. And therefore we must look for functions that have essential singularities at the point a. A satisfactory choice is

$$r(x) = e^{-\int_x^b \frac{dx}{|f(x)|}}. \tag{12}$$

With the substitution defined by this value of $r(x)$, the equation for $v(x)$ is

$$v(x) = \phi(x)e^{\int_x^b \frac{dx}{|f(x)|}} + \int_a^x K(x, \xi)\frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)} d\xi, \tag{13}$$

and the new kernel is

$$\bar{K}(x, \xi) = K(x, \xi) \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)}. \tag{14}$$

This new kernel $\bar{K}$ satisfies the conditions (a), (b), (c). For the integral

$$\int_a^x |\bar{K}(x, \xi)| d\xi$$

converges, $a \leqq x \leqq b$, except for a finite number of points $\lambda_1, \cdots, \lambda_t$ and is $< M$, $x \neq \lambda_1, \cdots, \lambda_t$.

Since $|K(x, \xi)| \leqq H$ we see, when we have defined the points $\alpha_1 \cdots \alpha_l$, that there is a length η such that

$$\int_y^{y+\eta} |\bar{K}(x, \xi)| d\xi < \epsilon, \qquad (x, y) \text{ and } (x, y+\eta) \text{ in } I_\delta.$$

And finally we can show that

$$\int_a^x |\bar{K}(x, \xi)| d\xi \leqq H < 1, \quad \begin{cases} a \leqq x \leqq b, \\ x \neq \alpha_1, \cdots, \alpha_l. \end{cases}$$

For

$$\int_a^x |K(x, \xi)| \left| \frac{e^{\int_x^b \frac{d\xi}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)} \right| d\xi \leqq H \int_a^x \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{|f(\xi)|} d\xi.$$

In this integral let

$$\zeta = \int_\xi^b \frac{d\xi}{|f(\xi)|} \qquad \text{and} \qquad z = \int_x^b \frac{dx}{|f(x)|};$$

then

$$d\zeta = -\frac{d\xi}{|f(\xi)|}.$$

When $\xi = a$, we have $\zeta = +\infty$, and when $\xi = x$, we have $\zeta = z$, so that

$$\int_a^x \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{|f(\xi)|} d\xi$$

becomes

$$\int_z^\infty e^{z-\zeta} d\zeta = \left[-e^{z-\zeta} \right]_z,$$

which has the value 1. Whence

$$\int_a^x |\bar{K}(x, \xi)| d\xi \leqq H.$$

Hence there is one and only one $v(x)$ which is a solution of (13) finite and continuous in t except for a finite number of points, provided that

$$\phi(x)e^{\int_x^b \frac{dx}{|f(x)|}}$$

remains finite as x approaches a. Hence by (10) there is at least one solution of the given equation (9), provided that

$$\phi(x)e^{\int_x^b \frac{dx}{|f(x)|}}$$

remains finite, namely,

$$u(x) = e^{-\int_x^b \frac{dx}{|f(x)|}} v(x). \tag{15}$$

It vanishes at the point a at least as sharply as

$$\text{const. } e^{-\int_x^b \frac{dx}{|f(x)|}}.$$

The case where $f(\xi)$ is discontinuous at a finite number of points is no more general than this that we have treated, for the discontinuities may be taken up by the $K(x, \xi)$. The case where $f(\xi)$ vanishes again in the interval is more complex. If it vanishes as a result of a finite jump, the problem may be treated by the previous analysis if we replace the equation (13) by

$$v(x) = \bar{\phi}(x) + \int_{a_1}^x K(x, \xi) \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)} v(\xi) d\xi,$$

where a_1 is this second vanishing point of $f(\xi)$ and where

$$\bar{\phi}(x) = \phi(x)e^{\int_x^{a_1} \frac{dx|}{|f(x)|}} + \int_a^{a_1} K(x, \xi) \frac{e^{\int_x^{a_1} \frac{dx}{|f(x)|} - \int_\xi^{a_1} \frac{d\xi}{|f(\xi)|}}}{f(\xi)} v(\xi) d\xi,$$

$v(\xi)$ being known in the interval (a, a_1) as a solution of the equation just studied.

§ 7. *A Generalization of Equation* (9).

An equation closely similar to (9) is

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)g(x)} u(\xi) d\xi, \tag{16}$$

where the continuous functions $f(x)$, $g(x)$, which vanish in t only at $x = a$, are such that the limit

$$\lim_{y=a} \int_y^x \frac{d\xi}{f(\xi)g(\xi)}$$

does not exist. If we multiply through by $g(x)$ we get the equation

$$u(x)g(x) = \phi(x)g(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi) d\xi,$$

and if we put $w(x) = u(x)g(x)$,

$$w(x) = \phi(x)g(x) + \int_a^x \frac{K(x, \xi)}{g(\xi)f(\xi)} w(\xi)d\xi.$$

This equation has a finite solution $w(x)$ continuous in t except for a finite number of points, if

$$\phi(x)g(x)e^{\int_x^b \frac{dx}{|f(x)||g(x)|}}$$

remains finite as x approaches a. Moreover, at the point a, $w(x)$ vanishes as strongly as

$$\text{const}\cdot e^{-\int_x^b \frac{dx}{|f(x)||g(x)|}}.$$

Hence the $u(x)$, which is given by

$$u(x) = \frac{w(x)}{g(x)},$$

vanishes at the point a as strongly as

$$\text{const}\cdot \frac{1}{g(x)} e^{-\int_x^b \frac{dx}{|f(x)||g(x)|}}.$$

§ 8. *A Generalization of Equation* (16).

Finally let us investigate the equation

$$(17) \qquad u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)u(\xi)}{g(x)f(\xi)\prod_{i=1}^p [\xi - \psi_i(x)]^{\lambda_i}} d\xi,$$

where

$$\sum_{i=1}^p \lambda_i = \lambda < 1$$

and the ψ_i are various arbitrary continuous functions.

This equation is equivalent to the equation

$$(18) \qquad w(x) = \theta(x) + \int_a^x \frac{\alpha K(x, \xi)}{h(\xi)\prod_{i=1}^p [\xi - \psi_i(x)]^{\lambda_i}} w(\xi)d\xi,$$

for (17) is reducible to (18) by the substitutions

$$w(x) = u(x)g(x), \qquad \theta(x) = \phi(x)g(x), \qquad h(x) = \alpha f(x)g(x);$$

and conversely, (18) is reducible to (17) by the inverse substitutions. We shall suppose, as we can without loss of generality, that $|K(x, \xi)| \leqq H < 1$. The α we shall leave undetermined for the present.

By the substitution $w(x) = r(x)v(x)$ the equation (18) is equivalent to

$$(19) \qquad v(x) = \frac{\theta(x)}{r(x)} + \int_a^x \frac{r(\xi)}{h(\xi)r(x)} \frac{\alpha K(x, \xi)}{\prod_{i=1}^p [\xi - \psi_i(x)]^{\lambda_i}} v(\xi)d\xi.$$

Let us denote by ν any positive number we please less than $1-\lambda$, and choose $r(x)$ so that

$$\int_a^x \left| \left(\frac{r(\xi)}{h(\xi)r(x)} \right)^{\frac{\nu+\lambda}{\nu}} \right| d\xi$$

shall be convergent. Such a choice is

$$r(x) = e^{-\int_x^b \frac{dx}{|h(x)|^{(\nu+\lambda)/\nu}}}.$$

For if we make the substitutions

$$z = \int_x^b \frac{dx}{|h(x)|^{(\nu+\lambda)/\nu}}, \qquad \zeta = \int_\xi^b \frac{d\xi}{|h(\xi)|^{(\nu+\lambda)/\nu}}, \qquad d\zeta = -\frac{d\xi}{|h(\xi)|^{(\nu+\lambda)/\nu}},$$

the given integral reduces to

$$-(e^z)^{\frac{\nu+\lambda}{\nu}} \int_{\lim_{x=a} z}^{z} (e^{-\zeta})^{\frac{\nu+\lambda}{\nu}} d\zeta = (e^z)^{\frac{\nu+\lambda}{\nu}} \frac{\nu}{\nu+\lambda} \int_{\frac{\nu+\lambda}{\nu} z}^{\frac{\nu+\lambda}{\nu} \lim_{x=a} z} e^{-\eta} d\eta = 1 - e^{\frac{\nu+\lambda}{\nu} z} \left(e^{-\frac{\nu+\lambda}{\nu} \lim_{x=a} z} \right).$$

But $z \leqq \lim_{x=a} z$ when $x \geqq a$, so that the second term of this last expression for the integral is less than or equal to 1, and for the given choice of $r(\xi)$ we have

$$\int_a^x \left| \frac{r(\xi)}{h(\xi)r(x)} \right|^{\frac{\nu+\lambda}{\nu}} d\xi < 1.$$

It must be remembered that $r(x)$, by its definition, depends on α.

Let us now consider the integral of the absolute value of the kernel of equation (19) and show that with this definition of $r(x)$ it can be made < 1 throughout t by properly choosing the constant α.

It is obvious that if a, b, m, n are any numbers whatever not negative,

$$a^m b^n \leqq a^{m+n} + b^{m+n};$$

and that if ν is any positive number

$$a^m b^n \leqq a^{m+n/\nu} + b^{m\nu+n}.$$

Moreover if a, m, $b_1, \cdots, b_p$, $n_1, \cdots, n_p$, are any numbers whatever not negative,

$$a^m b_1^{n_1} b_2^{n_2}, \cdots, b_p^{n_p} \leqq a^{m+\frac{n_1+\cdots+n_p}{\nu}} + b_1^{m\nu+n_1+\cdots+n_p} + b_2^{m\nu+n_1+\cdots+n_p} + \cdots + b_p^{m\nu+n_1+\cdots+n_p}.$$

Now

$$\int_a^x \left| \left(\frac{r(\xi)}{h(\xi)r(x)} \right) \frac{\alpha K(x,\xi)}{\prod_{i=1}^{p} \{\xi - \psi_i(x)\}^{\lambda_i}} \right| d\xi \leqq \alpha H \int_a^x \left| \frac{r(\xi)}{h(\xi)r(x)} \right| \prod_{i=1}^{p} \left| \frac{1}{[\xi - \psi_i(x)]^{\lambda_i}} \right| d\xi$$

$$\leqq \alpha H \int_a^x \left| \frac{r(\xi)}{h(\xi)r(x)} \right| \prod_{i=1}^{p} \left| \frac{1}{\xi - \psi_i(x)} \right|^{\lambda_i} d\xi.$$

The integrand now, however, is a product of which the factors are all positive. We have then, according to what we have just seen (putting $m = 1$),

$$\left|\frac{r(\xi)}{h(\xi)r(x)}\right| \prod_{i=1}^{p} \left|\frac{1}{\xi - \psi_i(x)}\right|^{\lambda_i} \leqq \left|\frac{r(\xi)}{h(\xi)r(x)}\right|^{1+\lambda/\nu} + \sum_{i=1}^{p} \left|\frac{1}{\xi - \psi_i(x)}\right|^{\nu+\lambda},$$

and the given integral is less than or equal to

$$\alpha H \int_a^x \left|\frac{r(\xi)}{h(\xi)r(x)}\right|^{\frac{\nu+\lambda}{\nu}} d\xi + \sum_{i=1}^{p} \int_a^x \left|\frac{1}{\xi - \psi_i(x)}\right|^{\nu+\lambda} d\xi.$$

Since $\nu + \lambda < 1$ we can choose

$$P \geqq \sum_{i=1}^{p} \int_a^x \left|\frac{1}{\xi - \psi_i(x)}\right|^{\nu+\lambda} d\xi, \qquad x \text{ in } t,$$

so that the integral of the absolute value of the kernel is

$$\leqq \alpha H(1 + P), \qquad x \text{ in } t,$$

the α being involved only explicitly; and by choosing

$$\alpha = \frac{1}{1 + P}$$

this becomes $\leqq H < 1$. There is then under these conditions a solution of equation (19) if $\theta(x)/r(x)$ remains finite.

In regard to (17) we know that there is one solution continuous except at a finite number of points provided that

$$\frac{\phi(x)g(x)}{r(x)}$$

remains finite as x approaches a.

The number of solutions of such equations as these — (9), (16), (17) — we have not investigated. Conceivably there may be, often in fact there are, other finite solutions.

HARVARD UNIVERSITY,
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EIN SEITENSTÜCK ZUR MOEBIUS'SCHEN GEOMETRIE DER KREISVERWANDTSCHAFTEN*

VON

H. BECK

Öfter wiederholt sich in der Mathematik, und insbesondere in der Geometrie, die Erscheinung, dass Gruppen von Lehrsätzen, die durch gleiche logische Prozesse verbunden sind, in sonst verschiedenen Theorieen auftreten. Es entstehen dann sogenannte Übertragungsprinzipe, für die der von S. LIE gefundene Zusammenhang zwischen Liniengeometrie und Kugelgeometrie das wohl berühmteste Beispiel abgibt. Wir erinnern hier an die Tatsache, dass die von MOEBIUS begründete Art von Kugelgeometrie nicht dem Gedankenkreise der Euklidischen Geometrie (aus dem MOEBIUS sie abgeleitet hat) eigentümlich ist, sondern dass sie in gleicher Weise in den verschiedenen Arten von Nicht-Euklidischer Geometrie auftritt. Für Erscheinungen dieser Art hat man in der Sprache der Gruppentheorie einen präzisen Ausdruck gefunden.

In der Kreisgeometrie von MOEBIUS zum Beispiel wird der Begriff des Kreises derart umgrenzt, dass er auch die geraden Linien umfasst, und es werden dann, bei konsequenter Ausgestaltung der Systematik, ausschliesslich solche Eigenschaften dieser "Kreise" betrachtet, die nicht zerstört werden können durch solche Punkttransformationen, die Kreise in Kreise überführen. Diese Transformationen bilden, wie MOEBIUS selbst schon gefunden hat, eine — nach unserer heutigen Ausdrucksweise — gemischte Gruppe (G_6, H_6), deren allgemeine Transformation von sechs Parametern abhängt.

Die kontinuierliche Gruppe G_6 enthält nun dreigliedrige Untergruppen, die nach LIE's Terminologie *ähnlich* sind zu den ebenfalls dreigliedrigen Bewegungsgruppen der Euklidischen und — je nach Wahl der Raumelemente (Punkt oder geeignetes Punktepaar) — der sphärischen oder der elliptischen, sowie der pseudosphärischen oder der hyperbolischen Geometrie. Danach erscheint es dann als vollkommen selbstverständlich, dass in jeder dieser verschiedenen Arten von Maassgeometrie, also in der Euklidischen Geometrie und in den $2 \cdot 2$ Hauptarten der Nicht-Euklidischen Geometrie an bestimmter Stelle die Lehrsätze von MOEBIUS' Kreisgeometrie auftreten müssen. Aber diese Lehrsätze haben, wiewohl sie unter einander die gleiche Verkettung aufweisen, doch nicht die gleiche

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Bedeutung; sie ordnen sich in verschiedene Gedankensysteme ein; sie können zwar oftmals mit ähnlichen, nicht aber mit den gleichen Worten beschrieben werden, wenigstens nicht, wenn man auf die ursprünglichen Definitionen zurückgeht: Ein *Kreis* in der sphärischen Geometrie zum Beispiel ist nicht Dasselbe, wie die mit dem gleichen Namen bezeichnete Figur der Euklidischen Geometrie. Eben darin ist das Interesse begründet, das solche Wiederholungen derselben Schlussketten haben können, trotzdem sie Wiederholungen sind.

In andern Fällen ist eine so weitgehende logische Aequivalenz nicht vorhanden. Die projektive Geometrie auf einer Kugel zum Beispiel ist sehr verschieden von der Geometrie auf einem Hyperboloïd. Erst, wenn man den elementaren Standpunkt der naïven Betrachtung mit einem andern vertauscht, offenbart sich auch hier eine weitgehende Übereinstimmung, zu deren Verständnis wiederum die Gruppentheorie den Schlüssel liefert. Denkt man sich nämlich die Gruppen der automorphen Kollineationen von Kugel und Hyperboloïd ins komplexe Gebiet erweitert (analytisch fortgesetzt), so sind diese *erweiterten* Gruppen wiederum zu einander *ähnlich* (nämlich zu einander komplex-projektiv).

Daraus ergibt sich, dass alle Lehrsätze der projektiven Geometrie, in denen reelle Figuren keine Sonderstellung einnehmen, von der Kugel auf das Hyperboloïd übertragen werden können, und umgekehrt. Ebenso verhält es sich mit LIE's Kugelgeometrie und der PLÜCKERschen Liniengeometrie. Man hat in solchen Fällen, wie erklärlich, bisher vielleicht zu viel auf das Gemeinsame der verschiedenen Disziplinen, als auf die doch auch vorhandenen und tiefgreifenden Unterschiede geachtet.

In der vorliegenden Untersuchung sollen nun die bekannten Beispiele solchen Zusammenhanges zwischen Theorieen verschiedenen Ursprungs durch ein weiteres vermehrt werden. Die vorzuführende Art von Konstruktionen verhält sich zu den Konstruktionen von MOEBIUS' Kreisgeometrie, die ja von der projektiven Geometrie auf einer Kugelfläche nicht wesentlich verschieden ist, ganz so, wie die Geometrie auf einem Hyperboloïd zur Geometrie auf der Kugel.

Das zugehörige Übertragungsprinzip gehört also zu der zweiten der beiden eben unterschiedenen Arten; erst im komplexen Gebiete fällt unsere Geometrie in logischer Hinsicht zusammen mit der von MOEBIUS; im Reellen sind sie beide sehr verschieden, unbeschadet des Umstandes, dass die tiefer liegenden Analogieen sich natürlich auch hier schon fühlbar machen.

Die *Geometrie der Zykeln*, von der wir reden wollen, entsteht aus der ebenen hyperbolischen Geometrie, wenn man als Raumelement den *Pfeil* betrachtet, nämlich ein wohlgeordnetes Paar von Punkten auf dem gewöhnlich als Kegelschnitt aufgefassten Ort der unendlich fernen Punkte. Diese Pfeile sind das Analogon zu den *Punkten* der MOEBIUSschen Geometrie. Sie werden den Transformationen einer gewissen Gruppe unterworfen, die zu der zuvor genannten MOEBIUS'schen Gruppe ein Seitenstück ist. Die den Kreisen von

MOEBIUS' Geometrie analogen Figuren endlich sind dann eben die weiterhin zu erklärenden Zykeln.

Da es uns nicht nur darauf ankommt, das Gemeinsame unserer Geometrie der Zykeln mit MOEBIUS' Kreisgeometrie hervortreten zu lassen, sondern uns auch daran gelegen ist, unsere Disziplin selbständig zu entwickeln und das Unterscheidende zu seinem Rechte kommen zu lassen, so stellen wir uns von vorneherein auf den Standpunkt der hyperbolischen Geometrie *des reellen Gebietes*. Alle vorkommenden Grössen und Figuren setzen wir als *reell* voraus.

§ 1. *Der Pfeil.*

Das absolute Polarsystem wählen wir in der Form:

$$x_1 y_1 - x_2 y_2 - x_3 y_3 = 0, \tag{1}$$

oder in Linienkoordinaten:

$$\mathfrak{x}_1 \mathfrak{y}_1 - \mathfrak{x}_2 \mathfrak{y}_2 - \mathfrak{x}_3 \mathfrak{y}_3 = 0. \tag{2}$$

Dann lässt sich das binäre Punktgebiet des absoluten Kegelschnitts auf folgende Weise darstellen:

$$x_1 : x_2 : x_3 = \lambda_1^2 + \lambda_2^2 : \lambda_2^2 - \lambda_1^2 : 2\lambda_1\lambda_2. \tag{3}$$

Jedem Werte von $\lambda_1 : \lambda_2$, mit Einschluss des uneigentlichen Wertes ∞, entspricht dann gerade ein unendlich ferner Punkt, und umgekehrt. Als positiv soll die Umlaufsrichtung auf dem absoluten Kegelschnitt bezeichnet werden, in der $\lambda_1 : \lambda_2$ wächst.

Als Raumelement wählen wir den *Pfeil.** Darunter soll verstanden werden die Figur zweier in eine bestimmte Reihenfolge gesetzter Punkte des absoluten Kegelschnitts. Ein Pfeil hat demnach einen *Anfangspunkt* und einen *Endpunkt*. Liegen beide getrennt, so heisst der Pfeil *eigentlich*. Sein *Träger* ist eine zugängliche Gerade. Sie trägt auch den *entgegengesetzten* Pfeil, d. i. den Pfeil, dessen Anfangspunkt und Endpunkt zusammenfallen mit Endpunkt und Anfangspunkt des ersten.

Fällt der Anfangspunkt eines Pfeils mit seinem Endpunkt zusammen, so reden wir von einem *Punktpfeil*. Ein Punktpfeil ist zu sich selbst entgegengesetzt. Die Gesamtheit der Punktpfeile fungiert mithin als Verzweigungsmannigfaltigkeit. Träger eines Punktpfeils ist eine Minimalgerade (Tangente des absoluten Kegelschnitts).

Eine unzugängliche Gerade trägt keine Pfeile.

Ein Pfeil kann durch ein Paar geordneter Zahlen ($\lambda = \lambda_1 : \lambda_2$, $\mu = \mu_1 : \mu_2$) *dargestellt werden*, wobei (vgl. Gl. 3) $\lambda_1 : \lambda_2$ seinen Anfangspunkt, $\mu_1 : \mu_2$ seinen

* Vgl. analoge Begriffsbildungen bei Hrn. E. STUDY: *Beiträge zur Nicht-Euklidischen Geometrie*; American Journal of Mathematics, vol. 29 (1907); *Über Nicht-Euklidische und Liniengeometrie*, Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 11 (1902), Bd. 15 (1906). Diesen Arbeiten ist die Terminologie grösstenteils nachgebildet.

Endpunkt bedeuten soll. Das System $(\mu_1 : \mu_2, \lambda_1 : \lambda_2)$ gibt dann den entgegengesetzten Pfeil. Verschwindet der Ausdruck

$$(\lambda\mu) = \lambda_1 \mu_2 - \lambda_2 \mu_1, \tag{4}$$

so liegt ein Punktpfeil vor. Die Zahlen $\lambda_1 : \lambda_2$, $\mu_1 : \mu_2$ sollen *Pfeilparameter* genannt werden.

Die Gesamtheit der ∞^2 eigentlichen Pfeile wird durch die ∞^1 Punktpfeile zu einem abgeschlossenen Kontinuum ergänzt.

Zwei Pfeile mit zusammenfallenden Anfangspunkten werden *α-parallel* genannt. Entsprechend heissen Pfeile *ω-parallel*, wenn sie gemeinsame Endpunkte besitzen.

Sind zwei Pfeile zu einem dritten α-parallel (ω-parallel), so sind sie es auch zu einander. Zwei Pfeile, die zugleich α-parallel und ω-parallel sind, fallen zusammen. Zwei Punktpfeile können parallel nur sein, wenn sie zusammen fallen. Durch einen Punkt, der dem absoluten Kegelschnitt nicht angehört, gibt es zu einem Pfeil stets gerade einen α-parallelen und einen ω-parallelen Pfeil.

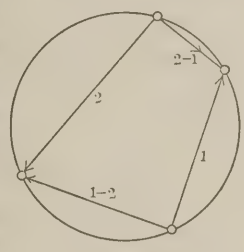

Fig. 1.

Es gibt stets einen einzigen Pfeil, der zu einem ersten Pfeil α-parallel und zu einem zweiten Pfeil ω-parallel ist. Er soll *Verbindungspfeil* (1–2) genannt werden (nicht zu verwechseln mit dem Verbindungspfeil (2–1)). Vgl. Fig. 1.

Die Konstruktion des Verbindens eines ersten Pfeils mit einem zweiten ist stets, *ohne irgend welche Ausnahmen* ausführbar.

Als Koordinaten der geraden Linie $\mathfrak{x}$, die den Pfeil (λ, μ) trägt, findet man:

$$\mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3 = -(\lambda_2 \mu_2 + \lambda_1 \mu_1) : \lambda_2 \mu_2 - \lambda_1 \mu_1 : \lambda_1 \mu_2 + \lambda_2 \mu_1 .$$

Setzt man noch:

$$\mathfrak{x}_0 = (\lambda\mu) = \lambda_1 \mu_2 - \lambda_2 \mu_1, \tag{5}$$

so dürfen die vier Verhältnisgrössen

$$\mathfrak{x}_0 : \mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3 = \lambda_1 \mu_2 - \lambda_2 \mu_1 : -(\lambda_2 \mu_2 + \lambda_1 \mu_1) : \lambda_2 \mu_2 - \lambda_1 \mu_1 : \lambda_1 \mu_2 + \lambda_2 \mu_1 \tag{6}$$

als *Pfeilkoordinaten* verwandt werden. Zwischen ihnen besteht dann die Relation:

$$\mathfrak{x}_0^2 + \mathfrak{x}_1^2 - \mathfrak{x}_2^2 - \mathfrak{x}_3^2 = 0. \tag{7}$$

Die Systeme $\mathfrak{x}_0 : \mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3$ und $-\mathfrak{x}_0 : \mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3$ ergeben entgegengesetzte Pfeile. Für Punktpfeile hat man: $\mathfrak{x}_0 = 0$.

Durch die *Koordinaten* eines Pfeils sind auch wieder dessen *Parameter* vollkommen bestimmt:

$$\begin{aligned} \lambda_1 : \lambda_2 &= \mathfrak{x}_3 + \mathfrak{x}_0 : \mathfrak{x}_2 - \mathfrak{x}_1 = \mathfrak{x}_1 + \mathfrak{x}_2 : -\mathfrak{x}_3 + \mathfrak{x}_0 \\ \mu_1 : \mu_2 &= \mathfrak{x}_3 - \mathfrak{x}_0 : \mathfrak{x}_2 - \mathfrak{x}_1 = \mathfrak{x}_1 + \mathfrak{x}_2 : -\mathfrak{x}_3 - \mathfrak{x}_0 \end{aligned} \tag{8}$$

Die Gleichung (7) zeigt:

Das Pfeilkontinuum lässt sich ausnahmslos eindeutig umkehrbar abbilden auf die Gesamtheit der Punkte einer singularitätenfreien geradlinigen Fläche zweiter Ordnung.

Diese Fläche, die als einschaliges Hyperboloïd angesehen werden darf, soll kurz *Parasphäre* genannt werden. Entgegengesetzten Pfeilen können zwei Punkte der Parasphäre zugeordnet werden, deren Verbindungsgerade den Mittelpunkt der Fläche trifft. Die Punktpfeile werden dann abgebildet auf den Schnitt der Parasphäre mit der uneigentlichen Ebene des R_3.

Die Pfeilparameter können mit den passend gewählten Parametern der Erzeugenden der Fläche identifiziert werden, sodass sich z. B. entsprechen:

Anfangspunkt eines Pfeils,	Link-seitige Erzeugende;
Endpunkt eines Pfeils,	Recht-seitige Erzeugende;
α-parallele (ω-parallele) Pfeile,	Punkte derselben link-(recht-)seitigen Erzeugenden.

§ 2. *Die parazyklische Gruppe.*

Zu dem erklärten Pfeilkontinuum gehört eine sechsgliedrige Transformationsgruppe, deren reelle Transformationen die Pfeile in der Weise unter einander vertauschen, dass die Anfangspunkte für sich und die Endpunkte für sich *projectiv* unter einander vertauscht werden:

$$(9)\quad \begin{aligned} \lambda_1' &= (\alpha_0+\alpha_3)\lambda_1 + (\alpha_2-\alpha_1)\lambda_2, & \mu_1' &= (\beta_0+\beta_3)\mu_1 + (\beta_2-\beta_1)\mu_2, \\ \lambda_2' &= (\alpha_2+\alpha_1)\lambda_1 + (\alpha_0-\alpha_3)\lambda_2. & \mu_2' &= (\beta_2+\beta_1)\mu_1 + (\beta_0-\beta_3)\mu_2. \\ N(\alpha) &= \alpha_0^2+\alpha_1^2-\alpha_2^2-\alpha_3^2 \neq 0, & N(\beta) &= \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 \neq 0. \end{aligned}$$

Wir nennen sie die *parazyklische* Gruppe und bezeichnen sie kurz als G_6. Die genannten reellen Transformationen umfassen vier getrennte Transformationsscharen, die durch folgende Ungleichungen charakterisiert sind:

$$N(\alpha)>0,\ N(\beta)>0;\ N(\alpha)>0,\ N(\beta)<0;$$
$$N(\alpha)<0,\ N(\beta)>0;\ N(\alpha)<0,\ N(\beta)<0.$$

Die erste Schar bildet mit jeder der drei übrigen zusammen eine Gruppe; die von den Transformationen der ersten und vierten Schar $[N(\alpha)\cdot N(\beta)>0]$ gebildete Gruppe werden wir die *engere* Gruppe nennen und mit G_6' bezeichnen.

Eine parazyklische Transformation verwandelt Punktpfeile im Allgemeinen nicht wieder in solche; sie zerstört die Eigenschaft zweier Pfeile, entgegengesetzt zu sein. Dagegen bleibt α-Parallelismus (und ω-Parallelismus) erhalten, denn es gelten die Gleichungen:

$$(\lambda' l') = N(\alpha)\cdot(\lambda l), \qquad (\mu' m') = N(\beta)\cdot(\mu m),$$

wo $(l_1 : l_2,\ m_1 : m_2)$ einen zweiten Pfeil bedeutet.

Die Gruppe G_6 kann leicht in Pfeil*koordinaten* dargestellt werden; man kann sie folgendermassen schreiben:

$$\mathfrak{x}'_i = c_{i0}\mathfrak{x}_0 + c_{i1}\mathfrak{x}_1 + c_{i2}\mathfrak{x}_2 + c_{i3}\mathfrak{x}_3. \qquad (i = 0, 1, 2, 3) \tag{10}$$

Darin haben die c_{ik} nachstehende Werte:

$$\begin{aligned} c_{00} &= \alpha_0\beta_0 + \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3, \\ c_{11} &= \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3, \\ c_{22} &= \alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 + \alpha_3\beta_3, \\ c_{33} &= \alpha_0\beta_0 - \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3, \end{aligned} \tag{10}$$

$$\begin{aligned} c_{01} &= -\alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2, & c_{10} &= \alpha_0\beta_1 - \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2, \\ c_{02} &= -\alpha_0\beta_2 + \alpha_2\beta_0 - \alpha_3\beta_1 + \alpha_1\beta_3, & c_{20} &= -\alpha_0\beta_2 + \alpha_2\beta_0 + \alpha_3\beta_1 - \alpha_1\beta_3, \\ c_{03} &= -\alpha_0\beta_3 + \alpha_3\beta_0 - \alpha_1\beta_2 + \alpha_2\beta_1, & c_{30} &= -\alpha_0\beta_3 + \alpha_3\beta_0 + \alpha_1\beta_2 - \alpha_2\beta_1. \end{aligned}$$

$$\begin{aligned} c_{23} &= \alpha_0\beta_1 + \alpha_1\beta_0 - \alpha_2\beta_3 - \alpha_3\beta_2, & c_{32} &= -\alpha_0\beta_1 - \alpha_1\beta_0 - \alpha_2\beta_3 - \alpha_3\beta_2, \\ c_{31} &= -\alpha_0\beta_2 - \alpha_2\beta_0 - \alpha_3\beta_1 - \alpha_1\beta_3, & c_{13} &= -\alpha_0\beta_2 - \alpha_2\beta_0 + \alpha_3\beta_1 + \alpha_1\beta_3, \\ c_{12} &= \alpha_0\beta_3 + \alpha_3\beta_0 + \alpha_1\beta_2 + \alpha_2\beta_1, & c_{21} &= \alpha_0\beta_3 + \alpha_3\beta_0 - \alpha_1\beta_2 - \alpha_2\beta_1. \end{aligned}$$

Es ist dann:

$$\mathfrak{x}'_0\mathfrak{y}'_0 + \mathfrak{x}'_1\mathfrak{y}'_1 - \mathfrak{x}'_2\mathfrak{y}'_2 - \mathfrak{x}'_3\mathfrak{y}'_3 = N(\alpha)\cdot N(\beta)\cdot(\mathfrak{x}_0\mathfrak{y}_0 + \mathfrak{x}_1\mathfrak{y}_1 - \mathfrak{x}_2\mathfrak{y}_2 - \mathfrak{x}_3\mathfrak{y}_3),$$

wo die $\mathfrak{y}$ die Koordinaten des Pfeils mit den Parametern $(l_1 : l_2,\ m_1 : m_2)$ sind. Die geometrische Bedeutung des Verschwindens der so gefundenen Simultaninvariante ergibt sich sofort aus der Gleichung:

$$\mathfrak{x}_0\mathfrak{y}_0 + \mathfrak{x}_1\mathfrak{y}_1 - \mathfrak{x}_2\mathfrak{y}_2 - \mathfrak{x}_3\mathfrak{y}_3 = 2(\lambda l)(\mu m).$$

Die parazyklische Gruppe lässt sich aber eleganter darstellen. Dabei wird sich zeigen, dass die den Gleichungen (10) anhaftenden Unsymmetrieen nicht vermieden werden können.

Wir bedienen uns eines Systems komplexer Zahlen in vier Einheiten mit folgender Multiplikationstafel:*

$$\begin{array}{cccc} e_0 & e_1 & e_2 & e_3 \\ e_1 & -e_0 & e_3 & -e_2 \\ e_2 & -e_3 & e_0 & -e_1 \\ e_3 & e_2 & e_1 & e_0 \end{array} \tag{11}$$

* Das System wird aus dem der Hamiltonschen Quaternionen durch eine lineare nicht reelle Transformation erhalten. Vgl. Enzyklopädie der mathematischen Wissenschaften, I A 4, p. 167, *Zweite Gestalt der Quaternionen.* Das dort unter VI^b angegebene System bedarf nur ganz geringfügiger Abänderung, die hier aber zweckmässig erscheint, um in die Form (11) überzugehen.

Es sei nun:

$$\dot{\alpha} = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \qquad \dot{\beta} = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3.$$

Weiter bedeute $\underline{\dot{\alpha}} = \alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3$ die zu $\dot{\alpha}$ *konjugierte* Quaternion. Dann ist, da e_0 der positiven reellen Einheit gleich gesetzt werden darf:

$$\dot{\alpha} \cdot \underline{\dot{\alpha}} = \underline{\dot{\alpha}} \cdot \dot{\alpha} = N(\alpha) \cdot e_0 = N(\alpha).$$

Ist $\dot{\gamma} = \dot{\alpha} \cdot \dot{\beta}$, so gelten auch die folgenden beiden Gleichungen:

$$\underline{\dot{\gamma}} = \underline{\dot{\beta}} \cdot \underline{\dot{\alpha}}, \qquad N(\gamma) = N(\alpha) \cdot N(\beta).$$

Man darf durch $\dot{\alpha}$ dividieren, solange $N(\alpha) \neq 0$.

Setzen wir noch:

$$\dot{\mathfrak{x}} = \mathfrak{x}_0 e_0 + \mathfrak{x}_1 e_1 - \mathfrak{x}_2 e_2 - \mathfrak{x}_3 e_3,$$

so stellt sich die parazyklische Gruppe durch die eine Formel dar:

$$\dot{\mathfrak{x}}' = \underline{\dot{\alpha}} \cdot \dot{\mathfrak{x}} \cdot \dot{\beta}. \tag{12}$$

Die beiden Grössensysteme α_i und β_i $(i = 0, 1, 2, 3)$ sollen *Parameter* der parazyklischen Transformation genannt werden. Jedes dieser Systeme darf, wie die Gleichungen (9) zeigen, noch mit einem beliebigen (nicht verschwindenden) Zahlenfaktor multipliziert werden, ohne dass die dargestellte Transformation sich ändert.

Es ist nun leicht, mehrere parazyklische Transformationen hintereinander auszuführen. Bedeutet nämlich $S_{\alpha\beta}$ die Transformation mit den Parametern α_i, β_i $(i = 0, 1, 2, 3)$, und ist (in WIENERscher Bezeichnung)

$$\mathfrak{x}\{S_{\alpha\beta}\}\mathfrak{x}'\{S_{\alpha'\beta'}\}\mathfrak{x}'' = \mathfrak{x}\{S_{\alpha''\beta''}\}\mathfrak{x}'',$$

so erhält man:

$$\dot{\mathfrak{x}}'' = \underline{\dot{\alpha}}' \cdot \dot{\mathfrak{x}}' \cdot \dot{\beta}' = \underline{\dot{\alpha}}' \cdot \underline{\dot{\alpha}} \cdot \dot{\mathfrak{x}} \cdot \dot{\beta} \cdot \dot{\beta}' = \underline{\dot{\alpha}}'' \cdot \dot{\mathfrak{x}} \cdot \dot{\beta}'',$$

wenn die neuen Quaternionen $\dot{\alpha}''$ und $\dot{\beta}''$ folgende Bedeutung haben:

$$\dot{\alpha}'' = \dot{\alpha} \cdot \dot{\alpha}', \qquad \dot{\beta}'' = \dot{\beta} \cdot \dot{\beta}'. \tag{13}$$

Die Gleichung:

$$N(\alpha'') \cdot N(\beta'') = N(\alpha) \cdot N(\beta) \cdot N(\alpha') \cdot N(\beta')$$

zeigt, wie die engere Gruppe G_6' zu Stande kommt.

Es gibt stets eine und nur eine parazyklische Transformation, die die Figur dreier wohlgeordneter Pfeile, von denen keine zwei parallel sind, in eine gegebene ebensolche Figur überführt.

§ 3. *Der Zykel.*

Die einfachsten Pfeilmannigfaltigkeiten sind die, deren Gleichungen (in Pfeil*parametern* dargestellt) linear oder bilinear sind.

Es gibt zwei Arten von *linearen* Pfeilmannigfaltigkeiten:

$$(p\lambda) = 0. \qquad (q\mu) = 0.$$

Die Gleichung $(p\lambda) = 0$ stellt ein *Büschel α-paralleler Pfeile* dar, oder wie wir kurz sagen wollen, ein *α-Büschel.* Entsprechend bedeutet die zweite Gleichung ein *ω-Büschel.* Beide Gebilde sind Träger binärer Gebiete.

Jeder Pfeil bestimmt ein einziges α-Büschel und ein einziges ω-Büschel. Ein α-Büschel und ein ω-Büschel durchdringen sich stets in einem einzigen Pfeil. Zwei getrennte α-Büschel (ω-Büschel) haben niemals einen Pfeil gemeinsam.

Die ∞^1 α-Büschel bilden eine einzige Klasse bereits gegenüber den Transformationen einer jeden der vier Scharen, ebenso die ∞^1 ω-Büschel. Ein α-Büschel kann aber durch eine Transformation von G_6 niemals in ein ω-Büschel übergeführt werden (doch vgl. § 4).

Verbindet man einen Pfeil mit sämtlichen Pfeilen eines α-Büschels (ω-Büschels), so entsteht ein α-Büschel (ein einziger Pfeil).

Verbindet man sämtliche Pfeile eines α-Büschels (ω-Büschels) mit einem Pfeil, so entsteht ein einziger Pfeil (ein ω-Büschel). —

Jede *bilineare* Pfeilmannigfaltigkeit ist in der folgenden Form enthalten:

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 = 0. \tag{14}$$

Alle durch solche Gleichungen dargestellten Gebilde sollen *Zykeln* genannt werden.

Der Zykel heisst *reduzibel*, sobald seine *Diskriminante*

$$\tfrac{1}{2}D_{aa} = a_{11}a_{22} - a_{12}a_{21} \tag{15}$$

verschwindet. Er besteht dann aus einem α-Büschel und einem ω-Büschel. Er besitzt einen *Mittelpfeil*, den Durchdringungspfeil der beiden Büschel.

Durch irgend drei getrennte Pfeile, von denen gerade zwei parallel sind, geht stets ein und nur ein reduzibler Zykel. Die ∞^2 reduziblen Zykeln bilden eine einzige Klasse gegenüber den Transformationen einer jeden der vier Scharen von G_6.

Zwei getrennte reduzible Zykeln durchdringen sich in zwei getrennten Pfeilen, oder sie haben ein ganzes α-Büschel (ω-Büschel) gemeinsam (*orthogonale* Zykeln, vgl. später § 5).

Liegt ein α-Büschel (ω-Büschel) nicht ganz in einem reduziblen Zykel, so haben beide Figuren stets einen einzigen Pfeil gemeinsam. —

Durch drei getrennte Pfeile, von denen keine zwei α-parallel oder ω-parallel sind, geht stets ein einziger irreduzibler Zykel.

Um mit diesem Begriff gleich eine Anschauung zu verbinden, betrachten wir, indem wir die Ergebnisse späterer Entwicklungen (§ 7) antizipieren, die verschiedenen Gestalten der irreduziblen Zykeln.

Zunächst ist zu nennen der absolute Kegelschnitt (als Ort aller Punktpfeile). Ferner jeder zugängliche Punkt (als Ort aller Pfeile, deren Träger durch ihn hindurchgehen) und jede zugängliche Gerade (als Ort der Pfeile, von deren Trägern sie senkrecht geschnitten wird).

Die übrigen irreduziblen Zykeln lassen sich in der Weise zu Paaren anordnen, dass von den beiden Zykeln eines Paares jeder aus dem anderen hervorgeht, wenn man jeden Zykelpfeil in den entgegengesetzten Pfeil verwandelt. Zu den beiden Zykeln eines Paares gehört demnach ein und dieselbe Geradenmannigfaltigkeit, und diese umhüllt eine irreduzible Kurve 2. Klasse (und 2. Ordnung), die den absoluten Kegelschnitt doppelt berührt (wobei noch die beiden Berührungspunkte zusammenfallen können). Dadurch, dass man diese Kurven (oder ihre im Unendlichen zusammenhängenden Zweige) *orientiert*, kann man die beiden Zykeln des zugehörigen Paares anschaulich unterscheiden.

Doch können von den angeführten Kurven nur diejenigen Zykeln liefern, die überhaupt zugängliche Tangenten besitzen. Indem wir für diese Kurven die in der hyperbolischen Geometrie üblichen Benennungen verwenden, erhalten wir folgende Zusammenstellung:

1. Zugängliche Kreise mit bestimmtem Umlaufssinn (Fig. 2).

2. Zugängliche Abstandskurven (LIEBMANN), *deren Zweige in entgegengesetztem Sinne durchlaufen werden* (Fig. 3).

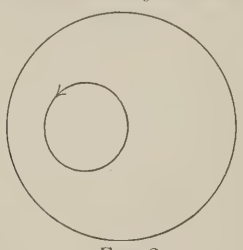

FIG. 2.

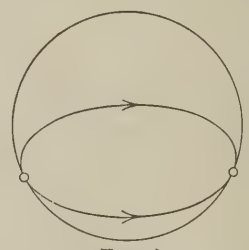

FIG. 3.

3. Zugängliche Grenzkurven, deren zugänglicher Zug in bestimmter Richtung durchlaufen wird (Fig. 4).

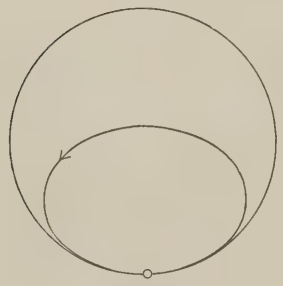

FIG. 4.

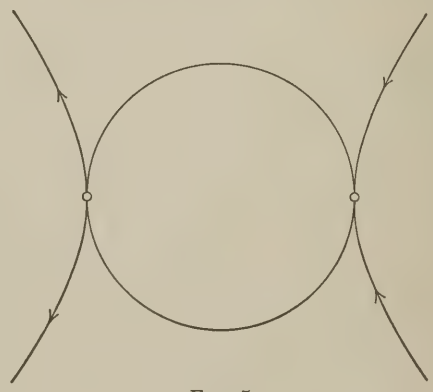

FIG. 5.

4. Unzugängliche Kurven zweiter Ordnung (und zweiter Klasse), die mit dem absoluten Kegelschnitt in *reeller* Doppelberührung liegen, und *deren Tangenten ins zugängliche Gebiet eindringen.* Die Orientierung ist aus der Figur 5 ersichtlich. —

Üben wir auf den durch die Gleichung (14) dargestellten Zykel die parazyklische Transformation $S_{\alpha\beta}$ aus, so reproduziert sich die Diskriminante bis auf den Faktor $N(\alpha)\cdot N(\beta)$:

Die ∞^3 irreduziblen Zykeln bilden eine einzige Klasse gegenüber den Transformationen der Gruppe G_6. Lässt man dagegen nur Transformationen von G_6' zu, so erweist sich das Vorzeichen der Diskriminante als wesentlich, und man hat *zwei* Klassen. Von den *eigentlichen* irreduziblen Zykeln ($D > 0$) sind dann zu unterscheiden die *uneigentlichen* ($D < 0$). Zu den uneigentlichen irreduziblen Zykeln gehören ausser den Geradenzykeln (§ 6) nur noch jene, bei denen die von den Pfeilen umhüllte Kurve zweiter Ordnung und Klasse im unzugänglichen Gebiet liegt (Fig. 5).

Ein α-Büschel (ω-Büschel) und ein irreduzibler Zykel haben stets einen einzigen gemeinsamen Pfeil, und umgekehrt:

*Eine Pfeilmannigfaltigkeit, die mit jedem α-Büschel und jedem ω-Büschel gerade einen Pfeil gemeinsam hat, ist ein irreduzibler Zykel.**

Zwei getrennte irreduzible Zykeln haben zwei, einen oder keinen gemeinsamen Pfeil. Ein reduzibler und ein irreduzibler Zykel durchdringen sich in einem oder in zwei Pfeilen.

Die Gesamtheit der ∞^3 Zykeln lässt sich durchaus eindeutig umkehrbar abbilden auf die Gesamtheit der Ebenen des R_3.

Den reduziblen Zykeln entsprechen dabei die Ebenen einer singularitätenfreien Fläche zweiter Klasse. Diese Fläche kann mit der Parasphäre identifiziert werden. Ferner lässt sich die Abbildung so einrichten, dass sich entspricht:

vereinigte Lage von Pfeil und Zykel,	vereinigte Lage von Bildpunkt auf der Parasphäre mit Bildebene.

Um diesen Gedanken auszuführen, beachten wir, dass sich die Gleichung (14) des Zykels in folgender Gestalt schreiben lässt:

$$[A\underset{\sim}{r}] = A_0\underset{\sim}{r}_0 + A_1\underset{\sim}{r}_1 + A_2\underset{\sim}{r}_2 + A_3\underset{\sim}{r}_3 = 0. \tag{16}$$

* Dieser Satz bedarf aber, um nicht falsch verstanden zu werden, einer Erläuterung. In der Tat hat ja jedes orientierte zugängliche Oval, als Ort von Pfeilen aufgefasst, mit jedem α-Büschel (ω-Büschel) gerade einen Pfeil gemeinsam; aber die Aufgabe, die Durchdringungspfeile einer solchen Pfeilmannigfaltigkeit mit einem α-Büschel (ω-Büschel) zu finden, führt auf *nicht-lineare* Gleichungen für die Parameter. Erst wenn man durch Einführung komplexer Pfeile den irreduziblen Zykel zu einer zweifach ausgedehnten Mannigfaltigkeit erweitert, und auch den Begriff des irreduziblen Zykels sachgemäss abändert, würde der Satz im Text ohne weitere Zusätze richtig sein.

Dabei haben die A nachstehende Bedeutung:

$$(17)\quad 2A_0 : 2A_1 : 2A_2 : 2A_3 = a_{12} - a_{21} : -(a_{11} + a_{22}) : a_{22} - a_{11} : a_{12} + a_{21}.$$

Die vier Verhältnisgrössen A_i sollen als *Zykelkoordinaten* verwandt werden. Deuten wir sie als Ebenenkoordinaten im R_3, so ist damit die eben erwähnte Abbildung vollzogen.

Man erkennt leicht die Richtigkeit folgender Gleichung:

$$(18)\quad 2\cdot(A_0B_0 + A_1B_1 - A_2B_2 - A_3B_3) = a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12},$$

wofür wir kürzer schreiben

$$(18)\qquad 2\cdot(AB) = D_{ab}.$$

Eigentlichen (uneigentlichen) irreduziblen Zykeln entsprechen also bei der Abbildung auf die Parasphäre solche Ebenen, deren Pole inbezug auf die Parasphäre innerhalb (ausserhalb)* der Fläche liegen.

Die parazyklische Gruppe geht bei dieser Abbildung über in eine sechsgliedrige projektive Gruppe des R_3, deren Transformationen die Parasphäre in Ruhe lassen. Jetzt erkennt man sofort den tieferen Unterschied zwischen den Transformationen der vier in § 2 erwähnten Scharen. Ist nämlich $N(\alpha)\cdot N(\beta) < 0$, so werden die beiden Gebiete, die wir soeben als Inneres und Äusseres der Parasphäre unterschieden haben, vertauscht.

Für reduzible Zykeln hat man (vgl. Gl. 15)

$$(AA) = \tfrac{1}{2}D_{aa} = 0.$$

§ 4. *Erweiterung der parazyklischen Gruppe.*

Gegenüber parazyklischen Transformationen besitzt die Figur von irgend vier Pfeilen *zwei unabhängige* absolute Invarianten, die wir als α-Doppelverhältnis und ω-Doppelverhältnis bezeichnen:

$$(1234)_\alpha = \frac{(\lambda_1\lambda_3)\cdot(\lambda_4\lambda_2)}{(\lambda_1\lambda_4)\cdot(\lambda_3\lambda_2)}; \qquad (1234)_\omega = \frac{(\mu_1\mu_3)\cdot(\mu_4\mu_2)}{(\mu_1\mu_4)\cdot(\mu_3\mu_2)}.$$

Dabei sind durch die Indizes jetzt nicht Pfeil*parameter*, sondern *Pfeile* unterschieden.

Für irgend vier Pfeile eines α-Büschels (ω-Büschels) ist das α-Doppelverhältnis (ω-Doppelverhältnis) unbestimmt.

Das α-Doppelverhältnis von vier Pfeilen eines irreduziblen Zykels ist gleich ihrem ω-Doppelverhältnis. Umgekehrt:

Ist das α-Doppelverhältnis von vier Pfeilen, von denen keine zwei α-parallel oder ω-parallel sind, gleich ihrem ω-Doppelverhältnis, so gehören die vier Pfeile einem irreduziblen Zykel an.

* Als *innerhalb* bezeichnen wir das Gebiet, in dem der Pol der uneigentlichen Ebene liegt.

Die Gruppe G_6 der parazyklischen Pfeiltransformationen erweitern wir noch durch Hinzufügung der *absoluten Spiegelung*,* d. i. der ausgezeichneten Transformation, die alle Pfeile umkehrt:

$$
(19)\qquad
\begin{gathered}
\lambda_1' : \lambda_2' = \mu_1 : \mu_2, \qquad \mu_1' : \mu_2' = \lambda_1 : \lambda_2, \\
\mathfrak{x}_0' : \mathfrak{x}_1' : \mathfrak{x}_2' : \mathfrak{x}_3' = -\mathfrak{x}_0 : \mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3, \\
\dot{\mathfrak{x}}' = \dot{\mathfrak{x}}.
\end{gathered}
$$

Aus den parazyklischen Transformationen gehen so hervor die *antiparazyklischen:*

$$
(20)\qquad
\begin{aligned}
\lambda_1' &= (\alpha_0 + \alpha_3)\mu_1 + (\alpha_2 - \alpha_1)\mu_2, & \mu_1' &= (\beta_0 + \beta_3)\lambda_1 + (\beta_2 - \beta_1)\lambda_2, \\
\lambda_2' &= (\alpha_2 + \alpha_1)\mu_1 + (\alpha_0 - \alpha_3)\mu_2, & \mu_2' &= (\beta_2 + \beta_1)\lambda_1 + (\beta_0 - \beta_3)\lambda_2, \\
N(\alpha) &\neq 0, & N(\beta) &\neq 0.
\end{aligned}
$$

Sie können vermittelst unserer Quaternionen in die kurze Formel zusammengefasst werden:

$$
(21)\qquad \dot{\mathfrak{x}}' = \dot{\underline{\alpha}} \cdot \dot{\mathfrak{x}} \cdot \beta.
$$

Die α_i, $\beta_i (i = 0, 1, 2, 3)$ sollen wieder die Parameter der Transformation heissen. Die Zusammensetzung irgend zweier Transformationen der gemischten Gruppe (G_6, H_6) bietet jetzt keine Schwierigkeit:

Es sei $S_{\alpha\beta}$ $(T_{\alpha\beta})$ die parazyklische (antiparazyklische) Transformation mit den Parametern α_i, β_i. Dann hat man (vgl. § 2):

$$
(22)\qquad
\begin{aligned}
&\left.\begin{aligned} S_{\alpha\beta} \cdot S_{\alpha'\beta'} &= S_{\alpha''\beta''} \\ T_{\alpha\beta} \cdot S_{\alpha'\beta'} &= T_{\alpha''\beta''} \end{aligned}\right\} \dot{\alpha}'' = \dot{\alpha} \cdot \dot{\alpha}', \quad \dot{\beta}'' = \dot{\beta} \cdot \dot{\beta}', \\
&\left.\begin{aligned} S_{\alpha\beta} \cdot T_{\alpha'\beta'} &= T_{\alpha''\beta''} \\ T_{\alpha\beta} \cdot T_{\alpha'\beta'} &= S_{\alpha''\beta''} \end{aligned}\right\} \dot{\alpha}'' = \dot{\beta} \cdot \dot{\alpha}', \quad \dot{\beta}'' = \dot{\alpha} \cdot \dot{\beta}'.
\end{aligned}
$$

Auch die Gesamtheit H_6 der antiparazyklischen Transformationen zerfällt in vier getrennte Scharen, die entsprechend, wie die von G_6 charakterisiert sind. Die Gruppe (G_6, H_6) enthält wieder eine sechsgliedrige Untergruppe (G_6', H_6'), charakterisiert durch die Ungleichung

$$
N(\alpha) \cdot N(\beta) > 0.
$$

Eine antiparazyklische Transformation vertauscht α-Parallelismus und ω-Parallelismus; sie vertauscht ferner das α-Doppelverhältnis mit dem ω-Doppelverhältnis.

Die Gruppe (G_6, H_6) umfasst alle Transformationen von Pfeilen, die Büschel paralleler Pfeile in ebensolche, und Zykeln in Zykeln überführen.

* Der Name wird weiterhin (§ 5) motiviert werden.

Die letzte Hälfte dieses Satzes kann auch folgendermassen präzisiert werden: Es gibt stets eine einzige *antiparazyklische* Transformation, die die Figur dreier wohlge ordneter Pfeile, von denen keine zwei parallel sind, in eine vorgegebene Figur derselben Eigenschaft überführt.

Gegenüber der Gruppe (G_6, H_6) gibt es nur noch *eine* Klasse von Büscheln paralleler Pfeile (vgl. § 3).

Bei der am Schlusse des vorigen Abschnitts erwähnten Abbildung der Zykeln auf den Ebenenraum geht die Gruppe (G_6, H_6) über in die Gruppe *aller* projektiven Transformationen der R_3, die die Parasphäre in Ruhe lassen, (G'_6, H'_6) in die Gruppe *aller* projektiven Transformationen des R_3, die ausserdem die beiden Seiten der Parasphäre nicht vertauschen.

§ 5. *Spiegelung am Zykel.*

Mit jedem *irreduziblen* Zykel A ist eine *involutorische antiparazyklische* Pfeiltransformation verbunden, die wir als *Spiegelung am Zykel* A bezeichnen:

$$(23)\quad \begin{aligned} \lambda'_1 &= (A_0 - A_3)\mu_1 + (A_1 - A_2)\mu_2, & \mu'_1 &= (A_0 + A_3)\lambda_1 + (A_2 - A_1)\lambda_2, \\ \lambda'_2 &= -(A_1 + A_2)\mu_1 + (A_0 + A_3)\mu_2, & \mu'_2 &= (A_2 + A_1)\lambda_1 + (A_0 - A_3)\lambda_2. \end{aligned}$$

Von zwei durch diese Transformation zugeordneten Pfeilen sagen wir: jeder liegt *invers* zum andern inbezug auf den Zykel A.

Liegen zwei Pfeile invers inbezug auf den irreduziblen Zykel A, so gehören ihre beiden Verbindungspfeile dem Zykel an.

Dieser Satz liefert eine höchst einfache Lösung der Aufgabe:

Einen Pfeil (λ, μ) an einem irreduziblen Zykel A zu spiegeln. (Fig. 6.)

Man zieht die beiden Zykelpfeile, die dem gegebenen parallel sind. Sie ergeben gerade noch einen Verbindungspfeil; dieser ist der gesuchte zu (λ, μ) inbezug auf A invers liegende (λ', μ'). Die Konstruktion versagt niemals.

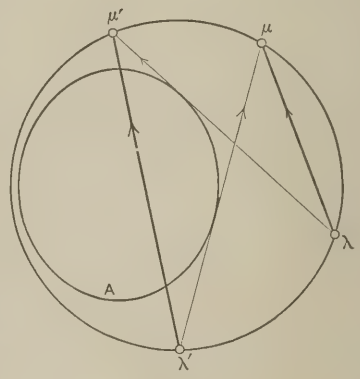

Fig. 6.

Jeder Pfeil liegt zu dem entgegengesetzten Pfeil invers inbezug auf den Zykel der Punktpfeile. Dieser wird später als *absoluter* Zykel bezeichnet werden, und daraus motiviert sich die Bezeichnung absolute Spiegelung (§ 4) für die ausgezeichnete Transformation, die alle Pfeile umkehrt.

Die Pfeile des spiegelnden Zykels selbst, und nur diese, sind ihre eigenen Spiegelbilder.

Das Spiegelbild eines α-Büschels ist ein ω-Büschel, das eines irreduziblen

(reduziblen) Zykels wieder ein irreduzibler (reduzibler) Zykel. Eigentliche (uneigentliche) Zykeln werden wieder in eigentliche (uneigentliche) übergeführt.

Bilden wir die Gleichungen (23) für einen *reduziblen* Zykel, so verschwinden die Transformationsdeterminanten. Von Spiegelungen an reduziblen Zykeln reden wir daher nicht.

Durch Zusammensetzung einer geraden (ungeraden) Anzahl von Spiegelungen an Zykeln erhält man alle parazyklischen (antiparazyklischen) Transformationen *der engeren Gruppe* (G'_6, H'_6).

Die Transformationen von (G_6, H_6), die der engeren Gruppe (G'_6, H'_6) nicht angehören, können nicht durch involutorische parazyklische oder antiparazyklische Pfeiltransformationen erhalten werden.

Wir stellen die Spiegelung am Zykel A nunmehr auch in *Pfeilkoordinaten* auf.

$$(24)\qquad \begin{aligned} -\mathfrak{x}'_0 &= 2\cdot A_0\cdot[A\mathfrak{x}]-(AA)\mathfrak{x}_0,\\ -\mathfrak{x}'_1 &= 2\cdot A_1\cdot[A\mathfrak{x}]-(AA)\mathfrak{x}_1,\\ -\mathfrak{x}'_2 &= -2\cdot A_2\cdot[A\mathfrak{x}]-(AA)\mathfrak{x}_2,\\ -\mathfrak{x}'_3 &= -2\cdot A_3\cdot[A\mathfrak{x}]-(AA)\mathfrak{x}_3. \end{aligned}$$

Bei Benutzung von *Zykel*koordinaten hat man:

$$(25)\qquad \begin{gathered} X'_i=(AA)X_i-2(AX)A_i, \qquad (i=0,1,2,3),\\ (X'X')=(AA)^2\cdot(XX), \qquad (X'Y')=(AA)^2\cdot(XY). \end{gathered}$$

Bei der Spiegelung an einem Zykel bleiben (ausser ihm selbst) ∞^2 Zykeln einzeln in Ruhe. Diese sollen zum spiegelnden Zykel *orthogonal* genannt werden. Ist ein irreduzibler Zykel Y orthogonal zu dem irreduziblen Zykel X, so ist auch X orthogonal zu Y. Diese Beziehung wird ausgedrückt durch eine symmetrische Gleichung (vgl. Gl. 18)

$$(XY)=0.$$

Diese Gleichung *definiert* die orthogonale Lage eines irreduziblen (oder reduziblen) Zykels zu einem *reduziblen*, wo die obige Erklärung versagt. Aus der Symmetrie dieser Gleichung folgt, dass man jetzt allgemein von *zu einander* orthogonalen Zykeln reden darf. Jetzt wird auch die bereits in § 3 gebrauchte Bezeichnung orthogonale Lage zweier reduzibler Zykeln gegeneinander motiviert; insonderheit darf jeder reduzible Zykel orthogonal zu sich selbst genannt werden.

Orthogonale Lage zweier Zykeln bleibt bei jeder parazyklischen und jeder antiparazyklischen Transformation erhalten, d. i. auch dann, wenn die Transformation nicht in der Gruppe (G'_6, H'_6) enthalten ist.

Die hier aufgetretenen Analogieen mit der Lehre von den MOEBIUS'schen

Kreisverwandtschaften erstrecken sich noch auf eine Reihe von weiteren Sätzen, bei deren Aufzählung wir uns kurz fassen werden. Nur abweichendes Verhalten soll besonders betont werden.

Die Figur aller ∞^2 Zykeln, die zu einem Zykel orthogonal liegen, bildet ein *Zykelbündel.* Der Zykel, zu dem alle Zykeln des Bündels orthogonal liegen, heisse *Mittelzykel* des Zykelbündels; er gehört dem Zykelbündel aber nur dann an, wenn er *reduzibel* ist. Wir klassifizieren die Zykelbündel:

1. Mittelzykel irreduzibel ∞^3. Eine Klasse.
   - *a.* Mittelzykel eigentlich ∞^3. Eine Klasse.
   - *b.* Mittelzykel uneigentlich ∞^3. Eine Klasse.
2. Mittelzykel reduzibel ∞^2. Eine Klasse.

Die Einteilung (1, 2, $\cdots$) soll sich (wie auch bei der Klassifikation der Zykel*büschel*) auf die Gruppe (G_6, H_6) beziehen, während die Einteilung (a, b) sich ergibt, wenn man nur Transformationen von (G'_6, H'_6) zulässt. Es gibt also gegenüber (G_6, H_6) *zwei*, gegenüber (G'_6, H'_6) *drei* Klassen von Zykelbündeln.

Zwei getrennte Zykelbündel durchdringen einander in einem *Zykelbüschel.* Diese werden klassifiziert durch Betrachtung der in ihnen enthaltenen reduziblen Zykeln.

1. Kein reduzibler Zykel. ∞^4. Eine Klasse gegenüber (G_6, H_6), zwei Klassen gegenüber (G'_6, H'_6):
   - *a.* Das Büschel besteht ganz aus eigentlichen Zykeln. ∞^4. Eine Klasse.
   - *b.* Das Büschel besteht ganz aus uneigentlichen Zykeln ∞^4. Eine Klasse.
2. Zwei getrennte reduzible Zykeln. ∞^4. Eine Klasse.
3. Ein (doppelt zählender) reduzibler Zykel. ∞^3. Eine Klasse gegenüber (G_6, H_6), zwei Klassen gegenüber (G'_6, H'_6):
   - *a.* Alle irreduziblen Zykeln sind eigentlich ∞^3. Eine Klasse.
   - *b.* Alle irreduziblen Zykeln sind uneigentlich ∞^3. Eine Klasse.
4. Das Büschel besteht ganz aus reduziblen Zykeln. ∞^1. Eine Klasse.

Jedem Zykelbüschel ist ein zweites *reziprok* zugeordnet:

In reziproken Zykelbüscheln liegt jeder Zykel des einen Büschels orthogonal zu sämtlichen Zykeln des anderen Büschels. Reziproke Zykelbüschel gehören demselben Haupttypus (aber in den Fällen 1. und 3. getrennten Untertypen) an. Jedes Büschel des Typus 4. ist zu sich selbst reziprok.

Bildet man (§ 3) die Zykeln auf die Ebenen des R_3 ab, so erscheinen die Zykelbündel als Punkte, die Zykelbüschel als gerade Linien.

§ 6. *Die Bewegungen und Umlegungen.*

Wir gehen jetzt zur *elementaren* Pfeilgeometrie über. Der Zykel der Punktpfeile wird als *absoluter Zykel* erklärt und es werden nur solche Transformationen von (G_6, H_6) zugelassen, die den absoluten Zykel in sich selbst

überführen. Diese werden dann *Bewegungen* oder *Umlegungen* genannt, jenachdem sie parazyklisch oder antiparazyklisch sind.

Eine Transformation von G_6 (H_6) ist eine Bewegung (Umlegung), wenn zwischen ihren Parametern die Gleichungen bestehen:

$$\beta_0 : \beta_1 : \beta_2 : \beta_3 = \alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 .$$

Die Bewegungen und Umlegungen sind also bereits in der Gruppe (G'_6, H'_6) enthalten.

Die Bewegungen und Umlegungen verwandeln entgegengesetzte Pfeile stets wieder in ebensolche. Sie umfassen *alle* Transformationen von (G_6, H_6), die diese Eigenschaft besitzen.

Nennen wir *antiparallel* zwei Pfeile, wenn der Endpunkt des einen mit dem Anfangspunkt des andern zusammenfällt, so gilt der Satz:

Antiparallelismus von Pfeilen bleibt bei jeder Bewegung und Umlegung erhalten.

Die Bewegungen und Umlegungen umfassen sämtliche Transformationen von (G_6, H_6), die antiparallele Pfeile stets wieder in ebensolche verwandeln.

Man kann die Bewegungen und Umlegungen durch die Quaternionengleichungen darstellen:

$$\dot{\mathfrak{x}}' = \underline{\dot{\alpha}} \cdot \dot{\mathfrak{x}} \cdot \dot{\alpha}, \qquad \dot{\mathfrak{x}}' = \underline{\dot{\alpha}} \cdot \underset{\sim}{\dot{\mathfrak{x}}} \cdot \dot{\alpha}. \tag{26}$$

Die Gleichungen (10) vereinfachen sich wesentlich:

$$\begin{gathered}
c_{01} = c_{02} = c_{03} = 0, \qquad c_{10} = c_{20} = c_{30} = 0, \\
c_{00} = \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2, \\
c_{11} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \\
c_{22} = \alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2, \\
c_{33} = \alpha_0^2 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2, \\
c_{23} = 2(\alpha_0\alpha_1 - \alpha_2\alpha_3), \qquad c_{32} = 2(-\alpha_0\alpha_1 - \alpha_2\alpha_3), \\
c_{31} = 2(-\alpha_0\alpha_2 - \alpha_3\alpha_1), \qquad c_{13} = 2(-\alpha_0\alpha_2 + \alpha_3\alpha_1), \\
c_{12} = 2(\alpha_0\alpha_3 + \alpha_1\alpha_2), \qquad c_{21} = 2(\alpha_0\alpha_3 - \alpha_1\alpha_2).
\end{gathered} \tag{27}$$

Übt man nach der Bewegung (Umlegung) $\dot{\alpha}$ die Bewegung oder Umlegung $\dot{\alpha}'$ aus, so gilt jetzt für die resultierende Transformation $\dot{\alpha}''$ *in allen Fällen* die Gleichung:

$$\dot{\alpha}'' = \dot{\alpha} \cdot \dot{\alpha}',$$

oder ausgeschrieben:

$$\begin{aligned}
\alpha_0'' &= \alpha_0\alpha_0' - \alpha_1\alpha_1' + \alpha_2\alpha_2' + \alpha_3\alpha_3', \\
\alpha_1'' &= \alpha_1\alpha_0' + \alpha_0\alpha_1' + \alpha_3\alpha_2' - \alpha_2\alpha_3', \\
\alpha_2'' &= \alpha_2\alpha_0' + \alpha_3\alpha_1' + \alpha_0\alpha_2' - \alpha_1\alpha_3', \\
\alpha_3'' &= \alpha_3\alpha_0' - \alpha_2\alpha_1' + \alpha_1\alpha_2' + \alpha_0\alpha_3'.
\end{aligned} \tag{28}$$

Ferner ist:

$$N(\alpha'') = N(\alpha) \cdot N(\alpha').$$

Aus dieser Gleichung schliessen wir:

Die Gesamtheit der Bewegungen zerfällt in zwei getrennte Schaaren:

1) $N(\alpha) > 0$. *Eigentliche* Bewegungen.

2) $N(\alpha) < 0$. *Uneigentliche* Bewegungen.

Die *uneigentlichen* Bewegungen kehren den Umlaufssinn auf dem absoluten Kegelschnitt, d. i. dem als Punktmannigfaltigkeit aufgefassten absoluten Zykel, um.

Die *Gruppe* der *eigentlichen* Bewegungen ist kontinuierlich. Der Umlaufssinn bleibt ungeändert.

Entsprechend haben wir bei den Umlegungen zu unterscheiden. *Eigentliche* Umlegungen lassen den Umlaufssinn ungestört $[N(\alpha) > 0]$.

Es bezeichne S (T) eine eigentliche, S' (T') eine uneigentliche Bewegung (Umlegung).

Drückt man dann durch die symbolische Gleichung

$$(SS) = S$$

den Tatbestand aus, dass zwei eigentliche Bewegungen wieder eine eigentliche Bewegung ergeben, so gelten auch folgende Gleichungen:

$$\begin{aligned} (SS) = S, \quad (SS') = S', \quad (ST) = T, \quad (ST') &= T'; \\ (S'S') = S, \quad (S'T) = T', \quad (S'T') &= T; \\ (TT) = S, \quad (TT') &= S'; \\ (T'T') &= S. \end{aligned}$$

Es gibt also folgende dreigliedrige reell-gemischte Gruppen:

$$\boxed{S, S', T, T'}, \quad \boxed{S, T}, \quad \boxed{S, T'}, \quad \boxed{S, S'}.$$

Bei einer Bewegung (Umlegung) werden die Anfangspunkte der Pfeile ebenso transformiert, wie die Endpunkte. Die Bewegungen und Umlegungen sind also auch Transformationen der Punkte des absoluten Kegelschnitts, d. i. des als Punktmannigfaltigkeit aufgefassten absoluten Zykels.

Betrachtet man bei dieser Auffassung jeden ursprünglichen unendlich fernen Punkt als Anfangspunkt, jeden transformierten als Endpunkt eines Pfeils, so entsteht eine Mannigfaltigkeit von ∞^1 Pfeilen, die ein *irreduzibler Zykel* sein muss, da sie offenbar mit jedem α-Büschel und mit jedem ω-Büschel gerade einen Pfeil gemeinsam hat.

Der Bewegung $\dot{\alpha}$ ist dabei derselbe Zykel zugeordnet, wie der Umlegung $\dot{\alpha}$; er hat die Koordinaten $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3$, also die Gleichung $\alpha_0 \mathfrak{x}_0 + \alpha_1 \mathfrak{x}_1 + \alpha_2 \mathfrak{x}_2 + \alpha_3 \mathfrak{x}_3 = 0$.

Die identische Bewegung, ebenso demnach die absolute Spiegelung, führt auf den absoluten Zykel $1:0:0:0$.

Die Theorie der Zykeln steht also in engem Zusammenhang mit der Lehre von den Bewegungen.

Wir betrachten jetzt die *involutorischen* Bewegungen und Umlegungen. Für sie muss, wenn wir von der identischen Bewegung und der absoluten Spiegelung absehen, der Parameter α_0 verschwinden.

1. *Die eigentlichen involutorischen Bewegungen.* $N(\alpha) > 0$. Kein Pfeil bleibt in Ruhe, dagegen werden ∞^1 Pfeile umgekehrt. Die Träger aller dieser Pfeile laufen durch den (zugänglichen) Punkt $\alpha_1 : \alpha_2 : \alpha_3$.

Umwendung um den Punkt $\alpha_1 : \alpha_2 : \alpha_3$.*

Der zugehörige Zykel besteht aus den erwähnten ∞^1 Pfeilen und soll *Punktzykel* genannt werden. Er besitzt keine Punktpfeile und gestattet die absolute Spiegelung. Er gehört zu den *eigentlichen* Zykeln (§ 3).

2. *Die uneigentlichen involutorischen Bewegungen.* $N(\alpha) < 0$. In Ruhe bleiben zwei entgegengesetzte Pfeile und damit auch die zugehörigen Verbindungs(Punkt-)pfeile. Auch hier werden ∞^1 Pfeile umgekehrt. Ihre Träger schneiden jetzt den Träger $\alpha_1 : -\alpha_2 : -\alpha_3$ des ersten Pfeilpaares senkrecht.

Umwendung um die Gerade $\alpha_1 : -\alpha_2 : -\alpha_3$.

Der zugehörige aus jenen ∞^1 Pfeilen und zwei Punktpfeilen bestehende Zykel soll *Geradenzykel* genannt werden. Er gestattet die absolute Spiegelung.

3. *Die eigentlichen involutorischen Umlegungen.* $N(\alpha) > 0$. Der (*eigentliche*) Punktzykel mit den Koordinaten $0 : \alpha_1 : \alpha_2 : \alpha_3$ bleibt *pfeilweise* in Ruhe.

Spiegelung am Punkt $\alpha_1 : \alpha_2 : \alpha_3$.

4. *Die uneigentlichen involutorischen Umlegungen.* $N(\alpha) < 0$. Der (*uneigentliche*) Geradenzykel mit den Koordinaten $0 : \alpha_1 : \alpha_2 : \alpha_3$ bleibt *pfeilweise* in Ruhe.

Spiegelung an der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$.

Die Spiegelungen an Punkten und Geraden fallen unter den allgemeineren Begriff: Spiegelung am Zykel. Damit ist also die Frage nach der Konstruktion dieser Transformationen bereits beantwortet. Entsprechendes gilt auch für die Konstruktion der Umwendungen.

§ 7. *Klassifikation der Bewegungen und der Zykeln.*

Jede von der identischen verschiedene Bewegung lässt sich auf ∞^1 *Arten darstellen als Aufeinanderfolge zweier Spiegelungen.*

* Statt *Umwendung um einen Punkt* pflegt man auch zu sagen: *Spiegelung am Punkt.* Wir sind hier aber gezwungen, eine Trennung in der Bezeichnung vorzunehmen, und verwenden das Wort Umwendung nur für Bewegungen, das Wort Spiegelung nur für Umlegungen.

Es sei

$$S_\alpha = T_\beta \cdot T_\gamma$$

Dann darf, wie man auf Grund der Gll. (28) leicht bestätigt, gesetzt werden:

$$\beta_0 : \beta_1 : \beta_2 : \beta_3 = 0 : \sigma_2\alpha_3 - \sigma_3\alpha_2 : -(\sigma_3\alpha_1 - \sigma_1\alpha_3) : -(\sigma_1\alpha_2 - \sigma_2\alpha_1),$$

$$\gamma_0 : \gamma_1 : \gamma_2 : \gamma_3 = 0 : \tau_2\alpha_3 - \tau_3\alpha_2 : -(\tau_3\alpha_1 - \tau_1\alpha_3) : -(\tau_1\alpha_2 - \tau_2\alpha_1).$$

Dabei sind die σ und τ durch folgende Relationen verbunden:

$$\left.\begin{aligned} \tau_1 &= \alpha_0\sigma_1 - \alpha_3\sigma_2 + \alpha_2\sigma_3, \\ \tau_2 &= -\alpha_3\sigma_1 + \alpha_0\sigma_2 + \alpha_1\sigma_3, \\ \tau_3 &= \alpha_2\sigma_1 - \alpha_1\sigma_2 + \alpha_0\sigma_3, \end{aligned}\right\} (\sigma_1 : \sigma_2 : \sigma_3 \neq \alpha_1 : \alpha_2 : \alpha_3).$$

Ferner ist

$$N(\alpha) = N(\beta) \cdot N(\gamma).$$

Eine *eigentliche* Bewegung kann zusammengesetzt werden aus zwei Spiegelungen an Punkten oder auch aus zwei Spiegelungen an Geraden.

Um eine *uneigentliche* Bewegung zu erhalten, hat man die Spiegelung an einem Punkt zusammenzusetzen mit der Spiegelung an einer Geraden.

Wir untersuchen jetzt die verschiedenen Fälle, indem wir mit den eigentlichen Bewegungen beginnen.

1. $\alpha_1^2 - \alpha_2^2 - \alpha_3^2 > 0$. Die beiden Geraden $\beta_1 : -\beta_2 : -\beta_3$ und $\gamma_1 : -\gamma_2 : -\gamma_3$ laufen durch den Punkt $\alpha_1 : \alpha_2 : \alpha_3$. Kein Pfeil bleibt in Ruhe, ebenso kein reduzibler Zykel, wohl aber der Punktzykel $0 : \alpha_1 : \alpha_2 : \alpha_3$.

Drehung um den Punkt $\alpha_1 : \alpha_2 : \alpha_3$.

Besonderer Fall: $\alpha_0 = 0$. Umwendung um den Punkt $\alpha_1 : \alpha_2 : \alpha_3$.

2. $\alpha_1^2 - \alpha_2^2 - \alpha_3^2 = 0$. Die beiden vorhin erwähnten Geraden sind jetzt parallel. In Ruhe bleibt ein Punktpfeil und ein reduziber Zykel, der die absolute Spiegelung gestattet.

Gleitung längs des Punktpfeils* $0 : \alpha_1 : -\alpha_2 : -\alpha_3$.

3. $\alpha_1^2 - \alpha_2^2 - \alpha_3^2 < 0$. Jetzt stellt das System $\alpha_1 : -\alpha_2 : -\alpha_3$ eine zugängliche Gerade dar, die Verbindungslinie der Punkte $\beta_1 : \beta_2 : \beta_3$ und $\gamma_1 : \gamma_2 : \gamma_3$, oder gemeinsame Normale der Geraden $\beta_1 : -\beta_2 : -\beta_3$ und $\gamma_1 : -\gamma_2 : -\gamma_3$. In Ruhe bleiben die beiden Pfeile auf der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$, sowie die zugehörigen Verbindungspfeile, ferner vier unschwer zu beschreibende reduzible Zykeln, endlich der Geradenzykel $0 : \alpha_1 : \alpha_2 : \alpha_3$.

Gleitung längs der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$.

Ist die Bewegung S_α uneigentlich, so sind die Mittelelemente der Spiegelungen T_β und T_γ Punkt auf der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$ und Gerade senkrecht

* Von Hrn. E. STUDY benutzt. Das sonst übliche Wort *Schiebung* wird besser vermieden, da es Analogieen suggeriert, die nicht existieren.

zu ihr. Es bleiben dieselben Gebilde in Ruhe, wie bei der eben besprochenen Gleitung.

Uneigentliche Gleitung längs der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$.

Besonderer Fall: $\alpha_0 = 0$. Unwendung um die Gerade $\alpha_1 : -\alpha_2 : -\alpha_3$.

In allen Fällen ist die Maassinvariante (Winkel, Öffnung*, Abstand) der Mittelelemente der beiden Spiegelungen gleich der halben Grösse der resultierenden Bewegung (Drehwinkel, Gleitöffnung,† Gleitstrecke).

Überall in diesem Abschnitt darf das Wort Spiegelung ersetzt werden durch Umwendung.

Wir geben jetzt im Zusammenhang die Parameter einer Bewegung von vorgeschriebenen üblichen Bestimmungsstücken an.

1. Drehung um den Punkt x durch den Winkel 2ϑ:

$$-\sqrt{x_1^2 - x_2^2 - x_3^2}\cot\vartheta : x_1 : x_2 : x_3.$$

2. Gleitung längs der Minimalgeraden $\mathfrak{x}$ durch die Öffnung 2ω:

$$-\frac{1}{\omega}\mathfrak{x}_1 : \mathfrak{x}_1 : -\mathfrak{x}_2 : -\mathfrak{x}_3.\dagger$$

3. Gleitung längs des (eigentlichen) Pfeils $\mathfrak{x}$ durch die Strecke 2η:

$$\mathfrak{x}_0 \coth\eta : \mathfrak{x}_1 : -\mathfrak{x}_2 : -\mathfrak{x}_3.$$

4. Uneigentliche Gleitung längs des Pfeils $\mathfrak{x}$ durch die Strecke 2η:

$$\mathfrak{x}_0 \operatorname{tgh}\eta : \mathfrak{x}_1 : -\mathfrak{x}_2 : -\mathfrak{x}_3.$$

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* Vgl. über diese Begriffe weiter unter die folgenden beiden Fussnoten.

† Hierzu bemerken wir noch Folgendes: Eine *invariante* Grösse, die durch zwei Punkte einer Minimalgeraden (oder, was dasselbe ist, durch zwei parallele Gerade) bestimmt wäre, gibt es natürlich nicht. Erklärt man die Öffnung zweier paralleler Gerader $\mathfrak{x}$ und $\mathfrak{x}'$ durch die Gleichung

$$-2\omega = [\mathfrak{x}, \mathfrak{x}'] = \frac{\mathfrak{x}_2\mathfrak{x}_3' - \mathfrak{x}_3\mathfrak{x}_2'}{\mathfrak{x}_1\mathfrak{x}_1' - \mathfrak{x}_2\mathfrak{x}_2' - \mathfrak{x}_3\mathfrak{x}_3'},$$

so ist zu beachten, dass diese Grösse invariant ist nur gegenüber den Bewegungen der im Texte unter (2) aufgeführten eingliedrigen Gruppe, die also nur einen einzigen unendlich fernen Punkt in Ruhe lassen, nämlich den unendlich fernen Schnittpunkt der beiden Parallelen. Sofern die Figur zweier getrennten parallelen Geraden, die in eine bestimmte Reihenfolge gesetzt sind, eine Bewegung der soeben beschriebenen Art völlig bestimmt, darf man von der Öffnung dieser Bewegung reden.

Invariant ist aber die *Öffnungshalbierende* zweier paralleler Geraden, eine dritte zugängliche parallele Gerade, die Symmetrieaxe der Figur der beiden ersten ist.

Will man den Begriff der Öffnung vermeiden, so empfiehlt sich etwa folgende Fragestellung: *Wie heissen die Parameter der Gleitung längs der Minimalgeraden* $L_1 : L_2$, *die die Minimalgerade* $M_1 : M_2$ *überführt in die Minimalgerade* $N_1 : N_2$? Man findet:

$$\frac{2(LM)(LN)}{(MN)} : L_2^2 + L_1^2 : L_2^2 - L_1^2 : 2L_1L_2,$$

wo $L_1 : L_2$ den Berührungspunkt der ersten Minimalgeraden darstellt (vgl. § 1, Gl. 3) und das Symbol (MN) die folgende Bedeutung hat:

$$(MN) = M_1N_2 - M_2N_1, \text{ usw.}$$

Mit der Einteilung der Bewegungen ist gleichzeitig bereits die Klassifikation der irreduziblen Zykeln (gegenüber Bewegungen und Umlegungen) vollzogen. Wir stellen kanonische Vertreter für die einzelnen Typen auf.

I. *Irreduzible Zykeln.* $(AA) \neq 0$.

a. *Eigentliche Zykeln.* $(AA) > 0$.

1. $A_1^2 - A_2^2 - A_3^2 > 0$.

α) $A_0 \neq 0$. *Drehzykel.* ∞^3. ∞^1 Klassen.

$$\cos\vartheta \cdot \mathfrak{x}_0 - \sin\vartheta \cdot \mathfrak{x}_1 = 0 \qquad (0 < \vartheta < \tfrac{1}{2}\pi).$$

β) $A_0 = 0$. *Punktzykel.* ∞^2. Eine Klasse.

$$\mathfrak{x}_1 = 0.$$

2. $A_1^2 - A_2^2 - A_3^2 = 0$.

α) Nicht alle A_i $(i = 1, 2, 3)$ verschwinden gleichzeitig. *Grenzzykel.* ∞^2. Eine Klasse.

$$\mathfrak{x}_0 + \mathfrak{x}_1 + \mathfrak{x}_2 = 0.$$

β) $A_1 = A_2 = A_3 = 0$. *Absoluter Zykel.* ∞^0.

$$\mathfrak{x}_0 = 0.$$

3. $A_1^2 - A_2^2 - A_3^2 < 0$. *Gleitzykel.* ∞^3. ∞^1 Klassen.

$$\cosh\eta \cdot \mathfrak{x}_0 - \sinh\eta \cdot \mathfrak{x}_3 = 0 \qquad (0 \neq \eta \neq \infty).$$

b. *Uneigentliche Zykeln.* $(AA) < 0$.

1. $A_0 \neq 0$. *Uneigentlicher Zykel* allgemeiner Art. ∞^3. ∞^1 Klassen.

$$\sinh\eta \cdot \mathfrak{x}_0 - \cosh\eta \cdot \mathfrak{x}_3 = 0 \qquad (0 \neq \eta \neq \infty).$$

2. $A_0 = 0$. *Geradenzykel.* ∞^2. Eine Klasse.

$$\mathfrak{x}_3 = 0.$$

II. *Reduzible Zykeln.* $(AA) = 0$.

1. $A_0 \neq 0$. Mittelpfeil eigentlich. ∞^2. Eine Klasse.

$$\mathfrak{x}_0 + \mathfrak{x}_3 = 0, \qquad \lambda_1 \mu_2 = 0.$$

2. $A_0 = 0$. Der Mittelpfeil ist Punktpfeil. ∞^1. Eine Klasse.

$$\mathfrak{x}_1 + \mathfrak{x}_2 = 0, \qquad \lambda_1 \mu_1 = 0.$$

Radius des Drehzykels, Grenzabstand des Grenzykels * und Abstand des

* Über die Invarianz dieses Ausdrucks gelten entsprechende Bemerkungen, wie in der vorigen Fussnote. Gegenüber den hyperbolischen Bewegungen, die nur einen einzigen unendlich fernen Punkt y in Ruhe lassen, hat ein zugänglicher Punkt x die Invariante $\{x, y\}$:

$$e^{\{x,y\}} = \frac{x_1 y_1 - x_2 y_2 - x_3 y_3}{\sqrt{x_1^2 - x_2^2 - x_3^2}}.$$

Gleitzykels (über die Gestalten dieser Gebilde vgl. § 3) hängen mit den Grössen 2ϕ der zugehörigen Bewegungen (Drehwinkel, Gleitöffnung, Gleitstrecke) zusammen durch die Gleichung

$$|r| = \log \epsilon \frac{e^{\frac{\epsilon\phi}{2}} + e^{-\frac{\epsilon\phi}{2}}}{e^{\frac{\epsilon\phi}{2}} - e^{-\frac{\epsilon\phi}{2}}},$$

wenn man der Reihe nach setzt:

$$\epsilon^2 = -1, \qquad \epsilon^2 = 0, \qquad \epsilon^2 = +1.$$

Im Falle $\epsilon^2 = -1$ gibt die Gleichung den bekannten Zusammenhang zwischen Lotlänge und Parallelenwinkel. —

Bezeichnet man die Zykeln, die die absolute Spiegelung gestatten, also die Punktzykeln, Geradenzykeln und den zweiten Typus der reduziblen Zykeln als *spezielle* Zykeln, so gilt der Satz:

Ein Zykelbüschel enthält einen einzigen speziellen Zykel, oder es besteht ganz aus solchen.

Im ersten Falle hat der spezielle Zykel einfache geometrische Bedeutung (Mittelpunkt des Drehzykels, gemeinsame Normale der beiden Pfeile, in denen sich zwei Zykel durchdringen, Scheitelpunkt des Grenzzykels u. s. w.).

Jeder zum absoluten Zykel orthogonale Zykel ist speziell.

§ 8. *Konstruktionen im Pfeilgebiet.*

Von der elementaren Pfeilgeometrie kehren wir noch einmal zu allgemeineren Betrachtungen zurück. Es ist zunächst zu zeigen, wie man beliebig viele Pfeile eines durch drei Pfeile bestimmten irreduziblen Zykels konstruieren kann. Daran schliessen wir noch einige andere Konstruktionsaufgaben an.

Irgend drei Pfeile, von denen keine zwei parallel sind, bestimmen, wie schon gesagt, eine Abbildung des absoluten Kegelschnitts auf sich selbst von folgender Eigenschaft:

Das Doppelverhältnis von irgend vier ursprünglichen Punkten ist gleich dem Doppelverhältnis der transformierten Punkte.

Deuten wir jeden ursprünglichen Punkt als Anfangspunkt, den transformierten als Endpunkt eines Pfeils, so erfüllen die ∞^1 auf diese Weise erhaltenen Pfeile den ganzen durch die drei gegebenen Pfeile bestimmten Zykel.

Ist also der Zykel pfeilweise bekannt, so ist dadurch die ganze Abbildung gegeben.

Man kann sich nun auf den Standpunkt stellen, dass nur die *speziellen* Zykeln

Es genügt, dieser Grösse, die wir als *Grenzabstand* des Punktes x gegen den Punkt y bezeichnen, positive Werte beizulegen. Der Ort der Punkte x, die gegen einen unendlich fernen Punkt gleichen Grenzabstand besitzen, ist eine *Grenzkurve*. Der Abstand zweier Grenzkurven, die zu ein und demselben unendlich fernen Punkte gehören, ist gleich der Differenz der Grenzabstände.

unmittelbar gegeben seien, und dann die Aufgabe stellen, *unter Benutzung nur von speziellen Zykeln* die Abbildung herzustellen und damit auch nicht-spezielle Zykeln zu konstruieren.

Das kann auf folgende Weise geschehen: (wir setzen voraus, dass der zu konstruierende Zykel nicht schon speziell ist).

1. Die Anfangspunkte (a, b, c) der drei Pfeile verbinden wir mit einem (nicht unendlich fernen, sonst aber) beliebigen Punkt P_1 Es entsteht ein spezieller Zykel, gegeben durch die drei Pfeile $\overrightarrow{aA}$ $\overrightarrow{bB}$, $\overrightarrow{cC}$.

2. Jetzt verbinden wir einen zweiten (nicht unendlich fernen) Punkt P_2 mit den Punkten A, B und C durch Pfeile. Der neue spezielle Zykel ist durch die Pfeile $\overrightarrow{A'A}$, $\overrightarrow{B'B}$, $\overrightarrow{C'C}$ gegeben.

3. Sind die Punkte P_1 und P_2 passend gewählt, so ist es stets möglich, einen dritten Punkt P_3 so zu finden, dass die orientierten Verbindungsgeraden $\overrightarrow{A'P_3}$, $\overrightarrow{B'P_3}$, $C\,P_3$ der Reihe nach durch die Endpunkte a', b', c' der drei gegebenen Pfeile laufen.

Wendet man diese Konstruktion auf einen vierten unendlich fernen Punkt d an, so erhält man den zugeordneten Punkt d' und somit einen vierten Pfeil des durch die drei gegebenen laufenden Zykels.

Die drei Punkte P_1, P_2, P_3 kann man z. B folgendermassen wählen: (Fig. 7)

$$P_1 = (ab, cc'), \qquad P_3 = (aa', bb'), \qquad P_2 = (ab, P_3c').$$

Die Punkte P existieren stets. Vorkommen kann es aber, dass der eine oder andre von ihnen unendlich fern liegt. Der Fall kann dann durch Einschieben noch eines vierten speziellen Zykels stets vermieden werden. Auf gleiche Weise beseitigt man unbequeme Lageverhältnisse.

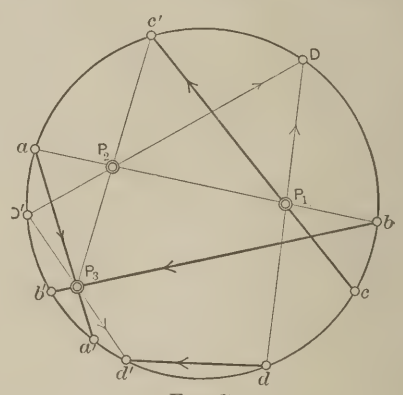

Fig. 7.

Benutzt sind nur die drei *linearen* Fundamentalkonstruktionen der ebenen *projektiven* Geometrie:

1) Zwei Punkte zu verbinden.

2) Zwei Gerade zum Schnitt zu bringen.

3) Den zweiten Schnittpunkt einer Geraden mit einer Kurve zweiter Ordnung zu finden.

Da wir von dem durch die drei Pfeile $\overrightarrow{aa'}$, $\overrightarrow{bb'}$, $\overrightarrow{cc'}$ gehenden Zykel auf Grund dieser linearen Konstruktionen jetzt beliebig viele Pfeile zeichnen können, erklären wir als

1. *Erste (lineare) Fundamentalkonstruktion der Pfeilgeometrie:*

Durch drei Pfeile, von denen keine zwei parallel sind, den Zykel zu legen.

Durch zweimalige Anwendung dieser Konstruktion löst man die Aufgabe:

2. *Einen Pfeil an einem irreduziblen Zykel zu spiegeln.*

Es ist zwar noch ein Pfeil mit einem andern zu *verbinden* (§ 1, Fig. 1); das ist aber keine neue geometrische Konstruktion, da der Pfeil eben als Punktepaar erklärt worden ist; den Träger hatten wir in den Begriff des Pfeils nicht mit aufgenommen.

3. *Es soll die parazyklische (antiparazyklische) Pfeiltransformation konstruiert werden, die drei Pfeile* $\overrightarrow{aA}$, $\overrightarrow{bB}$, $\overrightarrow{cC}$, *von denen keine zwei parallel sind, in die Pfeile* $\overrightarrow{a'A'}$, $\overrightarrow{b'B'}$, $\overrightarrow{c'C'}$ *überführt.*

Man konstruiert im ersten Falle die beiden Zykeln

$$\overrightarrow{aa'},\ \overrightarrow{bb'},\ \overrightarrow{cc'} \qquad \text{und} \qquad \overrightarrow{AA'},\ \overrightarrow{BB'},\ \overrightarrow{CC'};$$

im zweiten Falle dagegen die Zykeln

$$\overrightarrow{aA'},\ \overrightarrow{bB'},\ \overrightarrow{cC'} \qquad \text{und} \qquad \overrightarrow{Aa'},\ \overrightarrow{Bb'},\ \overrightarrow{Cc'}.$$

Das weitere Verfahren, zu einem vierten ursprünglichen Pfeil $\overrightarrow{dD}$ den transformierten zu finden, bedarf keiner Erläuterung mehr.

4. *Den zweiten Durchdringungspfeil zweier irreduzibler Zykeln zu zeichnen, wenn der erste gegeben ist.*

Ein Pfeil $\mathfrak{x}$ wird beliebig gewählt, aber so dass er keinem der beiden gegebenen Zykeln A und B angehört. Dann spiegeln wir ihn an den beiden Zykeln A und B und erhalten die Spiegelbilder $\mathfrak{y}$ und $\mathfrak{z}$. Kurz geschrieben:

$$\mathfrak{x}\{A\}\mathfrak{y}, \qquad \mathfrak{x}\{B\}\mathfrak{z}.$$

Jetzt wird der Zykel $\mathfrak{x}\mathfrak{y}\mathfrak{z}$ gezeichnet. Wird der gegebene Durchdringungspfeil I der beiden Zykeln A und B am Zykel $(\mathfrak{x}\mathfrak{y}\mathfrak{z})$ gespiegelt, so erhält man den zweiten gesuchten Durchdringungspfeil II:

$$\mathrm{I}\,\{\mathfrak{x}\mathfrak{y}\mathfrak{z}\}\,\mathrm{II}.$$

Damit die zuletzt angegebene Spiegelung am Zykel möglich ist, muss natürlich der Zykel $(\mathfrak{x}\mathfrak{y}\mathfrak{z})$ irreduzibel sein. So lässt sich aber der Pfeil $\mathfrak{x}$ stets wählen.

5. *Durch einen Pfeil* $\mathfrak{x}$ *den Zykel zu legen, der zu zwei getrennten irreduziblen Zykeln A und B orthogonal ist.*

$$\mathfrak{x}\{A\}\mathfrak{y}, \quad \mathfrak{x}\{B\}\mathfrak{z}.$$

Dann ist der Zykel $(\mathfrak{x}\mathfrak{y}\mathfrak{z})$ der gesuchte. Es gibt ∞^1 Lösungen, falls der reduzible Zykel, für den der Pfeil $\mathfrak{x}$ Mittelpfeil ist, mit A und B in einem Zykelbüschel liegt, in allen übrigen Fällen eine einzige Lösung. Das soeben angegebene Verfahren führt jedoch nicht zum Ziel, wenn der Pfeil $\mathfrak{x}$ einem der beiden gegebenen Zykeln angehört. Dann benutzt man am bequemsten Aufgabe 9.

6. *Den Zykel zu zeichnen, der zu zwei getrennten reduziblen Zykeln A und B orthogonal liegt und durch den Pfeil $\mathfrak{x}$ läuft.*

Man legt durch $\mathfrak{x}$ und die Mittelpfeile der beiden reduziblen Zykeln den Zykel. Es darf $\mathfrak{x}$ nicht bereits Mittelpfeil eines der beiden reduziblen Zykeln sein; dann gibt es ∞^1 Lösungen; ebenso wenn die Mittelpfeile der beiden gegebenen reduziblen Zykeln beide zum Pfeil $\mathfrak{x}$ α-parallel (ω-parallel) sind.

7. *Den Zykel zu zeichnen, der durch den Pfeil $\mathfrak{x}$ läuft, zu einem irreduziblen und zu einem reduziblen Zykel orthogonal ist.*

Man spiegelt den Pfeil $\mathfrak{x}$, sowie den Mittelpfeil des reduziblen Zykels am irreduziblen. Dadurch hat man vier Pfeile des gesuchten Zykels. Auch hier kann das Lösungsverfahren versagen (wenn $\mathfrak{x}$ dem irreduziblen Zykel angehört), ferner gibt es in zwei unschwer zu beschreibenden Fällen ∞^1 Lösungen.

8. *Den Zykel zu zeichnen, der durch einen Pfeil $\mathfrak{x}$ läuft und dem durch die beiden Zykeln A und B bestimmten Büschel angehört.*

Man benutzt (je nach der Beschaffenheit der beiden Zykeln A und B) eine der drei letzten Aufgaben und zeichnet zwei beliebige Zykeln orthogonal zu A und B. Sie mögen C und D heissen. Sie dürfen nicht reduzibel sein und dürfen nicht durch den gegebenen Pfeil laufen. Weiter:

$$\mathfrak{x}\{C\}\mathfrak{y}, \quad \mathfrak{x}\{D\}\mathfrak{z}.$$

Dann ist der Zykel $\mathfrak{x}\mathfrak{y}\mathfrak{z}$ der gesuchte.

9. *Den Orthogonalzykel dreier Zykeln A, B, C, die nicht einem Büschel angehören, zu finden.*

Man wählt einen Pfeil $\mathfrak{x}$, der keinem der drei Zykeln angehört, beliebig. Durch $\mathfrak{x}$ legt man die drei Zykeln, die der Reihe nach den Büscheln (B, C), (C, A), (A, B) angehören (vorige Aufgabe). Diese drei Zykeln durchdringen sich ausser in $\mathfrak{x}$ noch in einem Pfeil $\mathfrak{x}'$.

Diese Konstruktion wiederholt man mit einem zweiten Pfeil $\mathfrak{y}$ und erhält den Pfeil $\mathfrak{y}'$.

Die vier Verbindungspfeile $\overrightarrow{\mathfrak{x}-\mathfrak{x}'}$, $\overrightarrow{\mathfrak{x}'-\mathfrak{x}}$, $\overrightarrow{\mathfrak{y}-\mathfrak{y}'}$, $\overrightarrow{\mathfrak{y}'-\mathfrak{y}}$ gehören dem gesuchten Orthogonalzykel an. Durch die soeben angegebene Konstruktion lassen sich auch die oben (bei 5 und 7) erwähnten Fälle erledigen, wenn die dort aufgeführten Methoden versagen.

Als quadratische Konstruktion ist die folgende zu betrachten.

10. *Die beiden Durchdringungspfeile zweier Zykeln zu zeichnen.*

Die beiden Zykeln seien A und B. Man zeichnet zwei weitere Zykeln C und D beliebig, die zu beiden orthogonal sind. Dann bestimmen die drei Zykeln C, D und A einen Orthogonalzykel O. Dieser liegt mit A und B in einem Büschel. Die Durchdringungspfeile von A und B sind daher identisch mit denen von A und O. Die Aufgabe ist also etwas vereinfacht und es gilt als

11. *Zweite (quadratische) Fundamentalkonstruktion der Pfeilgeometrie: Die Durchdringungspfeile zweier orthogonaler Zykeln zu zeichnen.*

Sie ist in dem erklärten Gebiet, in dem es nur reelle Pfeile gibt, natürlich nicht immer lösbar.

§ 9. *Die Hauptarten Nicht-Euklidischer Geometrie der Ebene.*

Um der Pfeilgeometrie eine Stelle im Ganzen der Geometrie zuzuweisen, müssen wir etwas weiter ausholen und z. T. bekannte Dinge ausführen. Dabei werden sich dann gleichzeitig manche der eingeführten Benennungen motivieren.

Die Gesamtheit der eigentlichen reellen Punkte der Ebene kann bekanntlich auf zwei verschiedene Arten zu einem abgeschlossenen Kontinuum ergänzt werden.

A. Zunächst zum *projektiven* Kontinuum. Ein Punkt wird durch *drei* homogene *reelle* Koordinaten $x_1 : x_2 : x_3$ dargestellt. Die zugehörige *achtgliedrige* Transformationsgruppe ist *kontinuierlich:*

$$x'_i = c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3, \qquad (i = 1, 2, 3).$$

B. Das zweite Kontinuum wird dadurch erhalten, dass man das erste doppelt überdeckt:

Zyklisches Kontinuum.* Es hat den Zusammenhang einer singularitätenfreien Fläche zweiter Ordnung, die reelle Punkte, aber keine reellen Erzeugenden besitzt.

Der Punkt des zyklischen Kontinuums kann daher durch *eine komplexe* Zahl $\xi = \xi_1 : \xi_2$ dargestellt werden. Die zugehörige Gruppe ist *sechsgliedrig* (*zyklische* Gruppe = Gruppe der MÖBIUSschen Kreisverwandtschaften). Sie ist *gemischt* und besteht aus zwei getrennten Transformationsscharen:

$$\xi'_1 = a_{11}\xi_1 + a_{12}\xi_2, \qquad \xi'_1 = b_{11}\bar{\xi}_1 + b_{12}\bar{\xi}_2,$$
$$\xi'_2 = a_{21}\xi_1 + a_{22}\xi_2. \qquad \xi'_2 = b_{21}\bar{\xi}_1 + b_{22}\bar{\xi}_2.$$

Zyklische Transformationen. *Antizyklische* Transformationen.†

Die zyklische Gruppe ist enthalten in der unendlichen konformen Gruppe:

$$\xi' = f(\xi). \qquad \xi' = f(\bar{\xi}).$$

Konforme Transformationen. *Antikonforme* Transformationen.

A. Man betrachtet nun gewisse Untergruppen der *projektiven* Gruppe, indem man ein absolutes Gebilde (einen irreduziblen *Kegelschnitt*) einführt, und nach den Transformationen fragt, die es in Ruhe lassen. Diese bilden eine *kontinuierliche*‡ dreigliedrige Untergruppe und sollen *Bewegungen* genannt werden.

---

* Es ist das Kontinuum, mit dem man es in der Theorie der Funktionen einer komplexen Veränderlichen zu tun hat. Man könnte es vielleicht besser als *sphärisch* bezeichnen. Ebenso berechtigt wäre dann aber, wie wir noch sehen werden, der Name *pseudosphärisches* Kontinuum. In unserm Zusammenhang ist die im Text eingeführte Bezeichnung vorzuziehen.

† Durch $\bar{\xi}$ soll, wie üblich, der zu ξ konjugiert-komplexe Wert bezeichnet werden.

‡ Es ist zweckmässig, als *gemischt* nur solche Gruppen zu bezeichnen, deren Erweiterung ins komplexe Gebiet noch gemischt ist.

Der absolute Kegelschnitt besitzt keine reellen Punkte:

$$x_1^2 + x_2^2 + x_3^2 = 0.$$

Elliptische Bewegungen:

$$\dot{x}' = \underline{\dot{\alpha}} \cdot \dot{x} \cdot \dot{\alpha}.$$

Der absolute Kegelschnitt besitzt reelle Punkte:

$$x_1^2 - x_2^2 - x_3^2 = 0.$$

Hyperbolische Bewegungen:

$$\dot{x}' = \underline{\dot{\alpha}} \cdot \dot{x} \cdot \dot{\alpha}.$$

Dabei bedeutet:

$$\dot{\alpha} = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3,$$

$$\underline{\dot{\alpha}} = \alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3,$$

$$\dot{x} = \quad * \quad x_1 e_1 + x_2 e_2 + x_3 e_3.$$

Das benutzte System komplexer Zahlen ist

$$\begin{array}{cccc} e_0 & e_1 & e_2 & e_3 \\ e_1 & -e_0 & e_3 & -e_2 \\ e_2 & -e_3 & -e_0 & e_1 \\ e_3 & e_2 & -e_1 & -e_0 \end{array}$$

Erste Gestalt der Quaternionen (Hamiltonsche Quaternionen).

$$\begin{array}{cccc} e_0 & e_1 & e_2 & e_3 \\ e_1 & -e_0 & e_3 & -e_2 \\ e_2 & -e_3 & e_0 & -e_1 \\ e_3 & e_2 & e_1 & e_0 \end{array}$$

Zweite Gestalt der Quaternionen (vgl. § 2).

Die Transformationsdeterminante hat den Wert:

$$(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)^3;$$

Sie ist stets positiv.

Die Gruppe der reellen elliptischen Bewegungen bildet ein einziges Kontinuum.

$$(\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2)^3;$$

Sie kann das Vorzeichen wechseln.

Die Gruppe der reellen hyperbolischen Bewegungen umfasst zwei getrennte Kontinua:

1) $\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 > 0$. *Eigentliche* Bewegungen.

2) $\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 < 0$. *Uneigentliche* Bewegungen.*

B. Entsprechende Überlegungen stellt man für das *zyklische* Kontinuum an. Ein irreduzibler *Kreis* wird als absolutes Gebilde erklärt.† Er bleibt in Ruhe bei einer dreigliedrigen *gemischten* Gruppe. Diese heisst Gruppe der *Bewegungen* und *Umlegungen*. Die Bewegungen sind die *zyklischen*, die Umlegungen die *antizyklischen* Transformationen der fraglichen Eigenschaft.

* Manche Autoren nennen *Bewegungen* nur die, die wir als eigentliche Bewegungen bezeichnen. Unsere uneigentlichen Bewegungen werden öfter mit den Umlegungen der sphärischen Geometrie in Parallele gesetzt, was aber nicht zutreffend ist.

† Besser gesagt: Als absolutes Polarsystem wird das mit einer gewissen (nicht semidefiniten) binären Hermiteschen Form verbundene erklärt.

2. *Pseudosphärische Geometrie:*

$$x_0 : x_1 : x_2 : x_3 = i(\xi_1\bar{\xi}_2 - \xi_2\bar{\xi}_1) : \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 : \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2 : \xi_1\bar{\xi}_2 + \xi_2\bar{\xi}_1 .$$

$$x_0^2 - x_1^2 + x_2^2 + x_3^2 = 0 .$$

Es folgen jetzt unmittelbar folgende Sätze:

Die Gruppe der *sphärischen* Bewegungen ist *holomorph* zur Gruppe der *elliptischen* Bewegungen.

Die Gruppe der *sphärischen* Bewegungen und Umlegungen ist *meromorph* zur Gruppe der *elliptischen* Bewegungen.

Die Gruppe der *pseudosphärischen* Bewegungen ist *holomorph* zur Gruppe der *hyperbolischen* Bewegungen.

Die Gruppe der *pseudosphärischen* Bewegungen und Umlegungen ist *meromorph* zur Gruppe der *hyperbolischen* Bewegungen.

Der Meromorphismus kommt dadurch zustande, dass der Identität zugeordnet wird die ausgezeichnete Transformation

$$\dot{x}' = \underline{\dot{x}},$$

die beide Blätter der Ebene vertauscht.

Ein wesentlicher Unterschied zwischen sphärischer und pseudosphärischer Geometrie besteht in Folgendem:

Die vierte Koordinate x_0, durch deren Einführung aus dem Punkte $x_1 : x_2 : x_3$ des projektiven Kontinuums die beiden *übereinander liegenden* Punkte

$$x_0 : x_1 : x_2 : x_3 \qquad \text{und} \qquad x_0 : -x_1 : -x_2 : -x_3$$

des zyklischen Kontinuums hervorgehen, ist im *sphärischen* Falle *stets* reell; in der pseudosphärischen Geometrie indessen nur, solange $x_1^2 - x_2^2 - x_3^2$ nicht negativ ist.

In der pseudosphärischen Geometrie sind als reell (ausser den Punkten des absoluten Kreises) nur zu bezeichnen die Punkte des doppelt überdeckten "zugänglichen" Gebietes.

Während also die *sphärische Ebene* das ganze zyklische Kontinuum erfüllt, besteht die *pseudosphärische Ebene, die denselben Zusammenhang besitzt,* nur aus den Punkten einer doppelt zählenden kreisförmigen Scheibe, deren Ränder zusammenhängen.

§ 10. *Parasphärische Geometrie.*

Wir stellen jetzt die Frage, was für eine Geometrie entstehen würde, wenn man das *unzugängliche* Gebiet der hyperbolischen Ebene doppelt überdeckt, wobei wieder der absolute Kegelschnitt Verzweigungsmannigfaltigkeit sein soll.

Wir definieren eine vierte Koordinate x_0 jetzt durch die Gleichung

$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0 .$$

Der absolute Kreis besitzt keine reellen Punkte:

$$\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 = 0.$$

Sphärische Bewegungen:

$$\xi_1' = (\alpha_0 + i\alpha_3)\xi_1 + (\alpha_2 + i\alpha_1)\xi_2,$$
$$\xi_2' = (-\alpha_2 + i\alpha_1)\xi_1 + (\alpha_0 - i\alpha_3)\xi_2.$$

Sphärische Umlegungen:

$$\xi_1' = (-\alpha_2 - i\alpha_1)\bar{\xi}_1 + (\alpha_0 + i\alpha_3)\bar{\xi}_2,$$
$$\xi_2' = (-\alpha_0 + i\alpha_3)\bar{\xi}_1 + (-\alpha_2 + i\alpha_1)\bar{\xi}_2.$$

Der absolute Kreis besitzt reelle Punkte:

$$i(\xi_1\bar{\xi}_2 - \xi_2\bar{\xi}_1) = 0.$$

Pseudosphärische Bewegungen:

$$\xi_1' = (\alpha_0 - \alpha_3)\xi_1 + (\alpha_1 + \alpha_2)\xi_2,$$
$$\xi_2' = (\alpha_2 - \alpha_1)\xi_1 + (\alpha_0 + \alpha_3)\xi_2.$$

Pseudosphärische Umlegungen:

$$\xi_1' = (\alpha_0 - \alpha_3)\bar{\xi}_1 + (\alpha_1 + \alpha_2)\bar{\xi}_2,$$
$$\xi_2' = (\alpha_2 - \alpha_1)\bar{\xi}_1 + (\alpha_0 + \alpha_3)\bar{\xi}_2.$$

Die $\alpha_i (i = 0, 1, 2, 3)$ sind *reell* zu nehmen.

C. Man stellt sich jetzt die Aufgabe, den Punkt ξ des zyklischen Kontinuums durch *vier reelle* homogene Koordinaten $x_0 : x_1 : x_2 : x_3$ so darzustellen, dass der absolute Kreis der sphärischen (pseudosphärischen) Geometrie übergeht in den absoluten Kegelschnitt der elliptischen (hyperbolischen) Geometrie, dass ferner übereinander liegende Punkte beider Blätter der Ebene dargestellt werden durch die Systeme:

$$x_0 : x_1 : x_2 : x_3 \qquad \text{und} \qquad x_0 : -x_1 : -x_2 : -x_3.$$

Zwischen den x besteht dann eine quadratische Relation; das absolute Gebilde erscheint als Verzweigungsmannigfaltigkeit.* Setzt man jetzt:

$$\dot{x} = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_2,$$

so ergeben sich die einfachen Formeln:

Bewegungen: $\quad x' = \underline{\dot{\alpha}} \cdot \dot{x} \cdot \alpha,$

Umlegungen: $\quad \dot{x}' = \underline{\dot{\alpha}} \cdot \underline{\dot{x}} \cdot \dot{\alpha}.$

Bei Benutzung der ersten (zweiten) Gestalt der Quaternionen erhält man die sphärischen (pseudosphärischen) Bewegungen und Umlegungen.

Die x hängen mit den ξ und unter sich durch folgende Gleichungen zusammen:

1. *Sphärische Geometrie:*

$$x_0 : x_1 : x_2 : x_3 = \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 : \xi_1\bar{\xi}_2 + \xi_2\bar{\xi}_1 : i(\xi_1\bar{\xi}_2 - \xi_2\bar{\xi}_1) : \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2.$$

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0.$$

* Diesen Sachverhalt verkennt Hr. F. Kölmel: *Bewegungen und Umlegungen der Ebene bei projektiver Maassbestimmung*, Lahr bei Schauenburg, 1900, pag. 18. Wir weisen darauf hin, da Hr. Kölmel seine Behauptung, der absolute Kegelschnitt sei ebenfalls doppelt überdeckt, halten zu können glaubt. Auch in einer Strassburger Dissertation [A. Magener, *Anallagmatische Punktkoordinaten im Kugelgebüsch und ihre Anwendung auf die Nicht-Euklidische Geometrie*, 1906] werden tatsächlich bestehende Unterschiede verwischt. Der Verfasser hält es für unzulässig, die Punkte der von ihm zusammengefassten Punktepaare einzeln zu betrachten (!)

Ein Punkt $x_0 : x_1 : x_2 : x_3$ dieses Kontinuums (der *parasphärischen Ebene*, wie wir sagen wollen) kann dann durch *zwei reelle* Parameter (λ, μ) in folgender Weise dargestellt werden.

$$-x_0 : x_1 : x_2 : x_3 = \lambda_1\mu_2 - \lambda_2\mu_1 : \lambda_2\mu_2 + \lambda_1\mu_1 : \lambda_2\mu_2 - \lambda_1\mu_1 : \lambda_1\mu_2 + \lambda_2\mu_1 .$$

Die "parasphärischen Bewegungen und Umlegungen" werden, wenn man sich der Koordinaten x bedient, durch genau dieselben Gleichungen geliefert, wie die pseudosphärischen:

$$\dot{x}' = \underline{\dot{\alpha}} \cdot \dot{x} \cdot \dot{\alpha}, \qquad \dot{x}' = \underline{\dot{\alpha}} \cdot \underline{\dot{x}} \cdot \dot{\alpha}.$$

In den Parametern (λ, μ) lauten sie:

$$\lambda_1' = (\alpha_0 + \alpha_3)\lambda_1 + (\alpha_2 - \alpha_1)\lambda_2, \qquad \mu_1' = (\alpha_0 + \alpha_3)\mu_1 + (\alpha_2 - \alpha_1)\mu_2,$$
$$\lambda_2' = (\alpha_2 + \alpha_1)\lambda_1 + (\alpha_0 - \alpha_3)\lambda_2, \qquad \mu_2' = (\alpha_2 + \alpha_1)\mu_1 + (\alpha_0 - \alpha_3)\mu_2.$$

(Bewegungen)

$$\lambda_1' = (\alpha_0 + \alpha_3)\mu_1 + (\alpha_2 - \alpha_1)\mu_2, \qquad \mu_1' = (\alpha_0 + \alpha_3)\lambda_1 + (\alpha_2 - \alpha_1)\lambda_2,$$
$$\lambda_2' = (\alpha_2 + \alpha_1)\mu_1 + (\alpha_0 - \alpha_3)\mu_2, \qquad \mu_2' = (\alpha_2 + \alpha_1)\lambda_1 + (\alpha_0 - \alpha_3)\lambda_2.$$

(Umlegungen)

Die Übereinstimmung ist aber nicht nur formal; die parasphärischen Bewegungen und Umlegungen unterscheiden sich von den pseudosphärischen vielmehr nur durch die Objekte, nämlich durch die verschiedenen Ungleichungen, denen die zu transformierenden Veränderlichen zu genügen haben, während die Transformationsformeln dieselben sind.

Diese Übereinstimmung erstreckt sich aber nicht mehr auf umfassendere Gruppen. Die parasphärischen Bewegungen und Umlegungen lassen sich nämlich zu einer sechsgliedrigen Gruppe erweitern, (*parazyklische Gruppe*) die von der MOEBIUSschen, von uns *zyklisch* genannten Gruppe ganz verschieden ist.

Die *zyklische* Gruppe ist nämlich holomorph zu folgender Gruppe:

$$\dot{x}' = \underline{\dot{\alpha}} \cdot \dot{x} \cdot \dot{\beta}, \qquad \dot{x}' = \underline{\dot{\alpha}} \cdot \underline{\dot{x}} \cdot \dot{\beta}.$$

Zyklische Transformationen. *Antizyklische* Transformationen.

Dabei hat man $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ zu setzen und sich der *ersten* Gestalt der Quaternionen zu bedienen.*

Wählen wir nun wieder die zweite Gestalt der Quaternionen, setzen jetzt aber

$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0,$$

* Deutet man die x in den letzten Gleichungen als homogene Punktkoordinaten im R_3, so stellen diese Gleichungen die Gruppe der elliptischen Bewegungen und Umlegungen dar. Doch ist der Zusammenhang dieser Gruppe mit den sphärischen (pseudosphärischen) Bewegungen und Umlegungen viel komplizierter als bei den entsprechenden Verhältnissen der parazyklischen Gruppe, wie wir hier nicht näher ausführen können.

so liefern die Gleichungen die *parazyklische* Gruppe, die in den (λ, μ) durch die Gleichungen (9) und (20) ausgedrückt wird. Die beiden Transformationsscharen der Gruppe werden konsequent folgendermassen benannt:

$$\dot{x}' = \underline{\dot{\alpha}} \cdot \dot{x} \cdot \dot{\beta},$$

Parazyklische Punkttransformationen.

$$\dot{x}' = \underline{\dot{\alpha}} \cdot \underline{\dot{x}} \cdot \dot{\beta}.$$

Antiparazyklische Punkttransformationen.

Dem Aufbau der zur parazyklischen Gruppe gehörigen Geometrie stehen Schwierigkeiten nicht entgegen. Man wird es aber als Mangel empfinden, dass sie sich im unzugänglichen Gebiet abspielt. Diesen Mangel beseitigt die absolute Korrelation, indem sie aus der Gesamtheit der doppelt überdeckten *unzugänglichen Punkte* hervorgehen lässt die Gesamtheit der doppelt überdeckten *zugänglichen Geraden.* Die doppelte Überdeckung des Geradengebietes vollzieht sich nun vermöge des *Orientierungs*prozesses; die Gerade wird durch zwei eigentliche Pfeile (§ 1) ersetzt. Die Pfeilkoordinaten (§ 1) ergeben sich einfach:

$$\mathfrak{x}_0 : \mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3 = -x_0 : -x_1 : x_2 : x_3.$$

Die Pfeilgeometrie erweist sich somit als Bild einer Punktgeometrie, die an und für sich wenig Interesse zu bieten schien, andererseits aber, obwohl sie nicht die reelle Euklidische Geometrie zur Grenzlage hat, doch eine Lücke der vorhandenen Systeme Nicht-Euklidischer Geometrie ausfüllt.—

Unsere Pfeilgeometrie ist ferner eine von drei zusammengehörigen Disziplinen, die als *elliptische*, *hyperbolische* und *Euklidische Pfeilgeometrie* unterschieden werden können. Die hier allein behandelte würde als *hyperbolische* zu bezeichnen sein. Zur elliptischen Pfeilgeometrie gelangt man durch folgende Überlegungen:

Man kann auf der Kugel eine gegenseitig-eindeutige Zuordnung vornehmen zwischen den Punkten und den orientierten Hauptkreisen. Irgend ein Punkt liegt zu dem ihm entsprechenden orientierten Hauptkreis so, wie der Nordpol der Erdkugel zu dem von West nach Ost durchlaufenen Aequator. Gehen wir nun durch Zentralprojektion zur projektiv abgeschlossenen und dann doppelt überdeckten Ebene über, so werden die orientierten Hauptkreise zu orientierten Geraden, *Pfeilen.* Ein Pfeil kann dann also durch *einen komplexen* Parameter dargestellt werden. Dieses *elliptische* Pfeilkontinuum besitzt aber ganz anderen Zusammenhang wie das hyperbolische. Es gibt keine Verzweigungsmannigfaltigkeit. Ein Analogon zu den Punktpfeilen gibt es (im reellen Gebiet) nicht.

Die Gruppe der zyklischen Punkttransformationen geht über in eine Gruppe von zyklischen *Pfeil*transformationen. Die Formeln sind eben nur anders zu deuten.

Als Bewegungen und Umlegungen sind zu erklären diejenigen zyklischen bezw. antizyklischen Pfeiltransformationen, die entgegengesetzte Pfeile wieder in ebensolche überführen.

An die Stelle der bilinearen Formen in Pfeilparametern, deren Nullstellen in der hyperbolischen Pfeilgeometrie die Zykeln lieferten, treten binäre Hermitesche Formen. Den definiten Formen entsprechen *nullteilige* Zykeln, d. i. solche, die keine reellen Pfeile besitzen. Von den reduziblen Zykeln ist nur der Mittelpfeil übrig geblieben.

Verloren geht die Umkehrbarkeit der Beziehung zwischen Bewegungen und irreduziblen Zykeln. Ebenso verschwindet der Begriff: Parallelismus von Pfeilen und die daraus folgenden Tatsachen, z. B. auch die einfache Konstruktion der Spiegelung am Zykel.

Man darf daher wohl die *hyperbolische* Pfeilgeometrie als die interessantere erklären.

Beide Pfeilgeometrieen lassen sich nun ins komplexe Gebiet fortsetzen. Diese ihre Erweiterungen lassen sich dann geradezu identifizieren, wie wir noch kurz andeuten wollen.

Hyperbolische Geometrie:

Absoluter Kegelschnitt reell *mit reellen Punkten.*

$$x_1^2 - x_2^2 - x_3^2 = 0\,,$$

$$x_1 : x_2 : x_3 = \lambda_1^2 + \lambda_2^2 : \lambda_2^2 - \lambda_1^2 : 2\lambda_1\lambda_2\,,$$

Elliptische Geometrie:

Absoluter Kegelschnitt reell *ohne reelle Punkte.*

$$x_1^2 + x_2^2 + x_3^2 = 0\,,$$

$$x_1 : x_2 : x_3 = \lambda_1^2 - \lambda_2^2 : i(\lambda_1^2 + \lambda_2^2) : 2\lambda_1\lambda_2\,,$$

wobei der Parameter $\lambda_1 : \lambda_2$ jetzt alle möglichen *komplexen* Werte durchläuft. Gehört der Punkt y des absoluten Kegelschnitts zum Parameterwert $\mu_1 : \mu_2$, so kann der *komplexe* Pfeil mit dem Anfangspunkt x und dem Endpunkt y dargestellt werden durch die Koordinaten:

H. G. $$\mathfrak{x}_0 : \mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3 = \lambda_1\mu_2 - \lambda_2\mu_1 : -(\lambda_2\mu_2 + \lambda_1\mu_1) : \lambda_2\mu_2 - \lambda_1\mu_1 : \lambda_1\mu_2 + \lambda_2\mu_1\,,$$
$$\mathfrak{x}_0^2 + \mathfrak{x}_1^2 - \mathfrak{x}_2^2 - \mathfrak{x}_3^2 = 0\,.$$

E. G. $$\mathfrak{x}_0 : \mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3 = \lambda_1\mu_2 - \lambda_2\mu_1 : \lambda_1\mu_1 - \lambda_2\mu_2 : i(\lambda_1\mu_1 + \lambda_2\mu_2) : \lambda_1\mu_2 + \lambda_2\mu_1\,,$$
$$\mathfrak{x}_0^2 - \mathfrak{x}_1^2 - \mathfrak{x}_2^2 - \mathfrak{x}_3^2 = 0\,.$$

Die beiden Kontinua komplexer Pfeile lassen sich nun durchaus eindeutig-umkehrbar aufeinander abbilden, am bequemsten dadurch, dass man solche Pfeile einander zuordnet, die zu gleichen Parameterpaaren $\lambda_1 : \lambda_2$, $\mu_1 : \mu_2$ gehören. Wir erklären nun: Ein Pfeil heisst *reell*, wenn seine *Koordinaten* reell sind, d. i.

wenn seine Parameter *einzeln reell* sind:

$$\lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1 = 0, \quad \mu_1\bar{\mu}_2 - \mu_2\bar{\mu}_1 = 0.$$

Kontinuum der reellen *hyperbolischen* Pfeilgeometrie.

wenn seine Parameter *entgegengesetzt, reziprok* und *konjugiert-komplex* sind:

$$\lambda_1\bar{\mu}_1 + \lambda_2\bar{\mu}_2 = 0\,.$$

Kontinuum der reellen *elliptischen* Pfeilgeometrie.

Man erhält nun noch ein drittes ebenes Pfeilkontinuum, indem man nur die *eigentlichen* Geraden der Ebene doppelt überdeckt und den so gewonnenen *eigentlichen* Pfeilen die *uneigentliche* (unendlich ferne) *Gerade* als *uneigentlichen Pfeil* hinzufügt. Die zugehörige Geometrie mit *siebengliedriger* Gruppe umfasst die elementare Euklidische. Das neue Pfeilkontinuum darf daher als *Euklidisches* bezeichnet werden. Es hat den Zusammenhang eines einteiligen irreduziblen *Kegels* zweiter Ordnung. Auf weitere Mannigfaltigkeiten, die als zur Euklidischen Geometrie gehörige Pfeilkontinua erklärt werden können, wollen wir hier nicht eingehen. Das hier erwähnte findet man behandelt in der Dissertation von W. BLASCHKE.*

§ 11. *Die aequidistanten Pfeiltransformationen.*

Um den Zusammenhang zwischen der MOEBIUSschen Geometrie und der parazyklischen bis zu einem gewissen Grade zu erschöpfen, werfen wir noch einen kurzen Blick auf die Differentialgeometrie im Pfeilgebiet der hyperbolischen Ebene und zeigen das Vorhandensein einer *unendlichen* Gruppe von Pfeiltransformationen, die dieselbe Rolle spielt, wie in der MOEBIUSschen Geometrie die *konforme* Gruppe.

Wir führen *inhomogene* Pfeilparameter ein:

$$\lambda = \frac{\lambda_1}{\lambda_2}, \qquad \mu = \frac{\mu_1}{\mu_2}.$$

Dadurch erleiden die folgenden Entwicklungen Einschränkungen, mit deren Angabe wir uns aber nicht aufhalten wollen.

Ein Pfeil (λ, μ) *schneidet* einen benachbarten $(\lambda + d\lambda, \mu + d\mu)$, wenn $d\mu : d\lambda$ positiv ist. Zwei dem Pfeil (λ, μ) benachbarte Pfeile $(\lambda + d_1\lambda, \mu + d_1\mu)$ und $(\lambda + d_2\lambda, \mu + d_2\mu)$ bestimmen eine Strecke η:

$$\operatorname{tgh} \eta = \frac{d_1\lambda \cdot d_2\mu - d_2\lambda \cdot d_1\mu}{d_1\lambda \cdot d_2\mu + d_2\lambda \cdot d_1\mu}.$$

Ist $d_1\mu : d_1\lambda > 0$, $d_2\mu : d_2\lambda > 0$, so ist η die Länge der Strecke, die auf dem ursprünglichen Pfeil von den beiden benachbarten abgeschnitten wird.

Für $d_1\mu : d_1\lambda < 0$, $d_2\mu : d_2\lambda < 0$ bestimmt jeder benachbarte Pfeil mit dem ursprünglichen eine gemeinsame Normale; η ist dann der Abstand dieser beiden Geraden.

Ist endlich $d_1\mu : d_1\lambda > 0$, $d_2\mu : d_2\lambda < 0$, und bezeichnet η' den Abstand des Schnittpunktes $(\lambda, \mu; \lambda_1 + d_1\lambda, \mu + d_1\mu)$ von der gemeinsamen Normalen $(\lambda, \mu; \lambda + d_2\lambda, \mu + d_2\mu)$, so ist

$$\frac{d_1\lambda \cdot d_2\mu - d_2\lambda \cdot d_1\mu}{d_1\lambda \cdot d_2\mu + d_2\lambda \cdot d_1\mu} = \coth \eta'.$$

* W. BLASCHKE, *Untersuchungen über die Geometrie der Speere in der Euklidischen Ebene;* Monatshefte für Mathematik und Physik, 21. Jahrgang (1910). Herr BLASCHKE war so liebenswürdig, zu unserm Aufsatze die Figuren zu zeichnen.

Man erhält nun sofort den interessanten Satz:

Jede Pfeiltransformation

$$\lambda' = F(\lambda), \qquad \mu' = \phi(\mu), \quad (\textit{parakonforme} \text{ Transformation})$$

die α-parallele (ω-parallele) Pfeile in ebensolche überführt, ebenso jede Pfeiltransformation

$$\lambda' = F(\mu), \qquad \mu' = \phi(\lambda), \quad (\textit{antiparakonforme} \text{ Transformation})$$

die α-Parallelismus und ω-Parallelismus vertauscht, ist aequidistant.

Dabei bedeuten F und ϕ irgend welche differenzierbaren Funktionen von λ bezw. μ; aber sie müssen umkehrbar sein, d. i. es müssen durch die beiden im letzten Satze angegebenen Gleichungen wirklich nicht-ausgeartete Transformationen geliefert werden. Diese Transformationen lassen also den Abstand zweier Punkte (Punktzykel) invariant.

Man kann die eben erklärte, *parakonforme* Gruppe noch anders darstellen, wobei der Zusammenhang mit der konformen Gruppe evident wird.

Dazu bedienen wir uns eines komplexen Zahlsystems in zwei Haupteinheiten e_λ und e_μ, dessen Multiplikationsregel durch folgende Gleichungen definiert ist:

$$e_\lambda^2 = e_\lambda, \qquad e_\lambda \cdot e_\mu = e_\mu \cdot e_\lambda = 0, \qquad e_\mu^2 = e_\mu.$$

Dann bilden wir aus Pfeilparametern die komplexen Zahlen

$$z = \lambda \cdot e_\lambda + \mu \cdot e_\mu, \qquad \bar{z} = \mu \cdot e_\lambda + \lambda \cdot e_\mu,$$

deren zweite als zur ersten konjugiert-komplex bezeichnet werden darf. Funktionen dieser komplexen Variabeln werden definiert als Summen von Potenzreihen. Dann lässt sich die parakonforme Gruppe einfach schreiben:

$$z' = f(z), \qquad z' = f(\bar{z}).$$

Auch diese Schreibweise wollen wir noch etwas modifizieren. Setzt man nämlich

$$e_\lambda + e_\mu = 1, \qquad 2\xi = \lambda + \mu,$$

$$e_\lambda - e_\mu = \epsilon, \qquad 2\eta = \lambda - \mu,$$

so erhalten wir:

$$\xi' + \epsilon\eta' = f(\xi + \epsilon\eta), \qquad \xi' + \epsilon\eta' = f(\xi - \epsilon\eta),$$

Parakonforme Transformationen. *Antiparakonforme* Transformationen.

$$\epsilon^2 = +1.$$

Ebenso wie die konforme Gruppe von Punkttransformationen *winkeltreu* ist, ist also die parakonforme Gruppe von Pfeiltransformationen in gewissem Sinne *längentreu.*

In der elliptischen Pfeilgeometrie gedeutet, ist die konforme Gruppe aber ebenfalls längentreu.

Ein korrekter Ausbau dieser hier nur oberflächlich gestreiften Dinge bietet keinerlei Schwierigkeiten. Bemerkt werden muss aber, dass die Benutzung der hier aufgetretenen komplexen Grössen nicht immer vorteilhaft ist.

Wir können jetzt die Pfeilgeometrie in ihrer ganzen Struktur erkennen und stellen die aufgetretenen Gruppen denen der MOEBIUSschen Geometrie gegenüber.

2. ∞^∞ konforme Gruppe.	2. ∞^∞ Parakonforme Gruppe.
Konforme, Antikonforme } Trsf.	Parakonforme, Antiparakonforme } Trsf.
2. ∞^6 Zyklische Gruppe.	8. ∞^6 Parazyklische Gruppe.
Zyklische, Antizyklische } Trsf.	Parazyklische, Antiparazyklische } Trsf.

2. ∞^3	$2 \cdot 2.\ \infty^3$	4. ∞^3
Sphärische Geometrie.	Pseudosphärische Geometrie.	Parasphärische Geometrie.
Bewegungen	Bewegungen	Bewegungen
Umlegungen	Umlegungen	Umlegungen
∞^3	2. ∞^3	
Elliptische Geometrie.	Hyperbolische Geometrie.	
Bewegungen	Bewegungen	

Die aequidistanten Transformationen der Euklidischen Ebene können aus der parakonformen Gruppe durch einen Grenzübergang erhalten werden. Die fragliche unendliche Gruppe ist von Herrn G. SCHEFFERS bestimmt worden; sie führt auf das dritte System komplexer Zahlen in zwei Einheiten, welches von Herrn E. STUDY in der Geometrie der Dynamen verwandt ist.

HANNOVER, 2. *Mai*. 1909.

VECTOR INTERPRETATION OF SYMBOLIC DIFFERENTIAL PARAMETERS*

BY

LOUIS INGOLD

I. Introduction.

The purpose of this paper is to establish a relation between the symbolic theory of invariants of differential forms, due to the late Professor Maschke,† and the theory of extensive quantity (vectors), due to Grassmann. ‡

It will be shown that those symbolic expressions used by Maschke which lacked an interpretation in his theory, may be represented as vectors of the Grassmann type, and that all of Maschke's expressions, including his actual differential parameters, are expressible in the vector system. The theory of such vectors will be extended, new formulas in the symbolic theory will be obtained and applications to geometry will be made.

A preliminary section on space of two dimensions is given, in order that the well known geometric representation of vectors in two dimensions may serve as a basis for a compact treatment of the general theory.

The general theory is stated for three dimensions in concise form, proofs being omitted when their details are simple extensions of the two dimensional case; the obvious generalization to Euclidean space of n dimensions is then indicated.

The less obvious extension to surfaces in three dimensions and, generally, to arbitrary k-dimensional spaces imbedded in an Euclidean space of n dimensions is treated in more detail. This involves an application of the Grassmann theory to non-uniform spaces and furnishes an example of a vector system based on variable units.

The paper closes with a section dealing with differential parameters which involve second and higher derivatives of the symbols.

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* Presented to the Society (Chicago), March 30, 1907, in somewhat different form, under the title: *Vector theory, in terms of symbolic differential parameters.*

† Maschke, *A Symbolic Treatment of the Theory of Invariants of Quadratic Differential Quantics of n Variables*, Transactions of the American Mathematical Society, vol. 4 (1903), pp. 445–469.

‡ H. Grassmann, *Ausdehnungslehre*, 1862. References are to Grassmann's collected works edited by F. Engel.

II. The Two-dimensional Case.

1. *Comparison of the properties of vectors and symbols.* Given any binary quadratic differential form,

$$A \equiv E(x, y)\,dx^2 + 2F(x, y)\,dxdy + G(x, y)\,dy^2, \tag{1}$$

Maschke replaces the form by a symbolic square

$$A \equiv [f_{(1)}(x, y)\,dx + f_{(2)}(x, y)\,dy]^2 * \tag{2}$$

and writes symbolically

$$f_{(1)}^2(x, y) = E(x, y);\ f_{(1)}(x, y) f_{(2)}(x, y) = F(x, y);\ f_{(2)}^2(x, y) = G(x, y). \tag{3}$$

In the special case $E = G = 1$, $F = 0$, the differential form represents the square of the element of length in a plane; we have then

$$A = ds^2 = dx^2 + dy^2 = (f_{(1)}\,dx + f_{(2)}\,dy)^2, \tag{4}$$

where $f_{(1)}$ and $f_{(2)}$ are constant, and where

$$f_{(1)}^2 = 1, \qquad f_{(1)} f_{(2)} = 0, \qquad f_{(2)}^2 = 1. \tag{5}$$

The formulas (5) suggest the properties of so-called *inner* multiplication of a set of unit vectors in a plane. Thus if $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors along the x- and the y-axis respectively, and if $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2$ and $\boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2$ are any vectors, the *inner* product is

$$[\boldsymbol{a} \mid \boldsymbol{b}] = a_1 b_1 + a_2 b_2.\dagger \tag{6}$$

Then for inner multiplication

$$[\boldsymbol{e}_1 \mid \boldsymbol{e}_1] = \boldsymbol{e}_1^2 = 1, \qquad [\boldsymbol{e}_2 \mid \boldsymbol{e}_2] = \boldsymbol{e}_2^2 = 1, \qquad [\boldsymbol{e}_1 \mid \boldsymbol{e}_2] = [\boldsymbol{e}_2 \mid \boldsymbol{e}_1] = 0. \tag{7}$$

On account of this agreement, the *Maschke symbols* $f_{(1)}$ *and* $f_{(2)}$ *may be*

* Maschke, loc. cit., p. 448.

The general quadratic differential form

$$\sum_{i,j=1}^{n} a_{ij}\,dx_i\,dx_j$$

in n variables $x_1, x_2, \cdots, x_n$ is replaced by $[f_{(1)}\,dx_1 + f_{(2)}\,dx_2 + \cdots + f_{(n)}\,dx_n]^2$, so that symbolically $f_{(i)} f_{(j)} = a_{ij}$; for the special differential form

$$\sum_{i=1}^{n} dx_i^2$$

these equations become

$$f_{(i)} f_{(j)} = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases}$$

† The inner product is also defined to be $ab \cos \widehat{\boldsymbol{ab}}$, where a means the length of $\boldsymbol{a}$, i. e., $\sqrt{a_1^2 + a_2^2}$, $b = \sqrt{b_1^2 + b_2^2}$, and $\widehat{\boldsymbol{ab}}$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. This definition is equivalent to the one given in the text.

thought of as a set of unit vectors $\mathbf{e}_1$ *and* $\mathbf{e}_2$, *the multiplications in* (5) *being inner multiplications.*

It will be shown later that this correspondence of properties persists between the Maschke symbols and the vectors of Grassmann's theory in space of n dimensions, every essential property being identical in the two cases.

We proceed just now to show that many — in fact all — of the various symbolic expressions used by Maschke for two dimensions have a meaning in the vector theory, if we set $f_{(1)} = \mathbf{e}_1$ and $f_{(2)} = \mathbf{e}_2$ as suggested above; and that the relations existing between his symbolic expressions correspond to geometric relations.

2. *Notations.* It is found convenient in the symbolic theory to introduce other symbols, $\phi_{(1)}$, $\phi_{(2)}$ equivalent to $f_{(1)}, f_{(2)}$, in the sense that

$$\phi_{(1)}^2 = E, \qquad \phi_{(1)}\phi_{(2)} = F, \qquad \phi_{(2)}^2 = G,$$

according to equations (3). With the aid of two sets of symbols, $f_{(1)}, f_{(2)}$; $\phi_{(1)}, \phi_{(2)}$, expressions quadratic in the E, F, G may be represented symbolically; thus,

$$E^2 = f_{(1)}^2\phi_{(1)}^2, \qquad EF = f_{(1)}^2\phi_{(1)}\phi_{(2)}, \quad 2(EG - F^2) = (f_{(1)}\phi_{(2)} - f_{(2)}\phi_{(1)})^2, \quad \text{etc.}$$

When convenient, still other symbols are introduced, equivalent to f and ϕ, in the sense described above. It is agreed, however, that no more than two symbols of the same kind shall occur in the same product, and that in every product two symbols of like kind, i. e., two f's or two ϕ's, etc., occurring as factors are to be interpreted according to equations (3) [or equations (5) in the special case we are now considering]. Otherwise the symbols are subject to the same laws as ordinary algebraic quantities. The discriminant $EG - F^2$ of the general quadratic form (1) is denoted by $1/\beta^2$.* In general, β is a function of x and y; but $\beta = 1$ for the special differential form (4).

The symbols $f_{(1)}$, $f_{(2)}$ were regarded by Maschke as partial derivatives of a symbolic function $f(x, y)$, so that $f_{(1)} = \partial f/\partial x$, $f_{(2)} = \partial f/\partial y$.†

In the special case (4), since $f_{(1)}$ and $f_{(2)}$ are constant we may write $f = f_{(1)}x + f_{(2)}y$.

The Jacobian

$$\begin{vmatrix} U_{(1)} & V_{(1)} \\ U_{(2)} & V_{(2)} \end{vmatrix}$$

of any two functions U and V is of frequent occurrence and will be denoted briefly by $\{U, V\}$. Still more frequently occurs the expression $\beta\{U, V\}$,

* This notation is also used for the discriminant of the quadratic differential form in n variables.

† In general throughout this paper, subscripts in parentheses will denote partial differentiation. Thus $U(x, y)$ being any function of x and y, $U_{(1)} = \partial U/\partial x$, $U_{(2)} = \partial U/\partial y$. This differs slightly from the notation of MASCHKE, as he used subscripts without parentheses.

β being defined as above; this expression is denoted by (U, V); for our special case, since $\beta = 1$, we have $(U, V) = \{U, V\}$.*

If U and V are invariants of the differential form then (U, V) is again an invariant.†

3. *Differential parameters involving linear functions.* Consider the system of straight lines

$$a(x, y) = a_1 y - a_2 x = \text{const.}$$

parallel to the vector

$$\boldsymbol{a} = a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2.$$

Since $f_{(1)} = \boldsymbol{e}_1$ and $f_{(2)} = \boldsymbol{e}_2$ we may write $\boldsymbol{a} = (f, a)$.‡

Hence, *if a is any linear function, the differential parameter (f, a) may be interpreted as a vector parallel to the system of straight lines $a = const.$*

In particular, $(f, y) = \boldsymbol{e}_1$, $(f, -x) = \boldsymbol{e}_2$.

Other differential parameters expressed in symbolic form are $(f, a)(f, b)$, $(f, \phi)(f, a)$, $(f, \phi)(f, a)(\phi, b) = (a, b)$, where a and b are functions of x and y, and f and ϕ are symbolic functions of the differential form. We proceed to find their vector meaning when a and b are linear.

The product $(f, a)(f, b)$, when expanded, equals $a_1 b_1 + a_2 b_2$, where as above $a = a_1 y - a_2 x$, and $b = b_1 y - b_2 x$.

Comparing with the definition, p. 450, we see that this result is the *inner* product of the two vectors $\boldsymbol{a} = a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2$ and $\boldsymbol{b} = b_1 \boldsymbol{e}_1 + b_2 \boldsymbol{e}_2$. The product $(\phi, f)(\phi, a)$, when expanded, equals $-f_{(1)} a_2 + f_{(2)} a_1$. Interpreting $f_{(1)}, f_{(2)}$ as before, we see that this is a vector perpendicular to $\boldsymbol{a} = (f, a)$; with Grassmann we call this the *complement* § of the vector $\boldsymbol{a}$ and denote it by $|\boldsymbol{a}$; thus

$$(\phi, f)(f, a) = |\boldsymbol{a}.$$

Finally the expression‖

$$(f, \phi)(f, a)(\phi, b) = (a, b) = a_1 b_2 - a_2 b_1$$

is the so-called *outer* product ¶ of $\boldsymbol{a}$ and $\boldsymbol{b}$.

4. *Differential parameters involving arbitrary functions.* We investigate now the meaning of these same differential parameters when the functions a and b are arbitrary functions of x and y.

The equation of the tangent to the curve $a = \text{const.}$ at a given point x_0, y_0 on

* The same notation is used in the n-dimensional case; $\{a_1, \cdots, a_n\}$ represents the Jacobian of the functions $a_1(x_1, \cdots, x_n), \cdots, a_n(x_1, \cdots, x_n)$, and $(a_1, \cdots, a_n) = \beta\{a_1, \cdots, a_n\}$.

† For definitions of "Invariant of a differential quantic," "differential parameter," etc., see MASCHKE, loc. cit., p. 446.

‡ Notice that $a_{(1)} = \partial a / \partial x = -a_2$, and $a_{(2)} = \partial a / \partial y = a_1$.

§ For the general definition of complement see p. 457.

‖ Cf. MASCHKE, loc. cit., p. 453, equation (34).

¶ The outer product $[\boldsymbol{a}, \boldsymbol{b}]$ of two vectors is usually defined to be $A \cdot B \cdot \sin \widehat{\boldsymbol{ab}}$, where A and B are the lengths of $\boldsymbol{a}$ and $\boldsymbol{b}$. This is easily seen to be the same as $a_1 b_2 - a_2 b_1$.

the curve is

$$x\left(\frac{\partial a}{\partial x}\right)_{\substack{x=x_0\\y=y_0}} + y\left(\frac{\partial a}{\partial y}\right)_{\substack{x=x_0\\y=y_0}} = k = \text{const.},$$

or

$$x\,[\,a_{(1)}]_{\substack{x=x_0\\y=y_0}} + y\,[\,a_{(2)}]_{\substack{x=x_0\\y=y_0}} = k.$$

Let

$$L = x\,[\,a_{(1)}]_{\substack{x=x_0\\y=y_0}} + y\,[\,a_{(2)}]_{\substack{x=x_0\\y=y_0}};$$

then

$$(f,\,L) = (f,\,a)_{\substack{x=x_0\\y=y_0}} = f_{(1)}[\,a_{(2)}]_{\substack{x=x_0\\y=y_0}} - f_{(2)}[\,a_{(1)}]_{\substack{x=x_0\\y=y_0}};$$

but, by Art. 3, $(f,\,L)$ is directed parallel to the straight lines $L = \text{const.}$, and these straight lines are parallel to the tangent of that curve of the system $a = \text{const.}$ which passes through the point $x_0,\,y_0$; hence we have

THEOREM I. *If the symbols $f_{(1)}, f_{(2)}$ of the differential form $dx^2 + dy^2$ are interpreted as unit vectors along the x- and y-axes respectively, then the symbolic differential parameter $(f,\,a)$ may be interpreted as a vector function of position, tangent at each point of the plane to the curve $a = \text{const.}$ which passes through that point.*

As in the case where a and b were linear, the product $(f,\,a)(f,\,b)$ may be regarded as the *inner* product of the two vectors represented by $(f,\,a)$ and $(f,\,b)$. The expression $(\phi,\,f)(\phi,\,a)$ represents a vector perpendicular to $(f,\,a)$ and therefore normal to the curve $a = \text{const.}$; and the expression $(f,\,\phi)(f,\,a)(\phi,\,b)$ is the *outer* product of the vectors $(f,\,a)$, $(f,\,b)$.

The length* of the vector represented by $(f,\,a)$ is

$$\sqrt{(f,\,a)^2} = \sqrt{a_{(1)}^2 + a_{(2)}^2}.$$

The angle θ between two curves, $a = \text{const.}$, $b = \text{const.}$, is given by †

$$\cos\theta = \frac{(f,\,a)(f,\,b)}{\sqrt{(f,\,a)^2}\,\sqrt{(\phi,\,b)^2}} = \frac{a_{(1)}b_{(1)} + a_{(2)}b_{(2)}}{\sqrt{a_{(1)}^2 + a_{(2)}^2}\,\sqrt{b_{(1)}^2 + b_{(2)}^2}}$$

or by ‡

$$\sin\theta = \frac{(a,\,b)}{\sqrt{(f,\,a)^2}\,\sqrt{(\phi,\,b)^2}} = \frac{a_{(1)}b_{(2)} - a_{(2)}b_{(1)}}{\sqrt{a_{(1)}^2 + a_{(2)}^2}\,\sqrt{b_{(1)}^2 + b_{(2)}^2}}.$$

5. *Change of variables.* We obtain greater generality for the plane by changing to curvilinear coördinates. Let us make the substitution

$$u = u(x,\,y), \qquad v = v(x,\,y);$$

we have then

$$du = u_{(1)}dx + u_{(2)}dy, \qquad dv = v_{(1)}dx + v_{(2)}dy.$$

* Cf. p. 450, second footnote.

† Cf. p. 450, second footnote.

‡ Cf. p. 452, last footnote.

Solving for dx and dy, we have

$$dx = \frac{v_{(2)}du - u_{(2)}dv}{(u, v)}, \qquad dy = \frac{u_{(1)}dv - v_{(1)}du}{(u, v)}.$$

Substituting these values in the expression for ds, we have

$$ds = f_{(1)}dx + f_{(2)}dy = \frac{(f, v)}{(u, v)}du + \frac{(u, f)}{(u, v)}dv. \tag{8}$$

By Theorem I, the coefficients of du and dv represent vectors tangent at any point u_0, v_0 to the curves $v = \text{const.}$, $u = \text{const.}$, respectively.

Suppose that the vector function $f(x, y)$ transforms into the vector function $t(u, v)$; we have

$$f_{(1)} = t_{(u)}u_{(1)} + t_{(v)}v_{(1)}, \qquad f_{(2)} = t_{(u)}u_{(2)} + t_{(v)}v_{(2)}.$$

Solving for $t_{(u)}$ and $t_{(v)}$, we have

$$t_{(u)} = \frac{(f, v)}{(u, v)}, \qquad t_{(v)} = \frac{(u, f)}{(u, v)}. \tag{9}$$

The quantic giving length of arc

$$ds^2 = (f_{(1)}dx + f_{(2)}dy)^2 = dx^2 + dy^2$$

transforms into

$$(t_{(u)}du + t_{(v)}dv)^2 = Edu^2 + 2Fdudv + Gdv^2, \tag{10}$$

where $t_{(u)}^2 = E$, $t_{(u)}t_{(v)} = F$, $t_{(v)}^2 = G$, so that the function $t(u, v)$ into which $f(x, y)$ transforms is a symbolic function of the transformed differential quantic; hence

THEOREM II.* *If $A \equiv Edu^2 + 2Fdudv + Gdv^2$ is the differential form giving length of arc in a plane, the parameters u and v being arbitrary, then the derivatives $t_{(u)}$, $t_{(v)}$ of a symbolic function of the form A may be interpreted as vector functions of position, of length $\sqrt{E}$, $\sqrt{G}$, respectively, tangent at each point of the plane to the parameter curves which pass through that point.*

This gives rise to a vector system based on two *units*,

$$\boldsymbol{e}_1 = t_{(u)}, \qquad \boldsymbol{e}_2 = t_{(v)},$$

tangent to the parameter curves $v = \text{const.}$, $u = \text{const.}$, respectively; these *units* are themselves variable from point to point, both in length and in direction.

Again, remembering the factor $\beta = (EG - F^2)^{-\frac{1}{2}}$, we have

$$(f, a) = (t, a), \qquad (f, a)(f, b) = (t, a)(t, b),\dagger \quad \text{etc.}, \tag{11}$$

where the expressions on the right involve derivatives with respect to x and

* Cf. Theorem V.

† Since (f, a), (f, b), etc., are invariant expressions. See MASCHKE, loc. cit., p. 449.

y, while the expressions on the left involve derivatives with respect to u and v. These expressions, therefore, have always the same geometric interpretation, even when the parameters u, v are changed. In particular, the expression $(t, \tau)(t, a)$, where t and τ are equivalent symbols of the quantic

$$Edu^2 + 2Fdudv + Gdv^2,$$

still represents the *complement* of the vector (t, a), so that in this notation the operation of taking the complement is invariant.*

III. Vectors in Three Dimensions; Axioms.

6. *Fundamental notions.* In order to extend the preceding work to space of three dimensions, we shall explain briefly a vector system in that space, following in spirit Grassmann.

Let us start with three mutually perpendicular linear vectors, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, which are of unit length and lie along the three axes; equivalence of such linear vectors, their addition and their multiplication follow the rules for two dimensions. The expression $a\boldsymbol{e}_1 + b\boldsymbol{e}_2 + c\boldsymbol{e}_3$, where a, b, c are ordinary numbers, represents a unique vector defined by these rules, namely, the vector from the origin to the point a, b, c, or any parallel vector of equal length; conversely, any vector in space is expressible in that form.

In order to multiply vectors, we may proceed as in ordinary algebra, provided the fundamental products of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in pairs are defined; we shall introduce three of these products

$$(12) \qquad \boldsymbol{E}_1 = [\boldsymbol{e}_2, \boldsymbol{e}_3], \qquad \boldsymbol{E}_2 = [\boldsymbol{e}_3, \boldsymbol{e}_1], \qquad \boldsymbol{E}_3 = [\boldsymbol{e}_1, \boldsymbol{e}_2]$$

as new units, and call them *vectors of the second order.* Each may be thought of geometrically as a rectangle of which the factor vectors are sides, or as any equivalent area on any parallel plane. The other products are defined to be

$$(13) \quad [\boldsymbol{e}_3, \boldsymbol{e}_2] = -\boldsymbol{E}_1 = -[\boldsymbol{e}_2, \boldsymbol{e}_3], \qquad [\boldsymbol{e}_1, \boldsymbol{e}_3] = -\boldsymbol{E}_2, \qquad [\boldsymbol{e}_2, \boldsymbol{e}_1] = -\boldsymbol{E}_3.$$

The products used in (12) and (13) are called *outer* products. It is to be noted that the area of the rectangles mentioned is equal to the product of the lengths of the sides into the sine of the included angle,—a rule given in two dimensions on p. 452, footnote. This rule applied to the remaining products gives

$$(14) \qquad [\boldsymbol{e}_1, \boldsymbol{e}_1] = [\boldsymbol{e}_2, \boldsymbol{e}_2] = [\boldsymbol{e}_3, \boldsymbol{e}_3] = 0.$$

The outer product of *any* two vectors is obtained in terms of $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$ by

* The Grassmann definition of *complement* is given in terms of the *units.* In general (i. e., except for special transformations) the complement of a vector referred to a new system of units differs from the complement referred to the original units.

multiplying algebraically the expressions for the two vectors: thus if

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3, \qquad \boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3,$$

then

$$(15) \qquad [\boldsymbol{a}, \boldsymbol{b}]^* = (a_1b_2 - a_2b_1)\boldsymbol{E}_3 + (a_2b_3 - a_3b_2)\boldsymbol{E}_1 + (a_3b_1 - a_1b_3)\boldsymbol{E}_2.$$

It is usual to represent such a sum as occurs in (15) geometrically by a rule for addition of vectors of the *second* order, similar to the familiar parallelogram law for linear vectors; the parallelogram being replaced by a parallelopiped determined by the two addenda, and the sum being represented by a diagonal parallelogram of the parallelopiped; thus the expression $k_1\boldsymbol{E}_1 + k_2\boldsymbol{E}_2 + k_3\boldsymbol{E}_3$ may be represented by a certain parallelogram, or by an equivalent area on any parallel plane. It should be noted that the sign of this vector is reversed by reversing the sense in which the perimeter is directed. It follows that the outer product $[\boldsymbol{a}, \boldsymbol{b}]$ of any two linear vectors may be represented by the parallelogram which they determine, with its perimeter taken in the direction shown by $\boldsymbol{a}$.

The outer product of all three units might be represented in a similar fashion, as a vector of the *third* order; geometrically, a parallelopiped of edges $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$; it is more usual, however, to notice that all such vectors in space of three dimensions are numerical multiples of one another. It is customary, therefore, to characterize such a vector solely by the number which expresses its volume; we define, then,

$$(16) \qquad \begin{aligned} [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] &= [\boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1] = [\boldsymbol{e}_3, \boldsymbol{e}_1, \boldsymbol{e}_2] = 1, \\ [\boldsymbol{e}_3, \boldsymbol{e}_2, \boldsymbol{e}_1] &= [\boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_2] = [\boldsymbol{e}_2, \boldsymbol{e}_1, \boldsymbol{e}_3] = -1. \end{aligned}$$

The outer product of *any* three linear vectors can be obtained either by algebraic multiplication and use of (16)† or by computing the volume of the parallelopiped determined by the three vectors; thus if

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3, \quad \boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3, \quad \boldsymbol{c} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3,$$

then

$$(17) \qquad [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and, using parentheses as equivalent to brackets, we also write

$$(18) \qquad [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = ([\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{c}) = (\boldsymbol{a}, [\boldsymbol{b}, \boldsymbol{c}]).$$

On occasion, however, we shall distinguish temporarily 3-dimensional vectors from ordinary numbers.

* Cf. definition for two dimensions, p. 452.

† A product in which one of the units is repeated is zero, since the volume of the corresponding parallelopiped is zero.

7. *Complements; regressive and inner products.* Let $\boldsymbol{E}$ be the outer product of several of the unit vectors; we define the complement of $\boldsymbol{E}$, written $|\boldsymbol{E}$, to be the outer product of the unit vectors not used in forming $\boldsymbol{E}$, with a sign such that $[\boldsymbol{E}, |\boldsymbol{E}] = +1$.

It follows* that $|\boldsymbol{E}^{(i)} = [\boldsymbol{E}^{(i)}, \boldsymbol{E}^{(3-i)}]\boldsymbol{E}^{(3-i)}$, where $\boldsymbol{E}^{(3-i)}$ is the outer product of the unit vectors not used in forming $\boldsymbol{E}^{(i)}$, taken in *any* order.

The complement of any linear function of several vectors is defined to be the same linear function of their complements; thus†

$$|(A_1\boldsymbol{E}_1 + A_2\boldsymbol{E}_2) = A_1|\boldsymbol{E}_1 + A_2|\boldsymbol{E}_2. \tag{18}$$

The outer product of two vectors $\boldsymbol{A}, \boldsymbol{B}$ has been defined only for the case in which the sum of the orders of $\boldsymbol{A}$ and $\boldsymbol{B}$ does not exceed three. In case this sum is greater than three, the sum of the orders of $|\boldsymbol{A}$ and $|\boldsymbol{B}$ is less than three; hence in this case $[|\boldsymbol{A}, |\boldsymbol{B}]$ is defined. In terms of this product we define $[\boldsymbol{A}, \boldsymbol{B}]$.

Definition. If the sum of the orders of $\boldsymbol{A}$ and $\boldsymbol{B}$ exceeds three, then the outer product $[\boldsymbol{A}, \boldsymbol{B}]$ is a vector, $\boldsymbol{C}$, such that $|\boldsymbol{C} = [|\boldsymbol{A}, |\boldsymbol{B}]$; in this case the product is called *regressive;* when the sum of the orders of the factors is less than three, the product is called *progressive;* both are called outer products.

Definition. The outer product‡ $[\boldsymbol{A}, |\boldsymbol{B}]$ of a vector $\boldsymbol{A}$ and the complement of another vector, $\boldsymbol{B}$, is called the inner product of $\boldsymbol{A}$ and $\boldsymbol{B}$.§

8. *Vector formulas.* The following formulas enable us to determine the vector represented by a given regressive product. First we consider the regressive product of two units of the second order $([\boldsymbol{e}_i, \boldsymbol{e}_j], [\boldsymbol{e}_i, \boldsymbol{e}_k])$. We have by definition

$$\begin{aligned} |([\boldsymbol{e}_i, \boldsymbol{e}_j], [\boldsymbol{e}_i, \boldsymbol{e}_k]) &= ([\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k]\boldsymbol{e}_k, [\boldsymbol{e}_i, \boldsymbol{e}_k, \boldsymbol{e}_j]\boldsymbol{e}_j) \\ &= -[\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k]^2[\boldsymbol{e}_k, \boldsymbol{e}_j] = [\boldsymbol{e}_j, \boldsymbol{e}_k]. \end{aligned}$$

This by definition is the complement of $[\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k]\boldsymbol{e}_i$; hence

$$([\boldsymbol{e}_i, \boldsymbol{e}_j], [\boldsymbol{e}_i, \boldsymbol{e}_k]) = [\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k]\boldsymbol{e}_i.\| \tag{19}$$

Next consider the product $([\boldsymbol{e}_i, \boldsymbol{a}], [\boldsymbol{e}_i, \boldsymbol{e}_k])$ where $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$.

* Capital letters in black face type will be used to denote vectors of any order. If it is desired to specify the order an upper index in parentheses will be employed; thus $\boldsymbol{E}^{(i)}$ means a vector of the i-th order. Small black face letters invariably represent linear vectors.

† From the definition of complement and formula (18) it follows that $|(|\boldsymbol{A}) = \pm\boldsymbol{A}$.

‡ It is customary to omit the comma and regard the complement sign $|$ as the sign of inner multiplication.

§ This definition, and others of this section are in accord with the corresponding definitions for two dimensions in Art. 2. The definitions as stated here are at once extensible to n dimensions.

‖ We have here assumed i, j, k all different, but the formula holds if any two, or if all three are equal, for in that case both sides reduce to zero.

Substituting this expression for $\boldsymbol{a}$ we have

$$(20)\quad \begin{aligned}([\boldsymbol{e}_i, \boldsymbol{a}], [\boldsymbol{e}_i, \boldsymbol{e}_k]) &= a_i([\boldsymbol{e}_i, \boldsymbol{e}_i], [\boldsymbol{e}_i, \boldsymbol{e}_k]) + a_j([\boldsymbol{e}_i, \boldsymbol{e}_j], [\boldsymbol{e}_i, \boldsymbol{e}_k]),\\ &+ a_k([\boldsymbol{e}_i, \boldsymbol{e}_k], [\boldsymbol{e}_i, \boldsymbol{e}_k]) = a_j([\boldsymbol{e}_i, \boldsymbol{e}_j], [\boldsymbol{e}_k, \boldsymbol{e}_i]) = [\boldsymbol{e}_i, \boldsymbol{a}, \boldsymbol{e}_k]\boldsymbol{e}_i.\end{aligned}$$

Continuing we find, for any three vectors, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$,

$$(21)\qquad ([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{a}, \boldsymbol{c}]) = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{a}.$$

Formulas (19) and (20) are special cases of (21).

We are now able to see the geometric meaning of a regressive product $[\boldsymbol{A}, \boldsymbol{B}]$ in case $\boldsymbol{A}$ and $\boldsymbol{B}$ are vectors of the second order; let the planes in which $\boldsymbol{A}$ and $\boldsymbol{B}$ are supposed to lie be produced to meet, and let $\boldsymbol{a}$ be a vector in their line of intersection; then, by properly choosing $\boldsymbol{b}$ and $\boldsymbol{c}$, we may write $\boldsymbol{A} = [\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{B} = [\boldsymbol{a}, \boldsymbol{c}]$. By (21), then, we see that $[\boldsymbol{A}, \boldsymbol{B}]$ is a vector of the first order lying in (or parallel to) the line of intersection of the planes of $\boldsymbol{A}$ and $\boldsymbol{B}$.*

The inner product of two vectors

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i \boldsymbol{e}_i, \qquad \boldsymbol{b} = \sum_{i=1}^{3} b_i \boldsymbol{e}_i$$

may be found by applying the definition and multiplying out algebraically; thus

$$(22)\qquad [\boldsymbol{a}|\boldsymbol{b}] = \left[\sum_{i=1}^{3} a_i \boldsymbol{e}_i, \sum_{i=1}^{3} b_i|\boldsymbol{e}_i\right] = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] \sum_{i=1}^{3} a_i b_i = a_1b_1 + a_2b_2 + a_3b_3.$$

This is seen to be the extension to three dimensions of the definition of inner product given on p. 450.

Again, the inner product $([\boldsymbol{a}, \boldsymbol{b}]|\boldsymbol{c})$ may be found similarly to be

$$(a_1b_2 - a_2b_1)(c_1\boldsymbol{e}_2 - c_2\boldsymbol{e}_1) + (a_2b_3 - a_3b_2)(c_2\boldsymbol{e}_3 - c_3\boldsymbol{e}_2) + (a_3b_1 - a_1b_3)(c_3\boldsymbol{e}_1 - c_1\boldsymbol{e}_3).$$

If the inner product of this vector and a vector

$$\boldsymbol{d} = \sum_{i=1}^{3} d_i \boldsymbol{e}_i$$

be formed, the result is

$$\begin{aligned}\{([\boldsymbol{a}, \boldsymbol{b}]|\boldsymbol{c})|\boldsymbol{d}\} &= (a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1)\\ &+ (a_2b_3 - a_3b_2)(c_2d_3 - c_3d_2) + (a_3b_1 - a_1b_3)(c_3d_1 - c_1d_3)\end{aligned}$$

and this same result is found for the inner product $([\boldsymbol{a}, \boldsymbol{b}]|[\boldsymbol{c}, \boldsymbol{d}])$; hence

$$(23)\qquad ([\boldsymbol{a}, \boldsymbol{b}]|[\boldsymbol{c}, \boldsymbol{d}]) = \{([\boldsymbol{a}, \boldsymbol{b}]|\boldsymbol{c})|\boldsymbol{d}\}.$$

We proceed to obtain certain other fundamental formulas: let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be any three independent linear vectors, and $\boldsymbol{d}$ any other linear vector. Then we may

* In n dimensions if the product $[\boldsymbol{A}, \boldsymbol{B}]$ is progressive, it represents a vector in the least space containing $\boldsymbol{A}$ and $\boldsymbol{B}$; if regressive, a vector in the greatest space common to $\boldsymbol{A}$ and $\boldsymbol{B}$.

write

$$\boldsymbol{d} = l\boldsymbol{a} + m\boldsymbol{b} + n\boldsymbol{c},$$

where l, m, n are ordinary numbers. To obtain l, m, n, take the outer product of $\boldsymbol{d}$ with $[\boldsymbol{a}, \boldsymbol{b}]$, $[\boldsymbol{b}, \boldsymbol{c}]$, $[\boldsymbol{c}, \boldsymbol{a}]$, respectively. We thus obtain

$$(24) \qquad [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}] = n[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}], \quad [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}] = l[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}], \quad [\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{d}] = m[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}];$$

hence

$$(25) \qquad l = \frac{[\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}, \qquad m = \frac{[\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{d}]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}, \qquad n = \frac{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}.$$

Again, taking the outer product of $\boldsymbol{d}$ with $\boldsymbol{c}$, we have

$$[\boldsymbol{c}, \boldsymbol{d}] = l[\boldsymbol{c}, \boldsymbol{a}] - m[\boldsymbol{b}, \boldsymbol{c}];$$

hence

$$(26) \qquad \begin{aligned} ([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{c}, \boldsymbol{d}]) &= -l([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{a}, \boldsymbol{c}]) + m([\boldsymbol{b}, \boldsymbol{a}], [\boldsymbol{b}, \boldsymbol{c}]) \\ &= -l[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{a} - m[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{b}^* = [\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}]\boldsymbol{b} - [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]\boldsymbol{a}. \end{aligned}$$

Similarly

$$[\boldsymbol{a}, \boldsymbol{b}] = \frac{l}{m}[\boldsymbol{a}, \boldsymbol{d}] - \frac{n}{m}[\boldsymbol{a}, \boldsymbol{c}];$$

hence

$$(27) \qquad ([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{c}, \boldsymbol{d}]) = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{d}.$$

If we represent the two-dimensional vector $[\boldsymbol{c}, \boldsymbol{d}]$ in (26) as the complement of a linear vector $\boldsymbol{v}$, we have

$$(28) \qquad ([\boldsymbol{a}, \boldsymbol{b}] \mid \boldsymbol{v}) = [\boldsymbol{a} \mid \boldsymbol{v}]\boldsymbol{b} - [\boldsymbol{b} \mid \boldsymbol{v}]\boldsymbol{a}.$$

Taking the inner product of both sides of this equation by another vector $\boldsymbol{w}$, and using formula (23), we have

$$(29) \quad ([\boldsymbol{a}, \boldsymbol{b}] \mid [\boldsymbol{v}, \boldsymbol{w}]) = [\boldsymbol{a} \mid \boldsymbol{v}][\boldsymbol{b} \mid \boldsymbol{w}] - [\boldsymbol{b} \mid \boldsymbol{v}][\boldsymbol{a} \mid \boldsymbol{w}] = \begin{vmatrix} [\boldsymbol{a} \mid \boldsymbol{v}], & [\boldsymbol{a} \mid \boldsymbol{w}] \\ [\boldsymbol{b} \mid \boldsymbol{v}], & [\boldsymbol{b} \mid \boldsymbol{w}] \end{vmatrix}.$$

The vector formulas just given for three dimensions are readily extended to n dimensions. The extensions of formulas (27), (28), and (29) are written down here for reference. They may be proved by methods analogous to those just indicated for three dimensions.

Let $[\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k]$ and $[\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_r]$ be two vectors in a space of n dimensions, of orders k and r respectively, and suppose $k + r > n$, so that the product $([\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k], [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_r])$ is regressive. The extension of (27) may then be written

$$(27_n) \qquad \begin{aligned} &([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k], [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r]) \\ &\qquad = \sum_c [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k, \boldsymbol{b}_{c_1}, \cdots, \boldsymbol{b}_{c_{n-k}}][\boldsymbol{b}_{c_{n-k+1}}, \cdots, \boldsymbol{b}_{c_r}]^\dagger \end{aligned}$$

* Cf. formula (21), p. 458.

† *Ausdehnungslehre*, p. 83, no. 113.

where $c_1, \cdots, c_{n-k}$ stands for a combination of $n-k$ of the subscripts $1, 2, \cdots, r$, and $c_1, c_2, \cdots, c_r$ are taken in such order that $[\boldsymbol{b}_{c_1}, \boldsymbol{b}_{c_2}, \cdots, \boldsymbol{b}_{c_r}] = [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_r]$. We have also

$$(28_n)\qquad \begin{aligned}&([\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{k+r}] \mid [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_r])\\ &\qquad = \sum_c ([\boldsymbol{a}_{c_1}, \cdots, \boldsymbol{a}_{c_r}] \mid [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r])[\boldsymbol{a}_{c_{r+1}}, \cdots, \boldsymbol{a}_{c_{r+k}}],^*\end{aligned}$$

$$(29_n)\qquad ([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k] \mid [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_k]) = \begin{vmatrix} [\boldsymbol{a}_1 \mid \boldsymbol{b}_1], & [\boldsymbol{a}_1 \mid \boldsymbol{b}_2] \cdots [\boldsymbol{a}_1 \mid \boldsymbol{b}_k] \\ [\boldsymbol{a}_2 \mid \boldsymbol{b}_1], & [\boldsymbol{a}_2 \mid \boldsymbol{b}_2] \cdots [\boldsymbol{a}_2 \mid \boldsymbol{b}_k] \\ \vdots & \vdots \\ [\boldsymbol{a}_k \mid \boldsymbol{b}_1], & [\boldsymbol{a}_k \mid \boldsymbol{b}_2] \cdots [\boldsymbol{a}_k \mid \boldsymbol{b}_k] \end{vmatrix}.^\dagger$$

9. *Axioms.* The essential properties‡ used in what precedes may be stated in the following axioms, which will be seen to ensure the preceding facts for three dimensions, and which also furnish a sufficient basis for an immediate extension of all these ideas to space of n dimensions for a system of n mutually orthogonal linear unit vectors. The notation explained in footnote, p. 457, will be employed. Small letters in ordinary type denote ordinary numbers.

I. If $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are vectors of the r-th order, $r \leqq n$, then

(1) $l\boldsymbol{A} = \boldsymbol{A}l$;

(2) $l(m\boldsymbol{A}) = lm\boldsymbol{A}$;

(3) $0 \cdot \boldsymbol{A} = 0$;

(4) $l\boldsymbol{A} = 0$ implies $l = 0$ or $\boldsymbol{A} = 0$;

(5) $l\boldsymbol{A} + m\boldsymbol{B}$ is a uniquely determined vector of order r;

(6) $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$;

(7) $l\boldsymbol{A} + m\boldsymbol{A} = (l + m)\boldsymbol{A}$;

(8) $l(\boldsymbol{A} + \boldsymbol{B}) = l\boldsymbol{A} + l\boldsymbol{B}$.§

II. There exist n vectors of the first order, $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$, such that any linear relation, $l_1\boldsymbol{e}_1 + l_2\boldsymbol{e}_2 + \cdots + l_n\boldsymbol{e}_n = 0$, implies $l_1 = 0, l_2 = 0, \cdots, l_n = 0$.

Definition. k vectors of any order $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_k$ are said to be *independent* if a linear relation, $l_1\boldsymbol{A}_1 + l_2\boldsymbol{A}_2 + \cdots + l_k\boldsymbol{A}_k = 0$, implies $l_1 = 0, l_2 = 0, \cdots, l_k = 0$.

III. If $\boldsymbol{A}^{(i)}$ is a vector of the i-th order, and if $\boldsymbol{B}^{(j)}$ and $\boldsymbol{C}^{(j)}$ are vectors of the j-th order $(i + j \leqq n)$, then

(1) $[\boldsymbol{A}^{(i)}, \boldsymbol{B}^{(j)}]$ is a uniquely determined vector of the $(i+j)$-th order;

(2) $[\boldsymbol{A}^{(i)}, (\boldsymbol{B}^{(j)} + \boldsymbol{C}^{(j)})] = [\boldsymbol{A}^{(i)}, \boldsymbol{B}^{(j)}] + [\boldsymbol{A}^{(i)}, \boldsymbol{C}^{(j)}]$.

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* Ibid., p. 133, no. 173.

† Ibid., p. 135, no. 175.

‡ This system of axioms is compiled from GRASSMANN; all, however, are not given explicitly by him. Some are given as theorems; others are implied in his definitions. The system, as such, is new.

§ The property $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ can be proved by showing that $(\boldsymbol{A} + \boldsymbol{B}) - (\boldsymbol{B} + \boldsymbol{A}) = 0$, by use of (6), (7), (8); hence by (5), the result follows.

Definition. The expression $[\boldsymbol{A}^{(i)}, \boldsymbol{B}^{(j)}]$ is called the outer product of $\boldsymbol{A}^{(i)}$ and $\boldsymbol{B}^{(j)}$.

IV. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors of the first order

$$[l\boldsymbol{a}, m\boldsymbol{b}] = lm[\boldsymbol{a}, \boldsymbol{b}]; \tag{1}$$

$$[\boldsymbol{a}, \boldsymbol{b}] + [\boldsymbol{b}, \boldsymbol{a}] = 0. \tag{2}$$

V. If $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_i$ and $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_j$ are vectors of the first order and if $i + j \leqq n$, then

$$([\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_i], [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_j]) = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_i, \boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_j].$$

VI. n independent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ of the first order exist, such that $[\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n] = 1$.

Definition. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ be any n independent vectors of the first order; the products involving the different combinations of these vectors i at a time are called the *multiplicative combinations* of the i-th order of these vectors.

VII. If $\boldsymbol{A}^{(i)}$ is any vector of the i-th order, then n independent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ of the first order and

$$N_i = \frac{n!}{i!\,(n-i)!}$$

numbers, $l_1, l_2, \cdots, l_{N_i}$, exist such that $\boldsymbol{A}^{(i)} = l_1\boldsymbol{E}_1 + l_2\boldsymbol{E}_2 + \cdots + l_{N_i}\boldsymbol{E}_{N_i}$, where the $\boldsymbol{E}$'s are the multiplicative combinations of the i-th order of $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$.

Complements, regressive and inner products are defined for n dimensions precisely as on p. 457 for three dimensions. Applying the definition of inner multiplication to the units we h ve

$$[\boldsymbol{e}_i | \boldsymbol{e}_j] = \begin{matrix} 0 & (i \neq j)^* \\ 1 & (i = j), \end{matrix}$$

where the units are the $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ of Axiom VI.

IV. Interpretation of Symbolic Forms and Comparison with the Axioms.

10. *Definitions.* It will now be shown that symbolic differential parameters of the special form

$$dx^2 + dy^2 + dz^2 = (f_{(1)}dx + f_{(2)}dy + f_{(3)}dz)^2 \tag{0}$$

may be regarded as vectors of a three-dimensional space by interpreting $f_{(1)}$, $f_{(2)}$, $f_{(3)}$ as unit vectors along the x-, y-, and z-axes respectively. This will be done by showing that with suitable definition of outer product, etc., they satisfy the Axioms I–VII for $n = 3$.

Linear functions of any number of symbolic parameters of the type (f, a, b), where f is a symbolic function of the differential form (30), will be shown to be

* Compare with the formulas for $f_{(i)}f_{(j)}$, footnote, p. 450.

one-dimensional vectors; it should be noted that $f_{(1)}, f_{(2)}, f_{(3)}$ are themselves of this type, since $f_{(1)} = (f, y, z), f_{(2)} = (f, z, x), f_{(3)} = (f, x, y)$.

Also $(f, \phi, \psi)(\phi, \psi, a)$ is a linear function of parameters of this type since, when it is expanded, ϕ and ψ disappear according to equations in the footnote, p. 450.

Linear functions of parameters of the type (f, ϕ, a) involving two symbolic functions will be shown to be vectors of the second order.

For effecting transformations of certain symbolic expressions, the following determinant theorem, stated for determinants of the third order, is found useful.

Denote

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by $|a, b, c|$; then

$$|a,b,c|\cdot|d,e,f| = |d,b,c|\cdot|a,e,f| + |e,b,c|\cdot|d,a,f| + |f,b,c|\cdot|d,e,a|.^*$$

Let us use the notation

$$\boldsymbol{a} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, a) = a_{(1)}f_{(1)} + a_{(2)}f_{(2)} + a_{(3)}f_{(3)},$$
$$\boldsymbol{b} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, b) = b_{(1)}f_{(1)} + b_{(2)}f_{(2)} + b_{(3)}f_{(3)},$$
$$\boldsymbol{c} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, c) = c_{(1)}f_{(1)} + c_{(2)}f_{(2)} + c_{(3)}f_{(3)},$$

where a, b, c are functions of x, y, z.

We define outer products as follows:

$$[\boldsymbol{a}, \boldsymbol{b}] = \frac{1}{2!}(f, \phi, \psi)(f, a, b),$$

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = ([\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{c}) = (\boldsymbol{a}, [\boldsymbol{b}, \boldsymbol{c}]) = (a, b, c).^\dagger$$

We take Axioms $\mathrm{III}_{(2)}$ and $\mathrm{IV}_{(1)}$ as definitions of the expressions

$$[\boldsymbol{A}^{(i)}, (\boldsymbol{B}^{(j)} + \boldsymbol{C}^{(j)})] \qquad \text{and} \qquad [l\boldsymbol{a}, m\boldsymbol{b}].$$

11. *Comparison with the axioms.* We proceed to show that with these definitions Axioms I to VII are satisfied.

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* For determinants of the n-th order

$$|a_1, a_2, \cdots, a_n|\cdot|b_1, b_2, \cdots, b_n| = \sum_{i=1}^{n} |b_i, a_2, \cdots, a_n|\cdot|b_1, b_2, \cdots, b_{i-1}, a_1, b_{i+1}, \cdots, b_n|.$$

See MASCHKE, *Differential parameters of the first order*, Transactions of the American Mathematical Society, vol. 7 (1906), p. 70, equation (1). Cf. also E. PASCAL, Lincei Rendiconti, 1888.

† This definition is given in order to conform to the customary definition of $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ which makes this product a number rather than a vector of the third order. A definition more in accord with the other definitions given here would be

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = (f, \phi, \psi)(a, b, c)/\sqrt{3!}.$$

$\mathrm{I}_{(1)}\cdots\mathrm{I}_{(8)}$ are satisfied by the fundamental assumptions in regard to symbols.*

To show that II is satisfied we show that $f_{(1)}$, $f_{(2)}$, $f_{(3)}$ are linearly independent. Assume then a linear relation

$$pf_{(1)} + qf_{(2)} + rf_{(3)} = 0\,;$$

multiplying through by $f_{(1)}$, we have

$$pf_{(1)}^2 + qf_{(1)}f_{(2)} + rf_{(1)}f_{(3)} = 0\,.$$

But $f_{(1)}^2 = 1$, $f_{(1)}f_{(2)} = 0$, $f_{(1)}f_{(3)} = 0$. Hence the relation reduces to $p = 0$. Similarly, multiplying in turn by $f_{(2)}$, $f_{(3)}$ we find $q = 0$, $r = 0$. Hence $f_{(1)}$, $f_{(2)}$, and $f_{(3)}$ may be taken as the $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ of II.

$\mathrm{III}_{(2)}$ and $\mathrm{IV}_{(1)}$ are satisfied by definition.

$\mathrm{III}_{(1)}$ is satisfied by definition in case the two factors are of the first order; also in case the factors are one of the second order and one of the first order in the forms

$$\frac{1}{\sqrt{2!}}(f, \phi, \psi)(f, a, b)$$

and $(f, \phi, \psi)(\phi, \psi, a)/2$ respectively. There remains the case in which the factors are of the form (ϕ, ψ, a) and $(f, \phi, \psi)(\phi, \psi, b)$. In this case, since $(x, y, z) = 1$, we may write

$$(\phi, \psi, a)=(\phi, \psi, a)(x, y, z)=(\phi, \psi, x)(a, y, z)+(\phi, \psi, y)(a, z, x)+(\phi, \psi, z)(a, x, y)$$

by the determinant theorem. Now

$$(f, \phi, \psi)=(f, \phi, \psi)(x, y, z)=(f, y, z)(\phi, \psi, x)+(f, z, x)(\phi, \psi, y)+(f, x, y)(\phi, \psi, z).$$

Hence

$$\begin{aligned}(f, \phi, \psi)(f, y, z) &= (f, y, z)^2(\phi, \psi, x) + (f, z, x)(f, y, z)(\phi, \psi, y)\\ &\qquad + (f, x, y)(f, y, z)(\phi, \psi, z)\\ &= f_{(1)}^2(\phi, \psi, x) + f_{(1)}f_{(2)}(\phi, \psi, y) + f_{(1)}f_{(3)}(\phi, \psi, z) = (\phi, \psi, x);\end{aligned}$$

and similarly

$$(f, \phi, \psi)(f, z, x) = (\phi, \psi, y),$$

$$(f, \phi, \psi)(f, x, y) = (\phi, \psi, z).$$

Hence finally

$$\begin{aligned}(\phi, \psi, a) &= (a, y, z)\cdot(f, \phi, \psi)(f, y, z) + (a, z, x)\cdot(f, \phi, \psi)(f, z, x)\\ &\qquad + (a, x, y)\cdot(f, \phi, \psi)(f, x, y).\end{aligned}$$

Hence if

$$\boldsymbol{A} = \frac{1}{\sqrt{2}}(\phi, \psi, a) \qquad \text{and} \qquad \boldsymbol{B} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, b),$$

*See p. 451.

then

$$[\boldsymbol{A}, \boldsymbol{B}] = (a, y, z)(b, y, z) + (a, z, x)(b, z, x) + (a, x, y)(b, x, y).$$

$\mathrm{IV}_{(2)}$ is satisfied by a property of determinants.

V is satisfied by definition.

VI is satisfied by taking a, b, c, equal to x, y, z, respectively.

We shall prove that Axiom VII is satisfied by showing that every symbolic differential parameter of the type (f, a, b) can be expressed linearly in terms of (f, y, z), (f, z, x), (f, x, y) and that every symbolic differential parameter of the type (f, ϕ, a) can be expressed linearly in terms of the three parameters (f, ϕ, x), (f, ϕ, y), (f, ϕ, z).

Using the determinant theorem we have

$$(f, a, b) = (f, a, b)(x, y, z) = (x, a, b)(f, y, z) + (y, a, b)(f, z, x) + (z, a, b)(f, x, y);$$

and also

$$(f, \phi, a) = (f, \phi, a)(x, y, z) = (a, y, z)(f, \phi, x) + (a, z, x)(f, \phi, y) + (a, x, y)(f, \phi, z).$$

It is shown in the next article that $(f, \phi, x)/\sqrt{2}$, $(f, \phi, y)/\sqrt{2}$, $(f, \phi, z)/\sqrt{2}$ are the multiplicative combinations of the second order of $\boldsymbol{e}_{(1)} = f_{(1)}$, $\boldsymbol{e}_{(2)} = f_{(2)}$, $\boldsymbol{e}_{(3)} = f_{(3)}$. Axiom VII is, therefore, satisfied if we take

$$(f, y, z) = f_{(1)}, \qquad (f, z, x) = f_{(2)}, \qquad (f, x, y) = f_{(3)}$$

as the $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

The statement made at the head of this section is therefore justified.

The similar work for n dimensions is an easy generalization; hence we have

THEOREM III. *The symbols $f_{(1)}, f_{(2)}, \cdots, f_{(n)}$, of the special differential form*

$$\sum_{i=1}^{n} dx_1^2,$$

may be regarded as unit vectors along the coördinate axes; linear functions of parameters of the type $(f_1, f_2, \cdots, f_k, a_1, a_2, \cdots, a_{n-k})$ involving k symbolic functions $f_1, \cdots, f_k$, may then be interpreted as k-dimensional vectors lying in the space determined by $f_{(1)}, f_{(2)}, \cdots, f_{(n)}$.

In n dimensions we use the notation

$$\boldsymbol{a}_i = \frac{1}{(n-1)!}(\phi_1, \cdots, \phi_{n-1}, f)(\phi_1, \cdots, \phi_{n-1}, a_i) = \sum_{j=1}^{n} a_{i(j)} f_{(j)},$$

$$\boldsymbol{b} = \frac{1}{(n-1)!}(\phi_1, \cdots, \phi_{n-1}, f)(\phi_1, \cdots, \phi_{n-1}, b_i),$$

and define outer products as follows:

$$[\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k] = \frac{1}{(n-k)!\,\sqrt{k!}}(\phi_1, \cdots, \phi_{n-k}, f_1, \cdots, f_k)(\phi_1, \cdots, \phi_{n-k}, a_1, \cdots, a_k),$$

$$([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k], [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_j]) = \frac{1}{(n-k-j)!\,\sqrt{(k+j)!}}$$

$$\times(\phi_1, \cdots, \phi_{n-k-j}, f_1, \cdots, f_{k+j})(\phi_1, \cdots, \phi_{n-k-j}, a_1, \cdots, a_k, b_1, \cdots, b_j)$$

$$(k+j<n).$$

V. Formulas and Applications.

12. *General formulas.* The symbolic expressions for various combinations of vectors will now be obtained and the formulas will then be applied to obtain new relations connecting symbolic differential parameters. We first give different symbolic expressions for the units and their multiplicative combinations.

$$\boldsymbol{e}_1 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, x) = (f, y, z),$$

$$(31) \qquad \boldsymbol{e}_2 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, y) = (f, z, x),$$

$$\boldsymbol{e}_3 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, z) = (f, x, y).$$

$e_1 \pi \quad A_2 - I\,\xi_1 A_3 \bar{u}_n \xi_2 \xi_3$

$e_2 \pi \quad -I\,\xi_1 A_3 \bar{u}_n \xi_3 \xi_1$

$e_3 \pi = \quad -I\,\xi_1 A_3 \bar{u}_n \xi_1 \xi_2$

For,

$$(f, \phi, \psi) = (f, \phi, \psi)(x, y, z)$$

$$= (f, y, z)(\phi, \psi, x) + (f, z, x)(\phi, \psi, y) + (f, x, y)(\phi, \psi, z)$$

$e_3 \pi \quad I\,\xi_1 \Delta u_2$

and

$$\tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, x) = \tfrac{1}{2}\{2(f, y, z)\} = (f, y, z),$$

since

$$(\phi, \psi, x)(\phi, \psi, x) = 2$$

and

$$(\phi, \psi, x)(\phi, \psi, y) = (\phi, \psi, x)(\phi, \psi, z) = 0.$$

Similarly for the others.

By definition

$$\boldsymbol{e}_1 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, x).$$

For the units of the second order we have

$$[\boldsymbol{e}_2, \boldsymbol{e}_3] = |\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, y, z) = \frac{1}{\sqrt{2}}(\phi, \psi, x),$$

$$(32) \qquad [\boldsymbol{e}_3, \boldsymbol{e}_1] = |\boldsymbol{e}_2 = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, z, x) = \frac{1}{\sqrt{2}}(\phi, \psi, y),$$

$$[\boldsymbol{e}_1, \boldsymbol{e}_2] = |\boldsymbol{e}_3 = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, x, y) = \frac{1}{\sqrt{2}}(\phi, \psi, z).$$

$[e_2, e_3] \pi \quad I\,A_1 \xi_1 \xi_2 \Delta u_2 \Delta u_3$

$I\,A_1 \bar{u}_n \xi_1 \xi_2 q_1$

For, by definition, $|\boldsymbol{e}_1 = [\boldsymbol{e}_2, \boldsymbol{e}_3]$, where $\boldsymbol{e}_2 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, y)$ and $\boldsymbol{e}_3 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, z)$ and by definition, p. 462,

$$[\boldsymbol{e}_2, \boldsymbol{e}_3] = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, y, z);$$

that $(f, \phi, \psi)(f, y, z) = (\phi, \psi, x)$ is proved by writing

$$(f, \phi, \psi) = (f, \phi, \psi)(x, y, z),$$

etc., as in the proof of formulas (31).

By this same method it may be shown that

$\alpha\pi\ I\ \text{Ü}n\xi_1 A\text{Ü}n\alpha_1$
$\pi\ I.\xi_1\alpha_1$

$$(33)\quad \boldsymbol{a} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, a) = (a, y, z)\boldsymbol{e}_1 + (a, z, x)\boldsymbol{e}_2 + (a, x, y)\boldsymbol{e}_3;$$

for $(\phi, \psi, a) = (\phi, \psi, a)(x, y, z) = (\phi, \psi, x)(a, y, z)$, etc., and $\frac{1}{2}(f, \phi, \psi)(\phi, \psi, x) = \boldsymbol{e}_1$, etc.

Other important formulas follow:

$\alpha|\beta\ \pi\ I.\xi_1\alpha_1 I\xi_1\beta_1$
$= I.\alpha_1\beta_1$

$$(34)\qquad |\boldsymbol{a} = \frac{1}{\sqrt{2}}(\phi, \psi, a);^*$$

for by (33)

$$|\boldsymbol{a} = (a, y, z)|\boldsymbol{e}_1 + (a, z, x)|\boldsymbol{e}_2 + (a, x, y)|\boldsymbol{e}_3$$

$$= \frac{1}{\sqrt{2}}\{(a, y, z)(\phi, \psi, x) + (a, z, x)(\phi, \psi, y) + (a, x, y)(\phi, \psi, z)\},$$

and

$$\frac{1}{\sqrt{2}}(\phi, \psi, a) = \frac{1}{\sqrt{2}}(\phi, \psi, a)(x, y, z) = \frac{1}{\sqrt{2}}(a, y, z)(\phi, \psi, x),$$

etc., as above.

$$(35)\quad \begin{aligned}[\boldsymbol{a}|\boldsymbol{b}] = \tfrac{1}{2}(f, \phi, a)(f, \phi, b) &= (a, y, z)(b, y, z) + (a, z, x)(b, z, x) \\ &\quad + (a, x, y)(b, x, y) = a_{(1)}b_{(1)} + a_{(2)}b_{(2)} + a_{(3)}b_{(3)}.\end{aligned}$$

Proof. By (33)

$$\boldsymbol{b} = (b, y, z)\boldsymbol{e}_1 + (b, z, x)\boldsymbol{e}_2 + (b, x, y)\boldsymbol{e}_3;$$

hence

$$[\boldsymbol{a}|\boldsymbol{b}] = (b, y, z)[\boldsymbol{a}|\boldsymbol{e}_1] + (b, z, x)[\boldsymbol{a}|\boldsymbol{e}_2] + (b, x, y)[\boldsymbol{a}|\boldsymbol{e}_3)$$

and by definition, p. 462,

$$[\boldsymbol{a}|\boldsymbol{e}_1] = (a, y, z),$$

etc., where

$$\boldsymbol{a} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, a) \quad \text{and} \quad |\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, y, z);$$

hence

$$[\boldsymbol{a}|\boldsymbol{b}] = (a, y, z)(b, y, z) + (a, z, x)(b, z, x) + (a, x, y)(b, x, y).$$

* Except for a numerical factor the complement of any vector expressed in symbolic form is obtained by multiplying by a factor (f, ϕ, ψ) made up wholly of symbolic functions, including the symbolic functions which occur only once in the expression for the vector, or by leaving out such a factor in case it is present.

Again

$$\begin{aligned}(f, \phi, a)(f, \phi, b) &= \{(a, y, z)(f, \phi, x) + (a, z, x)(f, \phi, y) \\ &\quad + (a, x, y)(f, \phi, z)\}\{(b, y, z)(f, \phi, x) \\ &\quad + (b, z, x)(f, \phi, y) + (b, x, y)(f, \phi, z)\} \\ &= 2\{(a, y, z)(b, y, z) + (a, z, x)(b, z, x) \\ &\quad + (a, x, y)(b, x, y)\}\end{aligned}$$

and this proves formula (35).

$$(36) \qquad \begin{aligned}[\boldsymbol{a}, \boldsymbol{b}] &= \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, a, b) \\ &= (x, a, b)[\boldsymbol{e}_2, \boldsymbol{e}_3] + (y, a, b)[\boldsymbol{e}_3, \boldsymbol{e}_1] + (z, a, b)[\boldsymbol{e}_1, \boldsymbol{e}_2].\end{aligned}$$

$[a,b]\pi \ \frac{1}{\sqrt{2}} IA\,\xi_2\xi_3 A\,ab$

The proof is similar to the proof of (33).

$$(37) \qquad |[\boldsymbol{a}, \boldsymbol{b}] = [|\boldsymbol{a}|\boldsymbol{b}] = (f, a, b);$$

for from (36)

$$\begin{aligned}|[\boldsymbol{a}, \boldsymbol{b}] &= (x, a, b)\boldsymbol{e}_1 + (y, a, b)\boldsymbol{e}_2 + (z, a, b)\boldsymbol{e}_3 = (x, a, b)(f, y, z) \\ &\quad + (y, a, b)(f, z, x) + (z, a, b)(f, x, y) = (f, a, b).\end{aligned}$$

$$(38) \qquad ([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{c}, \boldsymbol{d}]) = (f, \phi, \psi)(\phi, a, b)(\psi, c, d).$$

Proof. By definition,

$$\begin{aligned}|([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{c}, \boldsymbol{d}]) &= (|[\boldsymbol{a}, \boldsymbol{b}]|[\boldsymbol{c}, \boldsymbol{d}]) = [\sum(x, a, b)\boldsymbol{e}_1, \sum(x, c, d)\boldsymbol{e}_1] \\ &= \begin{vmatrix}(x, a, b), (x, c, d) \\ (y, a, b), (y, c, d)\end{vmatrix}[\boldsymbol{e}_1, \boldsymbol{e}_2] + \begin{vmatrix}(y, a, b), (y, c, d) \\ (z, a, b), (z, c, d)\end{vmatrix}[\boldsymbol{e}_2, \boldsymbol{e}_3] \\ &\quad + \begin{vmatrix}(z, a, b), (z, c, d) \\ (x, a, b), (x, c, d)\end{vmatrix}[\boldsymbol{e}_3, \boldsymbol{e}_1];\end{aligned}$$

again,

$$\begin{aligned}(f, \phi, \psi)(\phi, a, b)(\psi, c, d) &= (\phi, a, b)\{(f, \phi, x)(\psi, y, z)(\psi, c, d) \\ &\quad + (f, \phi, y)(\psi, z, x)(\psi, c, d) + (f, \phi, z)(\psi, x, y)(\psi, c, d)\} \\ &= (\phi, a, b)\{(f, \phi, x)(x, c, d) + (f, \phi, y)(y, c, d) + (f, \phi, z)(z, c, d)\}.\end{aligned}$$

Now

$$\begin{aligned}(f, \phi, x)(\phi, a, b) &= (f, y, x)(\phi, z, x)(\phi, a, b) \\ &\quad + (f, z, x)(\phi, x, y)(\phi, a, b) = (y, a, b)(f, y, x) + (z, a, b)(f, z, x),\end{aligned}$$

with similar expressions for $(f, \phi, y)(\phi, a, b)$ and $(f, \phi, z)(\phi, a, b)$. Sub-

stituting, we have finally

$$
\begin{aligned}
(f, \phi, \psi)(\phi, a, b)(\psi, c, d) &= \{(y, a, b)(f, y, x) \\
&\quad + (z, a, b)(f, z, x)\}(x, c, d) + \{(z, a, b)(f, z, y) + (x, a, b)(f, x, y)\}(y, c, d) \\
&\quad + \{(x, a, b)(f, x, z) + (y, a, b)(f, y, z)\}(z, c, d) \\
&= (f, x, y)\begin{vmatrix} (x, a, b), & (x, c, d) \\ (y, a, b), & (y, c, d) \end{vmatrix} + (f, y, z)\begin{vmatrix} (y, a, b), & (y, c, d) \\ (z, a, b), & (z, c, d) \end{vmatrix} \\
&\quad + (f, z, x)\begin{vmatrix} (z, a, b), & (z, c, d) \\ (x, a, b), & (x, c, d) \end{vmatrix}.
\end{aligned}
$$

The expression found above for $|([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{c}, \boldsymbol{d}])$ is clearly the complement of this final expression for $(f, \phi, \psi)(\phi, a, b)(\psi, c, d)$; hence the formula is proved.

$$
([\boldsymbol{a}, \boldsymbol{b}] \,|\, \boldsymbol{c}) = (f, \phi, c)(\phi, a, b). \tag{39}
$$

For by (33) or (34)

$$
|\boldsymbol{c} = \Sigma(c, y, z)[\boldsymbol{e}_2, \boldsymbol{e}_3];
$$

hence

$$
\begin{aligned}
([\boldsymbol{a}, \boldsymbol{b}] \,|\, \boldsymbol{c}) &= (c, y, z)([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{e}_2, \boldsymbol{e}_3]) + (c, z, x)([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{e}_3, \boldsymbol{e}_1]) \\
&\qquad + (c, x, y)([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{e}_1, \boldsymbol{e}_2]) \\
&= (\phi, a, b)\Sigma(c, y, z)(f, \phi, \psi)(\psi, y, z) \qquad \text{by (38)}, \\
&= (\phi, a, b)\Sigma(f, \phi, x)(c, y, z) = (f, \phi, c)(\phi, a, b).
\end{aligned}
$$

$$
([\boldsymbol{a}, \boldsymbol{b}] \,|\, [\boldsymbol{c}, \boldsymbol{d}]) = (f, a, b)(f, c, d).^* \tag{40}
$$

For, by formula (37), $|[\boldsymbol{c}, \boldsymbol{d}] = \Sigma(x, c, d)\boldsymbol{e}_1$; hence

$$
\begin{aligned}
&([\boldsymbol{a}, \boldsymbol{b}] \,|\, [\boldsymbol{c}, \boldsymbol{d}]) \\
&\quad = (x, c, d)[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}_1] + (y, c, d)[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}_2] + (z, c, d)[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}_3] \\
&\quad = (x, c, d)(x, a, b) + (y, c, d)(y, a, b) + (z, c, d)(z, a, b);
\end{aligned}
$$

and $(f, a, b)(f, c, d)$ expanded gives the same.

We give below the extensions of the formulas (31)–(40) to n dimensions; the proofs are similar to the proofs given above for three dimensions.

$$
\begin{aligned}
\boldsymbol{e}_1 &= \frac{1}{(n-1)!}(f, \phi_1, \cdots, \phi_{n-1})(\phi_1, \cdots, \phi_{n-1}, x_i) \\
&= (x_1, \cdots, x_{i-1}, f, x_{i+1}, \cdots, x_n).
\end{aligned} \tag{31$_n$}
$$

* Except for a numerical factor the inner product of two vectors in symbolic notation is obtained by writing them both (or their complements) in terms of the same symbolic functions as far as possible.

$$(32_n)\quad \begin{aligned} &[\boldsymbol{e}_{c_1}, \cdots, \boldsymbol{e}_{c_k}] = \frac{1}{(n-k)!\sqrt{k!}} F_1 F_2 = \frac{1}{\sqrt{k!}}(f_1, \cdots, f_k, x_{c_{k+1}}, \cdots, x_{c_n}), \\ &F_1 = (f_1, \cdots, f_k, \phi_1, \cdots, \phi_{n-k}), \qquad F_2 = (\phi_1, \cdots, \phi_{n-k}, x_{c_1}, \cdots, x_{c_k}), \end{aligned}$$

where $c_1, \cdots, c_k$ stands for any combination of k of the subscripts $1, 2, \cdots, n$, and $c_{k+1}, \cdots, c_n$ are the remaining subscripts in such order that $(x_{c_1}, x_{c_2}, \cdots, x_{c_n}) = +1$.

$$(33_n)\quad \begin{aligned} \boldsymbol{a} = &\frac{1}{(n-1)!}(f, \phi_1, \cdots, \phi_{n-1})(\phi_1, \cdots, \phi_{n-1} a) \\ &\sum_{i=1}^{n}(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n)\boldsymbol{e}_i. \end{aligned}$$

$$(34_n)\quad |\boldsymbol{a} = \frac{1}{\sqrt{(n-1)!}}(f_1, \cdots, f_{n-1}, a).$$

$$(35_n)\quad [\boldsymbol{a}|\boldsymbol{b}] = \frac{1}{(n-1)!}(f_1, \cdots, f_{n-1}, a)(f_1, \cdots, f_{n-1}, b).$$

$$(36_n)\quad \begin{aligned} &[\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k] \\ &= \frac{1}{(n-k)!\sqrt{k!}}(\phi_1, \cdots, \phi_{n-k}, f_1, \cdots, f_k)(\phi_1, \cdots, \phi_{n-k}, a_1, \cdots, a_k) \\ &= \sum_c (a_1, \cdots, a_k, x_{c_1}, \cdots, x_{c_{n-k}})[\boldsymbol{e}_{c_{n-k+1}}, \cdots, \boldsymbol{e}_{c_n}]. \end{aligned}$$

$$(37_n)\quad \begin{aligned} |[\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k] &= (|[\boldsymbol{a}_1, \cdots, \boldsymbol{a}_i]|[\boldsymbol{a}_{i+1}, \cdots, \boldsymbol{a}_k]) \\ &= \frac{1}{\sqrt{(n-k)!}}(f_1, \cdots, f_{n-k}, a_1, \cdots, a_k). \end{aligned}$$

If $k + j > n$,

$$(38_n)\quad \begin{aligned} &([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k], [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_j]) = \frac{1}{(n-k)!\,(n-j)!\,\sqrt{(k+j-n)!}} M_1 M_2, \\ &M_1 = (\phi_1, \cdots, \phi_{n-k}, \psi_1, \cdots, \psi_{n-j}, f_1, \cdots, f_{k+j-n}), \\ &M_2 = (\phi_1, \cdots, \phi_{n-k}, a_1, \cdots, a_k)(\psi_1, \cdots, \psi_{n-j}, b_1, \cdots, b_j). \end{aligned}$$

$$(39_n)\quad \begin{aligned} &([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k+r}]|[\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r]) = \frac{1}{(n-k-r)!\sqrt{k!}} M_3, \\ &M_3 = (f_1, \cdots, f_k, \phi_{k+1}, \cdots, \phi_{n-r}, b_1, \cdots, b_r)(\phi_{k+1}, \cdots, \phi_{n-r}, a_1, \cdots, a_{k+r}), \end{aligned}$$

$$(40_n)\quad \begin{aligned} &([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_r]|[\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r]) \\ &= \frac{1}{(n-r)!}(\phi_1, \cdots, \phi_{n-r}, a_1, \cdots, a_r)(\phi_1, \cdots, \phi_{n-r}, b_1, \cdots, b_r). \end{aligned}$$

13. *Application to the symbolic invariant theory.* Results of the Grassmann theory may be employed to derive formulas in the symbolic theory. Three examples of this application are given below.*

1. From formula (27_n) follows by use of formula (38_n)

$$
\begin{aligned}
&(f_1, \cdots, f_{n-k}, \phi_1, \cdots, \phi_{n-j}, \psi_1, \cdots, \psi_{k+j-1})(f_1, \cdots, f_{n-k}, a_1, \cdots, a_k) \\
&\quad \times (\phi_1, \cdots, \phi_{n-j}, b_1, \cdots, b_j) = \frac{(n-k)!\,(n-j)!}{(2n-k-j)!} \\
(41) \qquad &\quad \times \sum_c (f_1, \cdots, f_{n-k}, \phi_1, \cdots, \phi_{n-j}, \psi_1, \cdots, \psi_{k+j-n}) \\
&\quad \times (f_1, \cdots, f_{n-k}, \phi_1, \cdots, \phi_{n-j}, b_{c_1}, \cdots, b_{c_{k+j-n}})(a_1, \cdots, a_k, b_{c_1'}, \cdots, b_{c'_{n-k}}).\dagger
\end{aligned}
$$

2. From formula (28_n) follows by use of formula (39_n)

$$
\begin{aligned}
&(f_1, \cdots, f_k, \phi_{k+1}, \cdots, \phi_{n-r}, b_1, \cdots, b_r)(\phi_{k+1}, \cdots, \phi_{n-r}, a_1, \cdots, a_{k+r}) \\
(42) \qquad &= \frac{(n-k-r)!}{(n-k)!\,(n-r)!} \sum_c (f_1, \cdots, f_{n-r}, a_{c_1}, \cdots, a_{c_r})(f_1, \cdots, f_{n-r}, b_1, \cdots, b_r) \\
&\qquad \times (\phi_1, \cdots, \phi_{n-k}, a_{c_1'}, \cdots, a_{c_k'})(\phi_1, \cdots, \phi_{n-k}, \psi_1, \cdots, \psi_k).
\end{aligned}
$$

3. From formula (29_n) follows by use of formulas (35_n) and (40_n)

$$
\begin{aligned}
&\frac{1}{(n-k)!}(f_1, \cdots, f_{n-k}, a_1, \cdots, a_k)(f_1, \cdots, f_{n-k}, b_1, \cdots, b_k) \\
(43) \qquad &= \frac{1}{\{(n-1)!\}^k} \begin{vmatrix} (f_1, a_1)(f_1, b_1), & (f_1, a_1)(f_1, b_2), & \cdots, & (f_1, a_1)(f_1, b_k) \\ (f_2, a_2)(f_2, b_1), & (f_2, a_2)(f_2, b_2), & \cdots, & (f_2, a_2)(f_2, b_k) \\ \cdot & \cdot & \cdot & \cdot \\ (f_k, a_k)(f_k, b_1), & (f_k, a_k)(f_k, b_2), & \cdots, & (f_k, a_k)(f_k, b_k) \end{vmatrix},
\end{aligned}
$$

where $(f_i, a_i) = (f_{i1}, f_{i2}, \cdots, f_{in-1}, a_i)$ and f_{ij} is a symbolic function equivalent to f.

These formulas can be shown to hold for the general differential form.

14. *Applications to geometry.* As heretofore, we regard the variables $x_1, x_2, \cdots, x_n$ as cartesian coördinates of a point in an n-dimensional space; we wish to determine how the k-dimensional vector $(f_1, \cdots, f_k, a_1, \cdots, a_{n-k})$ is related to the surfaces $a_1 = \text{const.}, a_2 = \text{const.}, \cdots, a_{n-k} = \text{const.}$

First we show that the $(n-1)$-dimensional vector $(f_1, \cdots, f_{n-1}, a)$ is, at

* Formulas (41) and (42) are new.

For formula (43) see MASCHKE, *Differential parameters of the first order*, loc. cit., p. 73.

† $c_1, \cdots, c_{k+j-n}$ stands for any combination of $k+j-n$ of the subscripts, $1, 2, \cdots, j$; $c_1', \cdots, c'_{n-k}$ are the remaining subscripts. In order to obtain the correct sign in each term, the order of the subscripts must be such that $c_1, \cdots, c_{k+j-n}, c_1', \cdots, c'_{n-k}$ may be obtained from $1, 2, \cdots, j$, by an even number of transpositions.

each point of space, tangent to one of the surfaces $a = \text{const.}$; for when a is linear, by formulas (33_n) and (34_n),

$$\frac{1}{\sqrt{(n-1)!}}(f_1, \cdots, f_{n-1}, a) = \sum_{i=1}^{n} a_{(i)} e_i,$$

is parallel to the planes $a = \text{const.}$

When a is not linear, then the linear space of $n-1$ dimensions, tangent at a given point $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ to the surface $a = \text{const.}$ which passes through that point, is given by

$$L \equiv \sum_{i=1}^{n} (a_{(i)})_{x=\bar{x}} x_i = \text{const.},$$

and clearly

$$(f_1, \cdots, f_{n-1}, a)_{x=\bar{x}} = (f_1, \cdots, f_{n-1}, L).$$

The linear space tangent to the intersection of $a_1 = \text{const.}$, $a_2 = \text{const.}$ will be the space common to the linear spaces tangent to the two hyper-surfaces. A vector lying in this space is found by taking the outer product of the vectors $(f_1, \cdots, f_{n-1}, a_1)$ and $(f_1, \cdots, f_{n-1}, a_2)$. By formula (37_n) this product may be written, except for a numerical factor, $(f_1, \cdots, f_{n-2}, a_1, a_2)$.

This argument can be repeated for three functions a_1, a_2, a_3, and again, for any number, $a_1, a_2, \cdots, a_{n-k}$; hence we have:

THEOREM IV. *If the derivatives $f_{(1)}, f_{(2)}, \cdots, f_{(n)}$ of a symbolic function of the differential form*

$$\sum_{i=1}^{n} dx_i^2$$

are interpreted as unit vectors along the coördinate axes, then

$$(f_1, f_2, \cdots, f_k, a_1, a_2, \cdots, a_{n-k})$$

may be interpreted as a k-dimensional vector, tangent to the k-dimensional spread which is common to $a_1 = \text{const.}$, $a_2 = \text{const.}$, $\cdots$, $a_{n-k} = \text{const.}$

VI. THE GENERAL DIFFERENTIAL FORM.

15. *Change of variables in three dimensions.* So far we have been considering differential parameters of a special type of differential form: namely, the differential form for length of arc in an Euclidean space; i. e., a form of the type

$$\sum_{i=1}^{n} dx_i^2,$$

or any form into which such a form can be transformed.

In this section it is shown that Theorem IV can be extended to apply to any space which can be considered a part of an Euclidean space (of higher dimensions); for example, to a surface lying in three dimensions.

Let us consider in an ordinary space of three dimensions the transformation

$$u = u(x, y, z), \qquad v = v(x, y, z), \qquad w = w(x, y, z).$$

We can find dx, dy, dz, in terms of du, dv, dw, by solving the equations

$$du = u_{(1)}dx + u_{(2)}dy + u_{(3)}dz,$$
$$dv = v_{(1)}dx + v_{(2)}dy + v_{(3)}dz,$$
$$dw = w_{(1)}dx + w_{(2)}dy + w_{(3)}dz.$$

If the resulting values are substituted in

$$ds = f_{(1)}dx + f_{(2)}dy + f_{(3)}dz$$

we obtain

$$ds = \frac{(f, v, w)}{(u, v, w)}du + \frac{(u, f, w)}{(u, v, w)}dv + \frac{(u, v, f)}{(u, v, w)}dw.$$

By Theorem IV the coefficients* of du, dv, dw may be regarded as linear vectors tangent to the intersections of $v =$ const., $w =$ const.; $w =$ const., $u =$ const.; and $u =$ const., $v =$ const., respectively. Let us denote these coefficients by $t_{(1)}$, $t_{(2)}$, $t_{(3)}$. It can be shown precisely as in the two-dimensional case that $t_{(1)}$, $t_{(2)}$, $t_{(3)}$ are the derivatives with respect to u, v, w of the function into which the symbolic function f transforms. Hence the transformed function $t(u, v, w)$ is a symbolic function of the differential quantic into which $dx^2 + dy^2 + dz^2$ transforms and the derivatives $t_{(u)}$, $t_{(v)}$, $t_{(w)}$ (i. e., $t_{(1)}$, $t_{(2)}$, $t_{(3)}$) may be interpreted as vector functions of position, tangent at each point of space to the parameter curves which pass through that point. The transformed differential form is given by $ds^2 = df^2 = dt^2 = (t_{(1)}du + t_{(2)}dv + t_{(3)}dw)^2$ where $t_{(1)} = (f, v, w)/(u, v, w)$, etc.

16. *The general two-dimensional case.* On the surface $w = a$ (a definite const.), $dw = 0$. In our formula for ds let us write a for w to indicate that only such values of x, y, z are to be used as will satisfy the equation $w = a$.

We have, then, on the surface

$$ds^2 = (t_{(1)}du + t_{(2)}dv)^2 = Edu^2 + 2Fdudv + Gdv^2$$

where

$$t_{(1)}^2 = E, \qquad t_{(1)}t_{(2)} = F, \qquad t_{(2)}^2 = G.\dagger$$

This gives rise to a vector system *on a surface*, having *variable units* $t_{(1)}$, $t_{(2)}$ which are tangent to the parameter curves of the surface. The usual vector expressions, as inner and outer products, and complements can be set up in terms

* Observe that the common denominator of these coefficients is an ordinary number, being the Jacobian of the functions u, v, w with respect to x, y, z.

† The function $f = f_{(1)} \cdot x + f_{(2)} \cdot y + f_{(3)} \cdot z$ represents the vector from the origin to the point x, y, z; hence $t(u, v)$ represents the vector from the origin to any point on the surface $w = a$.

of these variable units; these expressions take the same form as the corresponding expressions in the plane given on p. 453.

It should be noticed that here, since $w = a$ is an arbitrary surface, we are considering a perfectly general binary quadratic differential form; hence

THEOREM V. *If $A \equiv Edu^2 + 2Fdudv + Gdv^2$ is any binary quadratic differential form, the partial derivatives $\partial t/\partial u$, $\partial t/\partial v$ of a symbolic function of A may be interpreted as vectors, of length $\sqrt{E}$, $\sqrt{G}$ respectively, tangent to the parameter curves of the surface characterized by A.*

17. *The general k-dimensional case.* In the same way we consider in an Euclidean space of n dimensions, the sub-space of k dimensions determined by $a_1 = \text{const.}, \cdots, a_{n-k} = \text{const.}$ For arc length in the sub-space we shall have $ds^2 = (\Sigma t_{(i)} du_i)^2$, where

$$t_{(i)} = \frac{(u_1, u_2, \cdots, u_{i-1}, f, u_{i+1}, \cdots, u_k, a_1, \cdots, a_{n-k})}{(u_1, \cdots, u_k, a_1, \cdots, a_{n-k})}$$

and where $t(u_1, \cdots, u_k) = f(x_1, \cdots, x_n)$.

Precisely as in the three-dimensional cases it can be shown that $t_{(i)} = \partial t/\partial u_i$; hence we have

THEOREM VI. *If*

$$A \equiv \sum_{i,j=1}^{k} E_{ij} du_i du_j$$

is the differential form giving length of arc in any space of k dimensions belonging to an Euclidean space of n dimensions, then the partial derivatives $\partial t/\partial u_i, \cdots, \partial t/\partial u_k$ of a symbolic function of A may be interpreted as vectors tangent to the parameter curves of the sub-space characterized by A.

VII. SYMBOLIC DIFFERENTIAL PARAMETERS INVOLVING DERIVATIVES OF THE SECOND AND HIGHER ORDERS.

18. *Interpretation of the higher derivatives of the symbols.* Let the space R_n whose arc element is determined by

$$ds^2 = \sum_{i,j=1}^{n} E_{ij} du_i du_j = \left(\sum_{i=1}^{n} f_{(i)} du_i\right)^2$$

be supposed to lie in an Euclidean space S_r of r dimensions. The $f_{(i)}$ may then be interpreted as vectors in S_r tangent to R_n along the parameter curves. Since these tangent vectors lie in S_r, they may be expressed as linear functions of r constant, mutually orthogonal unit vectors, $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_r$, in S_r, the coefficients being functions of $u_1, \cdots, u_n$. The derivatives of the $f_{(i)}$ of any order, with respect to the variables $u_1, u_2, \cdots, u_n$, if they do not vanish, are again linear functions of the unit vectors, $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_r$. We have therefore

THEOREM VII. *If the space R_n whose arc element is determined by*

$$ds^2 = \sum_{i,j=1}^{n} E_{ij}\, du_i\, du_j = \left(\sum_{i=1}^{n} f_{(i)}\, du_i\right)^2$$

lies in an Euclidean space S_r of r dimensions, then the derivatives of the symbolic function f, of all orders, with respect to any of the variables $u_1, \cdots, u_n$, may be interpreted as vectors of the space S_r.

Since symbolic differential parameters involving derivatives of the E_{ij} to any order are expressible in terms of derivatives of the symbolic functions, it follows that all such differential parameters may be interpreted as vectors or as combinations of vectors of the space S_r.

19. *Second covariantive derivatives of the symbolic functions.* The second covariantive derivatives of the symbolic functions are given linearly in terms of the ordinary second derivatives and of the first derivatives by the formula

$$f^{(ij)} = f_{(ij)} - \frac{1}{(n-1)!}\phi_{(ij)}(\phi, \psi)(f, \psi)^* \qquad \left(f_{(ij)} = \frac{\partial^2 f}{\partial u_i \partial u_j}\right).$$

It follows that $f^{(ij)}$ may be interpreted as a vector. But we have always $f_{(k)} f^{(ij)} = 0$ † which shows that each $f^{(ij)}$ is orthogonal to all the vectors $f_{(k)}$. Hence

THEOREM VIII. *If the $f_{(k)}$ are interpreted as vectors in S_r tangent to R_n along the parameter curves, then the second covariantive derivatives $f^{(ij)}$ may be interpreted as normals to R_n in S_r.*

* MASCHKE, *A symbolic treatment*, etc., loc. cit., p. 457, equation (75).

† MASCHKE, loc. cit., p. 459, equation (84).

SURFACES WITH ISOTHERMAL REPRESENTATION OF THEIR LINES OF CURVATURE AND THEIR TRANSFORMATIONS.*

(SECOND MEMOIR)

BY

LUTHER PFAHLER EISENHART

Introduction.

In a former memoir with the same title † we established a transformation of surfaces with isothermal spherical representation of their lines of curvature into surfaces of the same kind, the transformation being such that lines of curvature on a surface and on a transform correspond, and the two surfaces constitute the envelope of a two-parameter family of spheres. This transformation was established by means of a theorem of MOUTARD, concerning partial differential equations of the LAPLACE type with equal invariants, ‡ and with the aid of a transformation of minimal surfaces discovered by THYBAUT. § In our discussion we did not take the equations of the transformation in the form given by THYBAUT, but in the form used by BIANCHI, || in which case the parametric curves on the minimal surfaces are the asymptotic lines. Such a minimal surface and its THYBAUT transform are the focal sheets of a W-congruence, that is, a congruence upon whose focal sheets the asymptotic lines correspond.

The present paper deals with the same transformations obtained by a very different method as a result of which the analysis is much simpler. In § 1 it is shown that when the lines of a W-congruence are subjected to the LIE line-sphere transformation, the congruence of spheres envelope two surfaces upon which the lines of curvature correspond. If S and S_1 denote these two surfaces, and Σ and Σ_1 the two focal sheets of the original W-congruence, the transformation from Σ to Σ_1 carries with it a transformation from S to S_1 without quadrature. In § 1 is determined the characteristic property which Σ must have in order that the lines of curvature on S may have isothermal spherical representation, and the equations of the surfaces are given in simple form. In § 2 the determination of

* Presented to the Society, February 26, 1910.

† Transactions of the American Mathematical Society, vol. 9 (1908), pp. 149–177.

‡ BIANCHI, *Lezioni*, vol. 2, p. 69.

§ *Sur la déformation du paraboloide, etc.*; Annales de l'École Normale (3), vol. 14 (1897), pp. 65–69.

|| *Lezioni*, vol. 2, pp. 334–338.

surfaces Σ_1 and S_1 of the same kind as Σ and S respectively and standing in the above mentioned relation to them is carried through, and the problem is shown to be reducible to the integration of an ordinary differential equation of the first order and to quadratures.

When S is a minimal surface (cf. § 3), the transformation is that of THYBAUT and the equations take the very form which occurs in his memoir.*

In § 4 it is shown that the transformation of § 2 is identical with that established in the first memoir, and in § 5 we prove the existence of a "theorem of permutability" which shows that the knowledge of two transforms S_1 and S_2 of S carries with it the determination of a surface S' of the same kind, which is a transform of both S_1 and S_2; and, moreover, S' may be obtained without quadrature.

§ 1. *A Surface* Σ *and its Lie transform* S.

The coördinates, ξ, η, ζ, of a surface Σ, referred to its asymptotic lines, may be given by the Lelieuvre formulas †

$$(1)\qquad \frac{\partial \xi}{\partial u} = \begin{vmatrix} \nu_2 & \nu_3 \\ \dfrac{\partial \nu_2}{\partial u} & \dfrac{\partial \nu_3}{\partial u} \end{vmatrix}, \qquad \frac{\partial \xi}{\partial v} = -\begin{vmatrix} \nu_2 & \nu_3 \\ \dfrac{\partial \nu_2}{\partial v} & \dfrac{\partial \nu_3}{\partial v} \end{vmatrix},$$

and similar equations for η and ζ, obtained by permuting the quantities ν_1, ν_2, ν_3, which are solutions of an equation of the form

$$(2)\qquad \frac{\partial^2 \theta}{\partial u \partial v} = M\theta,$$

where in general M is a function of u and v. Moreover, ν_1, ν_2 and ν_3 are proportional to the direction-cosines of the normal to Σ.

If x, y, z denote the coördinates and X, Y, Z the direction-cosines of the normal to the surface S which is obtained from Σ by the line-sphere transformation of Lie, these quantities are given by ‡

$$(3)\qquad \begin{aligned} x + iy &= -\zeta - \xi\frac{\nu_1\xi + \nu_2\eta}{\nu_2 + \nu_3\xi}, & X + iY &= \frac{2\nu_2\xi}{\nu_2 + \nu_3\xi}, \\ x - iy &= \frac{\nu_1 - \nu_3\eta}{\nu_2 + \nu_3\xi}, & X - iY &= \frac{2\nu_3}{\nu_2 + \nu_3\xi}, \\ z &= \frac{\nu_1\xi + \nu_2\eta}{\nu_2 + \nu_3\xi}, & Z &= \frac{\nu_3\xi - \nu_2}{\nu_2 + \nu_3\xi}. \end{aligned}$$

From these equations we determine the following values for the fundamental

* L. c.

† EISENHART, *Differential Geometry*, p. 193.

‡ DARBOUX, *Leçons*, vol. 1, p. 249.

quantities of S:

$$(4)\qquad \begin{aligned} E &= \frac{\left(\dfrac{\partial \zeta}{\partial u} - \xi \dfrac{\partial \eta}{\partial u} + \eta \dfrac{\partial \xi}{\partial u}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \qquad F = 0, \\ G &= -\frac{\left(\dfrac{\partial \zeta}{\partial v} - \xi \dfrac{\partial \eta}{\partial v} + \eta \dfrac{\partial \xi}{\partial v}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \end{aligned}$$

$$(5)\qquad \begin{aligned} D &= 2 \frac{\partial \xi}{\partial u} \frac{\dfrac{\partial \zeta}{\partial u} - \xi \dfrac{\partial \eta}{\partial u} + \eta \dfrac{\partial \xi}{\partial u}}{(\nu_2 + \nu_3 \xi)^2}, \qquad D' = 0, \\ D'' &= -2 \frac{\partial \xi}{\partial v} \frac{\dfrac{\partial \zeta}{\partial v} - \xi \dfrac{\partial \eta}{\partial v} + \eta \dfrac{\partial \xi}{\partial v}}{(\nu_2 + \nu_3 \xi)^2}, \end{aligned}$$

$$(6)\qquad \mathcal{E} = \frac{4\left(\dfrac{\partial \xi}{\partial u}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \qquad \mathcal{F} = 0, \qquad \mathcal{G} = \frac{-4\left(\dfrac{\partial \xi}{\partial v}\right)^2}{(\nu_2 + \nu_3 \xi)^2},$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are the coefficients of the linear element of the spherical representation of S. From these results we have incidentally the well-known fact that the lines of curvature on S are parametric.

However, we are concerned entirely with surfaces whose lines of curvature admit an isothermal spherical representation. From (6) it is seen that a necessary and sufficient condition for this is that *

$$(7)\qquad \frac{\partial \xi}{\partial u} - i \frac{\partial \xi}{\partial v} = 0,$$

or by means of (1),

$$(8)\qquad \frac{\partial}{\partial u} \log \frac{\nu_3}{\nu_2} + i \frac{\partial}{\partial v} \log \frac{\nu_3}{\nu_2} = 0.$$

It has been assumed that the parameters are isometric, which assumption does not affect the generality of our solution.

If we put

$$(9)\qquad u + iv = \alpha, \qquad u - iv = \beta,$$

equation (8) gives the result

$$(10)\qquad \nu_3 = \nu_2 A,$$

where A denotes an arbitrary function of α. Since ν_2 and ν_3 satisfy (2), which

* The case where (7) is replaced by $\partial \xi / \partial u + i \cdot \partial \xi / \partial v = 0$ leads to similar results.

by the transformation (9) becomes

$$\frac{\partial^2 \theta}{\partial \alpha^2} - \frac{\partial^2 \theta}{\partial \beta^2} = -iM\theta, \tag{11}$$

we must have

$$2A' \frac{\partial \nu_2}{\partial \alpha} + \nu_2 A'' = 0,$$

where the accents indicate differentiation with respect to the argument. From this equation we find

$$\nu_2 = \frac{\sqrt{B'}}{\sqrt{A'}}$$

where B denotes an arbitrary function of β, and where the accent denotes differentiation with respect to β. In consequence of (10), we have

$$\nu_1 = \frac{\sqrt{B'}}{\sqrt{A'}}\sigma, \qquad \nu_2 = \frac{\sqrt{B'}}{\sqrt{A'}}, \qquad \nu_3 = \frac{\sqrt{B'}}{\sqrt{A'}}A, \tag{12}$$

where in general σ is a function of α and β, being a solution of

$$\frac{\partial}{\partial \beta}\left[\frac{B'}{A'}\frac{\partial \sigma}{\partial \beta}\right] = \frac{\partial}{\partial \alpha}\left[\frac{B'}{A'}\frac{\partial \sigma}{\partial \alpha}\right], \tag{13}$$

which is obtained by expressing the fact that the functions (12) are solutions of equation (11).

§ 2. *The Determination of Σ_1 and of S_1.*

If θ_1 is a particular solution of equation (11), the functions $\bar{\nu}_i$, defined by*

$$\frac{\partial}{\partial \alpha}(\theta_1 \bar{\nu}_i) = -\begin{vmatrix} \theta_1 & \nu_i \\ \dfrac{\partial \theta_1}{\partial \beta} & \dfrac{\partial \nu_i}{\partial \beta} \end{vmatrix}, \qquad \frac{\partial}{\partial \beta}(\theta_1 \bar{\nu}_i) = -\begin{vmatrix} \theta_1 & \nu_i \\ \dfrac{\partial \theta_1}{\partial \alpha} & \dfrac{\partial \nu_i}{\partial \alpha} \end{vmatrix}, \tag{14}$$

determine a surface Σ_1, upon which the curves $u = \text{const.}$, $v = \text{const.}$, where u and v are given by (9), are the asymptotic lines; and Σ and Σ_1 are the focal surfaces of a W-congruence, formed by the lines joining corresponding points. Moreover, the coördinates, ξ_1, η_1, ζ_1, of Σ_1 are given by

$$\xi_1 - \xi = \nu_2 \bar{\nu}_3 - \nu_3 \bar{\nu}_2, \quad \eta_1 - \eta = \nu_3 \bar{\nu}_1 - \nu_1 \bar{\nu}_3, \quad \zeta_1 - \zeta = \nu_1 \bar{\nu}_2 - \nu_2 \bar{\nu}_1. \tag{15}$$

In order that Σ_1 may be a surface of the same kind as Σ of § 1, we must have

$$\bar{\nu}_1 = \frac{\sqrt{B_1'}}{\sqrt{A_1'}}\sigma_1, \quad \bar{\nu}_2 = \frac{\sqrt{B_1'}}{\sqrt{A_1'}}, \quad \bar{\nu}_3 = \frac{\sqrt{B_1'}}{\sqrt{A_1'}}A_1, \tag{16}$$

where A_1 and B_1 are functions of α and β alone to be determined and where, in general, σ_1 is a function of both α and β.

* EISENHART, *Differential Geometry*, pp. 417–419.

When the values for $\bar{\nu}_2$ and $\bar{\nu}_3$ are substituted in (14), we obtain four equations which are equivalent to

$$(17)\qquad \begin{aligned} \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} &= \frac{(\sqrt{A_1'})'}{\sqrt{A_1'}} + \frac{A_1'}{A-A_1}, \\ \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} &= \frac{(\sqrt{B'})'}{\sqrt{B'}} + \frac{\sqrt{B_1'}}{\sqrt{B'}}\frac{\sqrt{A'A_1'}}{A-A_1}, \end{aligned}$$

and

$$(18)\qquad \begin{aligned} \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} &= \sqrt{A'}\left(\frac{1}{\sqrt{A'}}\right)' + \frac{A'}{A-A_1}, \\ \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} &= -\frac{(\sqrt{B_1'})'}{\sqrt{B_1'}} + \frac{\sqrt{B'}}{\sqrt{B_1'}}\frac{\sqrt{A'A_1'}}{A-A_1}. \end{aligned}$$

The necessary and sufficient condition that these two values of $\partial\theta_1/\partial\alpha$ be equal is readily found to be

$$(19)\qquad \sqrt{A'A_1'} = \kappa_1(A-A_1),$$

where κ_1 denotes a constant.

In like manner the condition that the two values of $\partial\theta_1/\partial\beta$ be equivalent is reducible to

$$(20)\qquad \sqrt{B'B_1'} = \kappa_1(B-B_1).$$

In consequence of these results, equations (17) and (18) may be written

$$(21)\qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} = \frac{(\sqrt{A_1'})'}{\sqrt{A_1'}} + \kappa_1\frac{\sqrt{A_1'}}{\sqrt{A'}}, \qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} = \frac{(\sqrt{B'})'}{\sqrt{B'}} + \kappa_1\frac{\sqrt{B_1'}}{\sqrt{B'}},$$

and

$$(22)\qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} = -\frac{(\sqrt{A'})'}{\sqrt{A'}} + \kappa_1\frac{\sqrt{A'}}{\sqrt{A_1'}}, \qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} = -\frac{(\sqrt{B_1'})'}{\sqrt{B_1'}} + \kappa_1\frac{\sqrt{B'}}{\sqrt{B_1'}}.$$

When the expression (16) for $\bar{\nu}_1$ is substituted in (14), the resulting equations are reducible by means of (21) and (22) to

$$(23)\qquad \begin{aligned} \frac{\sqrt{A'}}{\sqrt{A_1'}}\frac{\partial\sigma_1}{\partial\alpha} + \kappa_1(\sigma_1-\sigma) + \frac{\sqrt{B'}}{\sqrt{B_1'}}\frac{\partial\sigma}{\partial\beta} &= 0, \\ \frac{\sqrt{B_1'}}{\sqrt{B'}}\frac{\partial\sigma_1}{\partial\beta} + \kappa_1(\sigma_1-\sigma) + \frac{\sqrt{A_1'}}{\sqrt{A'}}\frac{\partial\sigma}{\partial\alpha} &= 0. \end{aligned}$$

As these equations are linear, the determination of σ_1 requires quadratures only. Hence the problem of the determination of the surfaces Σ_1, when Σ is given in terms of its asymptotic lines, requires the solution of the ordinary differential equation (19) and quadratures.

When these funtions $\sigma_1, \xi_1, \eta_1, \zeta_1$ have been obtained from (23) and (15), we find the coördinates of a surface S_1 by means of equations analogous to (3). If x_1, y_1, z_1 denote the coördinates of S_1 and X_1, Y_1, Z_1 denote the direction cosines of the normal to S_1, we find that

$$
\begin{aligned}
& x_1 + iy_1 - (x + iy) = R\,[X_1 + iY_1 - (X + iY)], \\
(24) \qquad & x_1 - iy_1 - (x - iy) = R\,[X_1 - iY_1 - (X - iY)], \\
& z_1 - z = R(Z_1 - Z),
\end{aligned}
$$

where

$$
(25) \qquad R = \tfrac{1}{2}\,\frac{(A\sigma_1 - A_1\sigma)\xi + (A - A_1)\eta + \sigma_1 - \sigma}{A_1 - A}.
$$

Hence S and S_1 are the envelope of a two-parameter family of spheres whose radii are defined by (25). The coördinates, x_0, y_0, z_0, of the centers of these spheres are given by

$$
\begin{aligned}
& x_0 + iy_0 = x + iy - R(X + iY) = \xi\,\frac{\sigma_1 - \sigma}{A - A_1} - \zeta, \\
(26) \qquad & x_0 - iy_0 = x - iy - R(X - iY) = \frac{A_1\sigma - A\sigma_1}{A_1 - A}, \\
& 2z_0 = 2(z - RZ) = \frac{A_1\sigma - A\sigma_1}{A_1 - A}\,\xi + \eta + \frac{\sigma_1 - \sigma}{A_1 - A}.
\end{aligned}
$$

We shall refer to the above relation between S and S_1 as a transformation T_{κ_1}, thus indicating the constant κ_1 which appears in the formulas.

§ 3. *Transformation of Minimal Surfaces.*

Minimal surfaces are the only surfaces of negative curvature for which the following conditions are satisfied simultaneously

$$
E = G, \qquad \mathcal{E} = \mathcal{G}, \qquad F = 0, \qquad \mathcal{F} = 0.
$$

In consequence of (4) and (6), these conditions are equivalent to the following upon Σ:

$$
\left(\frac{\partial \xi}{\partial u}\right)^2 + \left(\frac{\partial \xi}{\partial v}\right)^2 = 0,
$$

$$
\left(\frac{\partial \zeta}{\partial u} - \xi\frac{\partial \eta}{\partial u} + \eta\frac{\partial \xi}{\partial u}\right)^2 + \left(\frac{\partial \zeta}{\partial v} - \xi\frac{\partial \eta}{\partial v} + \eta\frac{\partial \xi}{\partial v}\right)^2 = 0.
$$

If we replace the first of these equations by (7), we must replace the second by

$$
(27) \qquad \frac{\partial \zeta}{\partial \beta} - \xi\frac{\partial \eta}{\partial \beta} + \eta\frac{\partial \xi}{\partial \beta} = 0,
$$

if the functions D and D'', given by (5), are to differ in sign.

By means of (1) and (12) equation (27) is reducible to

$$\eta = \frac{1 + AB}{A'} \frac{\partial \sigma}{\partial \alpha} - \sigma B. \tag{28}$$

If this equation be differentiated with respect to α and the result equated to the value of $\partial \eta / \partial \alpha$ obtainable from (1) and (12), the resulting equation is reducible by means of (13) to

$$\frac{\partial}{\partial \beta} \log \left(\frac{B'}{A'} \frac{\partial \sigma}{\partial \beta} \right) = \frac{B'A}{1 + AB}.$$

From this we get, by integration,

$$\frac{B'}{A'} \frac{\partial \sigma}{\partial \beta} = f(\alpha)(1 + AB),$$

where the function f remains to be determined. Hence, integrating again with respect to β, we find

$$\sigma = A'f \int \frac{1 + AB}{B'} d\beta + f_1(\alpha). \tag{29}$$

When this value is substituted in (13), we obtain

$$fA - \left(\frac{f_1'}{A'} \right)' = \left[\frac{(A'f)'}{A'} \right]' \int \frac{d\beta}{B'} + \left\{ \left[\frac{(A'f)'}{A'} \right] A + A'f \right\}' \int \frac{B d\beta}{B'}. \tag{30}$$

Differentiating with respect to β, we have

$$\left[\frac{(A'f)'}{A'} \right]' + B \left\{ \left[\frac{(A'f)'}{A'} \right] A + A'f \right\}' = 0.$$

Hence the two terms of this equation must be zero, from which we get

$$A'f = c, \tag{31}$$

where c denotes a constant. When this value of f is substituted in (30), we obtain

$$f_1 = c \left[A \int \frac{A d\alpha}{A'} - \int \frac{A^2 d\alpha}{A'} \right] + c_1 A + c_2, \tag{32}$$

where c_1 and c_2 denote constants.

From (13) it is seen that if σ is a solution of this equation, so also is $\sigma + c_1 A + c_2$ where c_1 and c_2 are arbitrary constants. Hence for the present we shall put $c_1 = c_2 = 0$ and furthermore assume $c = 1$. We shall see presently that this choice of the constants does not restrict the generality of our solution. When these values are substituted in (29), we obtain

$$\sigma = \int \frac{1 + AB}{B'} d\beta + A \int \frac{A d\alpha}{A_1} - \int \frac{A^2 d\alpha}{A'}. \tag{33}$$

When this value and (12) are substituted in (1), we find

$$\xi = B,$$

$$(34) \qquad \eta = \int \frac{A\,d\alpha}{A'} + B\int \frac{A^2\,d\alpha}{A'} - B\int \frac{d\beta}{B'} + \int \frac{B\,d\beta}{B'},$$

$$\zeta = -\int \frac{d\alpha}{A'} - B\int \frac{A\,d\alpha}{A'} - B\int \frac{B\,d\beta}{B'} + \int \frac{B^2\,d\beta}{B'}.$$

It is readily found that these values satisfy equations (27) and (28).

From (3) we obtain by means of (12) and (34)

$$x = \frac{1}{2}\left(\int \frac{1-A^2}{A'}\,d\alpha + \int \frac{1-B^2}{B'}\,d\beta\right),$$

$$(35) \qquad y = -\frac{i}{2}\left(\int \frac{1+A^2}{A'}\,d\alpha - \int \frac{1+B^2}{B'}\,d\beta\right),$$

$$z = \int \frac{A\,d\alpha}{A'} + \int \frac{B\,d\beta}{B'},^*$$

and

$$(36) \qquad X = \frac{A+B}{1+AB}, \qquad Y = \frac{i(A-B)}{1+AB}, \qquad Z = \frac{AB-1}{1+AB}.$$

From (35) we find that the linear element of S is

$$(37) \qquad ds^2 = \frac{(1+AB)^2}{A'B'}\,d\alpha\,d\beta,$$

and from (36) that the linear element of the spherical representation is

$$(38) \qquad ds'^2 = \frac{4A'B'}{(1+AB)^2}\,d\alpha\,d\beta.$$

The minimal surface $\bar{S}$ adjoint to S is given by equations of the form

$$\frac{\partial \bar{x}}{\partial \alpha} = i\frac{\partial x}{\partial \alpha}, \qquad \frac{\partial \bar{x}}{\partial \beta} = -i\frac{\partial x}{\partial \beta}.$$

Hence

$$\bar{x} = \frac{i}{2}\left(\int \frac{A^2-1}{A'}\,d\alpha - \int \frac{B^2-1}{B'}\,d\beta\right),$$

$$(39) \qquad \bar{y} = \frac{1}{2}\left(\int \frac{A^2+1}{A'}\,d\alpha + \int \frac{B^2+1}{B'}\,d\beta\right),$$

$$\bar{z} = i\int \frac{A\,d\alpha}{A'} - i\int \frac{B\,d\beta}{B'}.$$

* Since the equations of any minimal surface can be given this form, the particular choice of the constants in equation (32) did not limit the generality of the discussion.

If we substitute the value (33) for σ and also

$$\sigma_1 = \int \frac{1 + A_1 B_1}{B_1'} d\beta + A_1 \int \frac{A_1 \, d\alpha}{A_1'} - \int \frac{A_1^2 d\alpha}{A_1'} \tag{40}$$

in equations (23), we find that these equations are identically satisfied in consequence of the relations (19) and (20). Hence the minimal surface S, defined by (35), and the minimal surface S_1, defined by equations of the same form as (35) but with A and B replaced by A_1 and B_1 respectively, constitute the envelope of a two-parameter family of spheres.

From (25) we find that the radius of the sphere is

$$R = \frac{(1 + AB)(1 + A_1 B_1)}{2\sqrt{A' B'} \cdot \sqrt{A_1' B_1'}}. \tag{41}$$

The coördinates of the center of the sphere are readily found by means of (26).

If $\bar{S}_1$ denotes the adjoint of $\bar{S}$, it is defined by equations analogous to (39). From these equations we find that

$$\begin{aligned} \bar{x} - \bar{x}_1 &= \frac{i}{2\kappa_1^2}\left(\frac{AA_1 - 1}{A - A_1} - \frac{BB_1 - 1}{B - B_1}\right), \\ \bar{y} - \bar{y}_1 &= \frac{1}{2\kappa_1^2}\left(\frac{AA_1 + 1}{A - A_1} + \frac{BB_1 + 1}{B - B_1}\right), \\ \bar{z} - \bar{z}_1 &= \frac{i}{2\kappa_1^2}\left(\frac{A + A_1}{A - A_1} - \frac{B + B_1}{B - B_1}\right). \end{aligned} \tag{42}$$

If X_1, Y_1, Z_1 denote the direction-cosines of the parallel normals to S_1 and $\bar{S}_1$, they are given by equations similar to (36). From these values and (42) we find

$$\Sigma X(\bar{x} - \bar{x}_1) = 0, \qquad \Sigma X_1(\bar{x} - \bar{x}_1) = 0.$$

Hence $\bar{S}$ and $\bar{S}_1$ are the focal sheets of the congruence formed by the joins of corresponding points. Moreover, the curves

$$\alpha + \beta = \text{const.}, \qquad \alpha - \beta = \text{const.}$$

on $\bar{S}$ and $\bar{S}_1$ are the asymptotic lines. Hence this congruence is a W-congruence, and the transformation from $\bar{S}$ to $\bar{S}_1$ is the transformation discovered by Thybaut.*

§ 4. *Identification of the Transformation of* § 2 *with a Former Transformation.*

In this section we shall show that the transformation of § 2 is identical with one formerly established in a different manner.†

* Cf. Introduction.

† *Surfaces with isothermal representation, etc.*, these Transactions, vol. 9 (1908), pp. 149–177. A reference to an equation (a) on page b of this memoir will be indicated by [(a) p. b].

Comparing (38) with [(2) p. 151], we have

$$e^{2\theta} = \frac{(1 + AB)^2}{4A'B'}.$$

When an equation for S_1 analogous to (38) is compared with [(27) p. 156], we find that

$$\frac{\phi^2}{\omega^2} = \frac{(1 + AB)^2(1 + A_1B_1)^2}{16A'A_1'B'B_1'}. \tag{43}$$

This identification is permissible in view of the results of § 3 and § 1 of the present and former memoirs respectively.

If we compare (37) with [(1), p. 151], we see that the coördinates of the minimal surface in § 3 are twice as large as in the former memoir. Moreover, the radius of the sphere, as given by [(30′), p. 157], is ϕ/ω, and consequently one half of R as given by (41) and (43). Furthermore, in § 5 of the former memoir it was shown that the case of minimal surfaces in § 1 of that memoir is a particular case of the general transformation of § 4 of the same memoir. Hence, in order to identify the two transformations, it is necessary and sufficient to show that R as given by (25) is equivalent to [cf. (48), p. 161]

$$R = \frac{2}{\omega}\left[m\phi W_1 + \left(\frac{\partial W}{\partial u}\frac{\partial \phi}{\partial u} - \frac{\partial W}{\partial v}\frac{\partial \phi}{\partial v}\right) - W(m\phi - \omega)\right], \tag{44}$$

where m is a constant and W, W_1 measure the distances from the origin to the tangent planes of a surface S and its transform S_1.

From (3) and similar equations for S_1 we have

$$W = -\frac{\eta + A\zeta}{1 + AB}, \qquad W_1 = -\frac{\eta_1 + A_1\zeta_1}{1 + A_1B_1}.$$

By means of (1), (9) and (12) we find

$$\frac{\partial W}{\partial \alpha} = \frac{A'(B\eta - \zeta)}{(1 + AB)^2}, \qquad \frac{\partial W}{\partial \beta} = \frac{AB'(\eta + A\zeta)}{(1 + AB)^2} + \frac{\sigma B'}{1 + AB}.$$

From [(25), p. 156] and [(6), p. 151], we get (using the notation of the present memoir),

$$\Sigma X_1 \frac{\partial X}{\partial \alpha} = -\frac{e^{-2\theta}}{m\phi}\frac{\partial \phi}{\partial \beta}, \qquad \Sigma X_1 \frac{\partial X}{\partial \beta} = -\frac{e^{-2\theta}}{m\phi}\frac{\partial \phi}{\partial \alpha},$$

which by means of (36) gives

$$\frac{\partial \phi}{\partial \alpha} = -\frac{m\phi[A(A_1B_1 - 1) + A_1 - A^2B_1]}{2A'(1 + A_1B_1)},$$

$$\frac{\partial \phi}{\partial \beta} = -\frac{m\phi[B(A_1B_1 - 1) + B_1 - A_1B^2]}{2B'(1 + A_1B_1)}.$$

When these values are substituted in (44), we get (25), provided that

$$m = \kappa_1^2.$$

Hence the two transformations are identical.

§ 5. *Theorem of Permutability.*

In this section we establish the following theorem of permutability:

*If a surface S with isothermal representation of its lines of curvature be transformed into two surfaces S_1 and S_2 of the same kind by transformations T_{κ_1} and T_{κ_2} respectively, there exists a surface S' which is the transform of S_1 and of S_2 by transformations T'_{κ_2} and T'_{κ_1} respectively; and this surface S' can be obtained without quadratures.**

In order that there may be such a transform S', there must exist functions $A^{(1)}$ and $B^{(1)}$ satisfying the following equations analogous to (19) and (20),

$$(45)\qquad \begin{aligned} &\sqrt{A'_1 A^{(1)'}} = \kappa'_1(A_1 - A^{(1)}), \qquad &&\sqrt{B'_1 B^{(1)'}} = \kappa'_1(B_1 - B^{(1)}),\\ &\sqrt{A'_2 A^{(1)'}} = \kappa'_2(A_2 - A^{(1)}), \qquad &&\sqrt{B'_2 B^{(1)'}} = \kappa'_2(B_2 - B^{(1)}), \end{aligned}$$

where for the present the constants κ'_1 and κ'_2 are undetermined. Also there must exist a function σ' satisfying equations analogous to (23), which by means of the latter can be given the form

$$(46)\qquad \begin{aligned} &\frac{\sqrt{A'_1}}{\sqrt{A^{(1)'}}}\frac{\partial\sigma'}{\partial\alpha} + \kappa'_1(\sigma' - \sigma_1) + \frac{\sqrt{B'}}{\sqrt{B^{(1)'}}}\left[\kappa_1(\sigma - \sigma_1) - \frac{\sqrt{A'_1}}{\sqrt{A'}}\frac{\partial\sigma}{\partial\alpha}\right] = 0,\\ &\frac{\sqrt{B^{(1)'}}}{\sqrt{B'_1}}\frac{\partial\sigma'}{\partial\beta} + \kappa'_1(\sigma' - \sigma_1) + \frac{\sqrt{A^{(1)'}}}{\sqrt{A'}}\left[\kappa_1(\sigma - \sigma_1) - \frac{\sqrt{B'}}{\sqrt{B'_1}}\frac{\partial\sigma}{\partial\beta}\right] = 0, \end{aligned}$$

and similar equations obtained by changing the subscripts 1 into 2.

Eliminating $\partial\sigma'/\partial\alpha$ and $\partial\sigma'/\partial\beta$ from these four equations, we obtain

$$(47)\qquad \begin{aligned} &(\kappa'_1\sqrt{B'_1} - \kappa'_2\sqrt{B'_2})\sigma' - (\kappa'_1\sqrt{B'_1}\sigma_1 - \kappa'_2\sqrt{B'_2}\sigma_2)\\ &\qquad + \frac{\sqrt{A^{(1)'}}}{\sqrt{A'}}[\kappa_1\sqrt{B'_1}(\sigma - \sigma_1) - \kappa_2\sqrt{B'_2}(\sigma - \sigma_2)] = 0,\\ &(\kappa'_1\sqrt{A'_2} - \kappa'_2\sqrt{A'_1})\sigma' - (\kappa'_1\sqrt{A'_2}\sigma_1 - \kappa'_2\sqrt{A'_1}\sigma_2)\\ &\qquad + \frac{\sqrt{B'}}{\sqrt{B^{(1)'}}}[\kappa_1\sqrt{A'_2}(\sigma - \sigma_1) - \kappa_2\sqrt{A'_1}(\sigma - \sigma_2)] = 0. \end{aligned}$$

* Cf. former memoir, l. c., p. 166.

By means of (19) and (20) equations (45) are reducible to

$$\sqrt{A^{(1)'}} = \frac{\kappa_1' \kappa_2'}{\kappa_1 \kappa_2} \sqrt{A'} \frac{\kappa_1 \sqrt{A_2'} - \kappa_2 \sqrt{A_1'}}{\kappa_2' \sqrt{A_1'} - \kappa_1' \sqrt{A_2'}}, \tag{48}$$

and the same equation in the B's. When these values are substituted in (47), it is found that the two equations are consistent if and only if

$$\kappa_1' = \pm \kappa_1, \qquad \kappa_2' = \pm \kappa_2 \tag{49}$$

or

$$\kappa_1' = \pm \kappa_2, \qquad \kappa_2' = \pm \kappa_1, \tag{50}$$

where the same sign must be used in every case.

When the values (49) are used we get

$$\sqrt{A^{(1)'}} = \mp \sqrt{A'}, \qquad \sigma' = \sigma,$$

so that the case (49) is to be excluded.

For the values (50) equation (48) is reducible to

$$A^{(1)} = \frac{\kappa_1^2 A_2 (A - A_1) - \kappa_2^2 A_1 (A - A_2)}{\kappa_1^2 (A - A_1) - \kappa_2^2 (A - A_2)}, \tag{51}$$

and the two expressions (47) may be given the form

$$\begin{aligned} \sigma' = {} & \frac{\kappa_1^2 (B - B_1) - \kappa_2^2 (B - B_2)}{\kappa_1^2 (A - A_1) - \kappa_2^2 (A - A_2)} \frac{A_1 - A_2}{B_1 - B_2} \sigma \\ & + \frac{\kappa_1^2 - \kappa_2^2}{B_1 - B_2} \frac{\sigma_2 (A - A_1)(B - B_2) - \sigma_1 (A - A_2)(B - B_1)}{\kappa_1^2 (A - A_1) - \kappa_2^2 (A - A_2)}. \end{aligned} \tag{52}$$

Hence the theorem is established if these values satisfy (46), as they can be shown to do.

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ON THE BASE OF A RELATIVE NUMBER-FIELD, WITH AN APPLICATION TO THE COMPOSITION OF FIELDS*

BY

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It is a well known fact that in an algebraic number-field of degree n there exist n integers $\alpha_1, \cdots, \alpha_n$ such that every integer of the field can be expressed in the form

$$\omega = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n,$$

where $x_1, x_2, x_3, \cdots, x_n$ are rational integers.

If we now consider a field relative to a given subfield of this field, from the nature of the proof of the above fact it is evident that, if the subfield in question is such that the number of classes of ideals in this field is one, and r the degree of the larger field relative to its subfield, then there exist r integers $\beta_1, \beta_2, \cdots, \beta_r$ such that every integer in the larger field can be expressed in the form

$$\omega = y_1\beta_1 + y_2\beta_2 + \cdots + y_r\beta_r,$$

where now $y_1, y_2, \cdots, y_r$ are integers in the subfield.

When, however, the number of classes of the subfield is greater than one, this is not the case. SOMMER † has shown that in this case for a field which is of the second degree relative to a subfield of the second degree the four numbers composing the base may be taken to be $\omega_1, \omega_2, \beta_1\Omega, \beta_2\Omega$, where ω_1, ω_2 is the base of the subfield and β_1, β_2 the base of an ideal in the subfield and Ω a number of the larger field, not necessarily an integer, but such that $\beta_1\Omega$ and $\beta_2\Omega$ are integers in this field.

In the first part of this paper I establish a similar form for the base of any algebraic number-field relative to any subfield, and in the last part I apply this to the study of the discriminant of the field.

The Base of a Relative Field.

Let K be an algebraic number-field of degree N which contains the subfield k of degree n. We then know that $r = N/n$ is the degree of K relative to k, and if ϑ be any integer which determines the field K, then ϑ is the root of an

* Presented to the Society (Chicago), December 31, 1909.

† SOMMER, *Vorlesungen über Zahlentheorie*, p. 299.

equation

$$x^r + \alpha_1 x^{r-1} + \alpha_2 x^{r-2} + \cdots + \alpha_r = 0, \tag{1}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_r$ are integers in k. We therefore know that every number of K can be expressed in the form

$$\omega = \rho_1 + \rho_2 \vartheta + \rho_3 \vartheta^2 + \cdots + \rho_r \vartheta^{r-1}, \tag{2}$$

where $\rho_1, \rho_2, \cdots, \rho_r$ are numbers of the subfield k. Then, in the same manner as when we consider a field relative to the domain of rational integers, we can show that every integer in K can be expressed in the form

$$\omega = \frac{A_1 + A_2 \vartheta + A_3 \vartheta^2 + \cdots + A_r \vartheta^{r-1}}{D_k(\vartheta)}, \tag{3}$$

where $A_1, A_2, \cdots, A_r$ are integers in k and $D_k(\vartheta)$ is the relative discriminant of ϑ with respect to k.*

Let us now consider all integers of K for which $A_{\tau+1} = A_{\tau+2} = \cdots = A_r = 0$, where τ is one of the integers $1, 2, 3, \cdots, r$. These integers we can indicate as follows

$$\omega_\tau^{(i)} = \frac{O_{1\tau}^{(i)} + O_{2\tau}^{(i)} \vartheta + O_{3\tau}^{(i)} \vartheta^2 + \cdots + O_{\tau\tau}^{(i)} \vartheta^{\tau-1}}{D_k(\vartheta)} \quad (i = 1, 2, 3, \cdots). \tag{4}$$

It is evident that any integer of the type (4) when multiplied by an integer of k gives another integer of the same type, and the sum of any number of such integers is also an integer of this type. But this means that the product of any one of the coefficients $O_{\tau\tau}^{(i)}$ by an integer of k gives another of these integers and, moreover, the sum of any number of them is also another integer of the same set. Hence, since all the integers $O_{\tau\tau}^{(i)}$ belong to k, they form an ideal in this field, which ideal is the highest common ideal factor of all these coefficients. This ideal we shall designate by O_τ.

Thus, for $\tau = 1, 2, 3, \cdots, r$ we get r ideals $O_1, O_2, O_3, \cdots, O_r$ in k. But since $\omega_1^{(i)} = O_{11}^{(i)}/D_k(\vartheta)$ is a number of k and, moreover, from our assumption an integer, and since evidently all integers of k are thus represented, k being a subfield of K, and r being the lowest degree of the equation which ϑ satisfies in k, we conclude that $O_{11}^{(i)}$ $(i = 1, 2, 3, \cdots)$ represent all multiples $D_k(\vartheta)$ in k and therefore the ideal O_1 is the principal ideal $[D_k(\vartheta)]$. Moreover, since $\vartheta \cdot \omega_{\tau-1}^{(i)}$ evidently is an integer of K of the type (4), we conclude that all the integers of $O_{\tau-1}$ also belong to O_τ, and hence that $O_{\tau-1}$ is divisible by O_τ. Therefore, in the sequence $O_1, O_2, O_3, \cdots, O_r$ each ideal is a factor of all that precede it in the sequence, and hence all of them are factors of $D_k(\vartheta)$. Therefore, the integer $D_k(\vartheta)$ belongs to each one of these ideals. But we know that if we choose arbitrarily from an ideal any integer

* HILBERT, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 4 (1894–95), p. 180.

there exists another integer such that the given ideal is the greatest common divisor of these two integers. Hence each of the above ideals must contain an integer such that the ideal in question is the greatest common divisor of this integer and $D_k(\vartheta)$. If we then let this integer in O_τ be the coefficient $O_{\tau\tau}^{(1)}$ we can write

$$O_\tau = [O_{\tau\tau}^{(1)},\ D_k(\vartheta)] \qquad (\tau = 1, 2, 3, \cdots, r).$$

Let us now put $D_k(\vartheta)/O_\tau = I_\tau$. Evidently I_τ is relatively prime to $O_{\tau\tau}^{(1)}$ and, hence, there exists in k an integer β such that

$$O_{\tau\tau}^{(1)} \cdot \beta \equiv 1 \qquad (I_\tau).$$

Hence, if we indicate by $\bar{\omega}_\tau$ the numerator in the expression for $\omega_\tau^{(1)}$, we have the following congruence

$$\bar{\omega}_\tau \cdot \beta = (O_{1\tau}^{(1)} + O_{2\tau}^{(1)}\vartheta + O_{3\tau}^{(1)}\vartheta^2 + \cdots + O_{\tau\tau}^{(1)}\vartheta^{\tau-1}) \cdot \beta \equiv \bar{\Omega}_\tau \qquad (I_\tau),$$

where

$$\bar{\Omega}_\tau = A_{1\tau} + A_{2\tau}\vartheta + A_{3\tau}\vartheta^2 + \cdots + A_{\tau-1\,\tau}\vartheta^{\tau-2} + \vartheta^{\tau-1}.$$

We observe that in $\bar{\Omega}_\tau$ the coefficient of $\vartheta^{\tau-1}$ is unity. But since $\omega_\tau^{(1)} = \bar{\omega}_\tau / D_k(\vartheta)$ is an integer, it follows that $\bar{\omega}_\tau$ is divisible by $D_k(\vartheta)$ and hence by I_τ. From the last congruence it follows that also $\bar{\Omega}_\tau$ is divisible by I_τ.

The quotient $\Omega_\tau = \bar{\Omega}_\tau / D_k(\vartheta)$ is not necessarily an integer in K, but $D_k(\vartheta) \cdot \Omega_\tau$ and $O_{\tau\tau}^{(1)}\Omega_\tau$ are evidently both integers in K.

Let

$$\omega = \frac{\beta_1 + \beta_2\vartheta + \beta_3\vartheta^2 + \cdots + \beta_r\vartheta^{r-1}}{D_k(\vartheta)} \tag{5}$$

be any integer in K. From what we have seen above β_r belongs to the ideal $O_r = [O_{rr}^{(1)},\ D_k(\vartheta)]$ and, hence, there exist in k two integers λ_r and μ_r such that

$$\lambda_r O_{rr}^{(1)} + \mu_r D_k(\vartheta) = \beta_r. \tag{6}$$

But then

$$\beta_r \Omega_r = \lambda_r O_{rr}^{(1)} \Omega_r + \mu_r D_k(\vartheta) \cdot \Omega_r,$$

which is evidently an integer in K and, moreover, this integer when expressed in the form (4) has the same coefficient of ϑ^{r-1} as ω in (5). Hence $\omega - \beta_r\Omega_r$ is an integer of the type

$$\frac{\beta_1' + \beta_2'\vartheta + \beta_3'\vartheta^2 + \cdots \beta_{r-1}'\vartheta^{r-2}}{D_k(\vartheta)}.$$

In the same manner as above we can determine λ_{r-1} and μ_{r-1} such that

$$\beta_{r-1}' = \lambda_{r-1} O_{r-1\,r-1}^{(1)} + \mu_r D_k(\vartheta), \tag{6^a}$$

and $\beta_{r-1}'\Omega_{r-1}$ is then an integer of K of the type (4) when $\tau = r - 1$ and the coefficient of ϑ^{r-2} is β_{r-1}'.

Hence we have

$$\omega - \beta_r \Omega_r - \beta'_{r-1} \Omega_{r-1} = \frac{\beta''_1 + \beta''_2 \vartheta + \cdots + \beta''_{r-2} \vartheta^{r-3}}{D_k(\vartheta)},$$

which is again an integer.

By continuing in this manner we evidently shall at last have

$$\omega - \beta_r \Omega_r - \beta'_{r-2} \Omega_{r-1} - \cdots - \beta_1^{(r-1)} \Omega_1 = 0,$$

whence

$$\omega = \beta_1^{(r-1)} \Omega_1 + \beta_2^{(r-2)} \Omega_2 + \cdots + \beta_r \Omega_r, \tag{7}$$

where, for all values of s from 1 to r, $\beta_s^{(r-s)}$ is an integer belonging to the ideal O_s. Now ω is any integer of K and, hence, every integer of K can be represented in the form (7). Moreover, since $O_{\tau\tau}^{(1)} \Omega_\tau$ and $D_k(\vartheta) \Omega_\tau$ are both integers in K, from (6) and (6^a) and the similar expressions for the remaining $\beta_s^{(r-s)}$ we can conclude that when $\beta_s^{(r-s)}$ belongs to the ideal O_s all numbers represented by (7) are integers of the field K.

If we now let $(i_{1\tau}, i_{2\tau}, \cdots, i_{n\tau})$ be the base of the ideal O_τ, then evidently the N integers $i_{\lambda\tau} \Omega_\tau (\lambda = 1, 2, 3 \cdots n;\ \tau = 1, 2, 3 \cdots r)$ form a base of all the integers in the field K. By replacing in this base $\Omega_1, \Omega_2, \Omega_3, \cdots, \Omega_r$ by their relative conjugates with respect to the field k we get the basis of the respective relative conjugate fields.

The Relative Discriminant of K.

Let us now apply this result to the relative discriminant of the field K with respect to k. The relative discriminant is defined as the greatest common factor of the squares of all the determinants formed from the matrix of the base of K and its relative conjugates. It is easily seen that each of the determinants can be written

$$i_{\lambda_1\tau_1} \cdot i_{\lambda_2\tau_2} \cdots i_{\lambda_r\tau_r} \begin{vmatrix} \Omega_{\tau_1}, & \Omega_{\tau_2}, & \cdots, & \Omega_{\tau_r} \\ \Omega_{\tau_1}^{(1)}, & \Omega_{\tau_2}^{(1)}, & \cdots, & \Omega_{\tau_r}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \Omega_{\tau_1}^{(r-1)}, & \Omega_{\tau_2}^{(r-1)}, & \cdots, & \Omega_{\tau_r}^{(r-1)} \end{vmatrix}, \tag{8}$$

where the upper index is used to designate the relative conjugates. When two of the subscripts τ are alike the expression (8) vanishes. Where no two are alike the determinant factor in (8) is, aside from sign, the same for all determinants of the matrix, and hence the determinant

$$|\Omega_{\tau_k}^{(i)}|$$

must be a common factor of all the determinants of the matrix. Moreover, all the numbers of the ideal product $O_1 \cdot O_2 \cdot O_3 \cdots O_r$ can be represented linearly by means of the products

$$\pi = i_{\lambda_1\tau_1} \cdot i_{\lambda_2\tau_2} \cdots i_{\lambda_r\tau_r}$$

and hence the greatest common factor of the products π must be a factor of the product $O_1 \cdot O_2 \cdots O_r$. But each π is divisible by $O_1 \cdot O_2 \cdots O_r$, there being in π one number from each of these ideals, and hence their highest common factor must be divisible by $O_1 \cdot O_2 \cdots O_r$. Thus $O_1 \cdot O_2 \cdot O_2 \cdots O_r$ is itself the highest common factor of the products π, and hence the relative discriminant of K is

$$D_k = O_1^2 \cdot O_2^2 \cdot O_3^2 \cdots O_r^2 \,|\, \Omega_{\tau_k}^{(i)} |^2,$$

which is an ideal in the subfield k. Since $\Omega_\tau^{(i)} = \bar{\Omega}_\tau^{(i)} / D_k(\vartheta)$ we can write

$$D_k = \frac{O_2^2 \cdot O_3^2 \cdots O_r^2}{D_k(\vartheta)^{2r-2}} \,|\, \bar{\Omega}_\tau^{(i)} |^2$$

remembering that O_1 is the principal ideal $[D_k(\vartheta)]$. Again since

$$\bar{\Omega}_\tau^{(i)} = A_1 + A_2 \vartheta^{(i)} + \cdots + \vartheta^{(i)^{\tau-1}},$$

we have

$$D_k = \frac{O_2^2 \cdot O_3^2 \cdots O_r^2}{D_k(\vartheta)^{2r-2}} D_k(\vartheta).$$

If as before we put $D_k(\vartheta) / O_\tau = I_\tau$, we have

$$D_k = \frac{D_k(\vartheta)}{I_2^2 \cdot I_3^2 \cdots I_r^2}.$$

It is easily seen from the previous deduction that the ideals $I_1, I_2, \cdots, I_r$ are those factors of the relative discriminant of ϑ for which as moduli ϑ is the root of a congruence of lower degree than r in k.

The Discriminant of a Field Formed by the Composition of Two Fields.

We shall next make an application of the relative discriminant of a field in the form found above to determine the nature of the discriminant of a field formed by the composition of two fields, in the case where the degree of the compounded field is equal to the product of the degrees of the two fields from which it is formed, divided by the degree of their greatest common subfield.

The problem of the composition of fields has been studied by HENSEL,* but his results are of a different nature from those obtained by the method here used.

I shall suppose that the field K of degree N is formed by composition from the two fields k_1 and k_2, of degrees n_1 and n_2 respectively. Moreover, let κ be the greatest common subfield to k_1 and k_2 and let the degree of κ be ν. If we let P_1 and P_2 be the degrees of k_1 and k_2 relative to κ, and r_1 and r_2 the degrees of K relative to k_1 and k_2, respectively, we have

$$n_1 = \nu P_1, \qquad n_2 = \nu P_2, \qquad N = r_1 n_1 = r_2 n_2.$$

* HENSEL: *Journal für die reine und angewandte Mathematik*, vol. 105 (1889), p. 329; vol. 120 (1899), p. 99.

But by hypothesis $N = n_1 \cdot n_2/\nu = P_2 n_1$. From this and $N = r_1 n_1$ we conclude that $r_1 = P_2$, and in the same way $r_2 = P_1$. Therefore $N = r_1 \cdot r_2 \cdot \nu$.

I shall indicate by D, d_1, d_2, δ respectively the discriminants of K, k_1, k_2, κ, and the relative discriminant by using as subscript the symbol of the field relative to which it is taken. I shall, moreover, indicate by $N_1(\ )$, $N_2(\ )$, $N(\ )$ the norms taken respectively in the fields k_1, k_2, κ and by N_{k_1}, N_{k_2}, N_κ the relative norms.

From what we have seen above we know that $r_1 \cdot r_2$ is the relative degree of K with respect to κ.

If now α_2 is a number which determines k_2, then α_2 will determine K relative to k_1, and we can therefore write

$$D_{k_1} = \frac{O_{12}^2 \cdot O_{13}^2 \cdots O_{1r_1}^2}{D_{k_1}(\alpha_2)^{2r_1-2}} D_{k_1}(\alpha_2), \tag{9}$$

$$d_{2\kappa} = \frac{O_2^2 \cdot O_3^2 \cdots O_{r_1}^2}{D_\kappa(\alpha_2)^{2r_1-2}} D_\kappa(\alpha_2), \tag{10}$$

where $d_{2\kappa}$ is the relative discriminant of k_2 with respect to κ. But since $P_2 = r_1$ we have

$$D_\kappa(\alpha_2) = D_{k_1}(\alpha_2),$$

because the equation in k_1, which α_2 satisfies, is then the same as the equation in κ, the subfield of k_1. Now $O_{1\tau}$ is an ideal in k_1 and O_τ is an ideal in κ, and since κ is a subfield of k_1 from the definition of $O_{1\tau}$ and O_τ, it follows that O_τ is contained in $O_{1\tau}$ and, hence, $O_{1\tau}$ is a factor of O_τ. Hence $D_\kappa(\alpha_2)/O_\tau = I_\tau$ is a factor of $D_\kappa(\alpha_2)/O_{1\tau} = I_{1\tau}$. If we therefore put

$$\frac{I_{12} \cdot I_{13} \cdots I_{1r_1}}{I_2 \cdot I_3 \cdots I_{r_1}} = T,$$

we see that T contains all the ideal factors of $D_\kappa(\alpha_2)$ in k_1 for which α_2 will satisfy a congruence of degree less than r_1 in k_1 but not in the κ.

Hence from (9) and (10)

$$d_{2\kappa} = T^2 D_{k_1}.$$

Taking the norm in k_1, we get

$$N_1(d_{2\kappa}) = N_1(T^2) N_1(D_{k_1}),$$

or since $d_{2\kappa}$ is an ideal in κ,

$$N(d_{2\kappa}^{r_2}) = N_1(T^2) \cdot N_1(D_{k_1}).$$

But then from the known relation

$$d_2 = \delta^{r_1} N(d_{2\kappa})^* \tag{11}$$

* BACHMANN, *Allgemeine Arithmetik der Zahlenkörper*, p. 452.

we have

$$\frac{d_2^{r_2}}{\delta^{r_1 r_2}} = N_1(T^2) \cdot N_1(D_{k_1}). \tag{12}$$

Similar to (11) we have (Bachmann, loc. cit.)

$$D = d_1^{r_1} \cdot N_1(D_{k_1}),$$

and from this we finally have

$$D = \frac{d_1^{r_1} \cdot d_2^{r_2}}{\delta^{r_1 \cdot r_2} \cdot N_1(T^2)}. \tag{13}$$

From (12) we see that every prime factor of $N_1(T^2)$ must be a factor of d_2, and if in the previous work we interchange α_1 and α_2, and k_1 and k_2 we should get another ideal T' and $N_2(T'^2)$ would be such that every prime factor of this would be a factor of d_1. But as in either case the final result (13) would have to be the same, we see that

$$N_1(T^2) = N_2(T'^2),$$

and this therefore contains only such prime factors as are common to d_1 and d_2.

URBANA, ILL.

THE STRAIN OF A NON-GRAVITATING SPHERE OF VARIABLE DENSITY*

BY

L. M. HOSKINS

§ 1. *Introductory.*

The mathematical problem of the strain of an elastic sphere is of special physical interest because of its relation to the question of the elastic yielding of the earth to disturbing forces. The cases of the problem which have been solved, however, involve assumptions which depart from what is known or thought probable in the case of the earth. The solutions of LAMÉ and KELVIN apply to a homogeneous body acted upon by bodily forces which are known functions of the coördinates. This case disagrees with the facts as regards the earth in two particulars: (1) The earth is certainly far from homogeneous in density and is very likely far from homogeneous in elastic properties. (2) The important bodily forces of self-gravitation cannot be expressed as functions of the coördinates in advance of the solution, since the changes in these forces caused by the strain are of the same order of importance as the external disturbing forces; it is only by assuming incompressibility that it becomes possible to express the gravitational forces without the use of unknown functions.

In a previous paper † was given a solution of the problem for a compressible sphere initially of uniform density, taking account of the changes in the gravitational forces caused by the strain.

In the present paper will be found a solution for the case of a non-gravitating sphere of which the density is a given function of the distance from the center, the elastic moduli being assumed to be uniform in value throughout the body.

By combining these two cases, and introducing the additional assumption that the elastic moduli are functions of the distance from the center, we should have perhaps the nearest possible approximation to a completely general statement of the problem presented by the actual earth. While there is no difficulty in forming the differential equations for this general problem in a manner analogous to that employed in the two cases which have been solved, a solution of these general equations has not thus far been obtained.

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† Transactions of the American Mathematical Society, vol. 11 (1910), p. 203.

The case treated in this paper appears, however, to possess some interest and importance, being a generalization in one important particular of the problem which formed the basis of the original estimates of the rigidity of the earth.

§ 2. *Statement of Problem.*

The problem here treated is the following:

To determine the strain of an elastic sphere of which the density is a given function of the distance from the center, under the action of bodily forces having a potential, but free from stress on the bounding surface. The potential may be any function which, when expressed in polar coördinates, is developable as to θ and ϕ in a series of spherical surface harmonics.

§ 3. *Equations of Equilibrium of Volume-Element.*

Using polar coördinates r, θ, ϕ, let u_r, u_θ, u_ϕ be the elastic displacements, ϖ_r, ϖ_θ, ϖ_ϕ the rotation-components, Δ the cubical dilatation, F_r, F_θ, F_ϕ the components of bodily force per unit mass, $\lambda + \frac{2}{3}\mu$, μ the bulk-modulus and rigidity-modulus, ρ the density.

The three equations of equilibrium for an element of volume are the following:

$$(1) \qquad (\lambda + 2\mu)\frac{\partial \Delta}{\partial r} - \frac{2\mu}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(\varpi_\phi \sin\theta) - \frac{\partial \varpi_\phi}{\partial\phi}\right] + \rho F_r = 0,$$

$$(2) \qquad (\lambda + 2\mu)\frac{1}{r}\frac{\partial \Delta}{\partial\theta} - \frac{2\mu}{r \sin\theta}\left[\frac{\partial \varpi_r}{\partial\phi} - \frac{\partial}{\partial r}(r\varpi_\phi \sin\theta)\right] + \rho F_\theta = 0,$$

$$(3) \qquad (\lambda + 2\mu)\frac{1}{r \sin\theta}\frac{\partial \Delta}{\partial\phi} - \frac{2\mu}{r}\left[\frac{\partial}{\partial r}(r\varpi_\theta) - \frac{\partial \varpi_r}{\partial\theta}\right] + \rho F_\phi = 0;$$

in which

$$(4) \qquad \Delta = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 u_r \sin\theta) + \frac{\partial}{\partial\theta}(r u_\theta \sin\theta) + \frac{\partial}{\partial\phi}(r u_\phi)\right],$$

$$(5) \qquad 2\varpi_r = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(r u_\phi \sin\theta) - \frac{\partial}{\partial\phi}(r u_\theta)\right],$$

$$(6) \qquad 2\varpi_\theta = \frac{1}{r \sin\theta}\left[\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(r u_\phi \sin\theta)\right],$$

$$(7) \qquad 2\varpi_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right].$$

Let W denote the potential of the bodily forces, and assume that

$$(8) \qquad W = \Sigma R_i S_i,$$

in which S_i denotes a spherical surface harmonic of order i, and R_i a function of r. Then

$$
(9) \qquad \begin{aligned} F_r &= \Sigma \frac{dR_i}{dr} S_i, \\ F_\theta &= \Sigma \frac{R_i}{r} \frac{\partial S_i}{\partial \theta}, \\ F_\phi &= \Sigma \frac{R_i}{r \sin\theta} \frac{\partial S_i}{\partial \phi}. \end{aligned}
$$

A solution of equations (1), (2) and (3) may be formed by putting together solutions corresponding to the separate terms in the value of W. We therefore confine the following discussion to the solution corresponding to a single term $R_i S_i$.

We now assume

$$
(10) \qquad u_r = u S_i, \qquad u_\theta = v \frac{\partial S_i}{\partial \theta}, \qquad u_\phi = \frac{v}{\sin\theta} \frac{\partial S_i}{\partial \phi},
$$

in which u and v are functions of r only. Substituting these values in equations (4), $\cdots$, (7), and making use of the differential equation

$$
(11) \qquad \frac{1}{\sin^2\theta} \frac{\partial^2 S_i}{\partial \phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S_i}{\partial \theta}\right) + i(i+1) S_i = 0
$$

which is satisfied by any spherical surface harmonic of order i, we find

$$
(12) \qquad \Delta = \left[\frac{1}{r^2} \frac{d(r^2 u)}{dr} - i(i+1)\frac{v}{r}\right] S_i,
$$

$$
(13) \qquad 2\varpi_r = 0,
$$

$$
(14) \qquad 2\varpi_\theta = \frac{1}{r}\left[u - \frac{d(rv)}{dr}\right] \frac{1}{\sin\theta} \frac{\partial S_i}{\partial \phi},
$$

$$
(15) \qquad 2\varpi_\phi = -\frac{1}{r}\left[u - \frac{d(rv)}{dr}\right] \frac{\partial S_i}{\partial \theta}.
$$

If the foregoing values of F_r, F_θ, F_ϕ, Δ, ϖ_r, ϖ_θ and ϖ_ϕ be substituted in (1), (2) and (3), these are freed from θ and ϕ. Equations (2) and (3) in fact become identical, and there remain to be satisfied two ordinary differential equations for determining u, v as functions of r. Writing R for R_i, these equations are the following:

$$
(16) \qquad \begin{aligned} &(\lambda + 2\mu) \frac{d}{dr}\left(\frac{1}{r^2} \frac{d(r^2 u)}{dr} - i(i+1)\frac{v}{r}\right) \\ &\qquad + \frac{i(i+1)\mu}{r^2}\left(\frac{d(rv)}{dr} - u\right) + \rho \frac{dR}{dr} = 0, \end{aligned}
$$

$$(17)\qquad (\lambda+2\mu)\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right)+\mu\frac{d}{dr}\left(\frac{d(rv)}{dr}-u\right)+\rho R=0.$$

It is now convenient to replace r by $x=r/a$ as independent variable, and to introduce y and z, defined by the equations

$$(18)\qquad y=\frac{\Delta}{S_i}=\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r},$$

$$(19)\qquad z=\frac{2\varpi_\phi}{\partial S_i/\partial\theta}=\frac{1}{r}\left(\frac{d(rv)}{dr}-u\right),$$

thus reducing the differential equations to the following:

$$(20)\qquad (\lambda+2\mu)\frac{dy}{dx}+i(i+1)\mu\frac{z}{x}+\rho\frac{dR}{dx}=0,$$

$$(21)\qquad (\lambda+2\mu)y+\mu\frac{d(xz)}{dx}+\rho R=0.$$

§ 4. *Surface Conditions.*

We now have to express the condition that the bounding surface is free from stress, that is,

$$(22)\qquad \widehat{rr}=0,\qquad \widehat{r\theta}=0,\qquad \widehat{r\phi}=0\qquad (x=1).$$

Using (10) in the general formulas for these stresses, we find

$$(23)\qquad \widehat{rr}=\lambda\Delta+2\mu\frac{\partial u_r}{\partial r}=\left(\lambda y+2\mu\frac{du}{dr}\right)S_i,$$

$$(24)\qquad \widehat{r\theta}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)=\mu\left(\frac{dv}{dr}-\frac{v}{r}+\frac{u}{r}\right)\frac{\partial S_i}{\partial\theta},$$

$$(25)\qquad \widehat{r\phi}=\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)=\mu\left(\frac{dv}{dr}-\frac{v}{r}+\frac{u}{r}\right)\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi},$$

so that equations (22) reduce to the two

$$(26)\qquad \lambda y+2\mu\frac{du}{dr}=0,$$

$$(27)\qquad \frac{dv}{dr}-\frac{v}{r}+\frac{u}{r}=0,$$

to be satisfied when $x=1$.

§ 5. *Solution of Differential Equations.*

By eliminating successively z and y between equations (20) and (21) we obtain the two equations

$$(28)\quad \left(x^2\frac{d^2}{dx^2}+2x\frac{d}{dx}-i(i+1)\right)\left((\lambda+2\mu)y+\rho R\right)-\frac{d}{dx}\left(x^2R\frac{d\rho}{dx}\right)=0,$$

$$(29)\quad \mu\left(x^2\frac{d^2z}{dx^2}+2x\frac{dz}{dx}-i(i+1)z\right)+xR\frac{d\rho}{dx}=0.$$

The solutions of these equations are the following:

$$(30)\quad \begin{aligned}(2i+1)[(\lambda+2\mu)y+\rho R]&=x^i\int x^{-i-1}\frac{d}{dx}\left(x^2R\frac{d\rho}{dx}\right)dx\\ &\quad -x^{-i-1}\int x^i\frac{d}{dx}\left(x^2R\frac{d\rho}{dx}\right)dx,\end{aligned}$$

$$(31)\quad (2i+1)\mu z=-x^i\int x^{-i-1}\left(xR\frac{d\rho}{dx}\right)dx+x^{-i-1}\int x^i\left(xR\frac{d\rho}{dx}\right)dx,$$

in which, for the general solutions, each integral must be understood to involve an arbitrary constant. These constants are here omitted, since they are accounted for in the complementary solutions to be obtained presently. Modifying by integration by parts, we find

$$(32)\quad (\lambda+2\mu)y=-\frac{1}{2i+1}\left[(i+1)x^i\int\rho\frac{d}{dx}\left(\frac{R}{x^i}\right)dx+\frac{i}{x^{i+1}}\int\rho\frac{d(Rx^{i+1})}{dx}dx\right],$$

$$(33)\quad \mu z=\frac{1}{2i+1}\left[x^i\int\rho\frac{d}{dx}\left(\frac{R}{x^i}\right)dx-\frac{1}{x^{i+1}}\int\rho\frac{d(Rx^{i+1})}{dx}dx\right].$$

These values of y and z may now be substituted in (18) and (19), giving two simultaneous linear differential equations for determining u and v. The solutions of these equations may conveniently be expressed as follows:

From (18) and (19) we readily obtain the equations

$$(34)\quad \frac{d}{dx}(iv-u)+\frac{i+2}{x}(iv-u)=a(iz-y),$$

$$(35)\quad \frac{d}{dx}[(i+1)v+u]-\frac{i-1}{x}[(i+1)v+u]=a[(i+1)z+y],$$

of which the solutions are

$$(36)\quad iv-u=\frac{a}{x^{i+2}}\int(iz-y)x^{i+2}dx,$$

$$(i+1)v+u=ax^{i-1}\int\frac{(i+1)z+y}{x^{i-1}}\,dx. \tag{37}$$

These are easily solved for u and v.

The forms of the complementary solutions are easily seen by considering the constants of integration in (32), (33), (36) and (37). The values of the constants must be determined so as to satisfy (18), (19), (20) and (21). The complementary solutions are thus found to be the following:

$$\begin{aligned} y &= C_i x^i + C_{-i-1}x^{-i-1}, \\ z &= D_i x^i + D_{-i-1}x^{-i-1}, \\ \frac{u}{r} &= A_i x^i + A_{-i-1}x^{-i-1} + A_{i-2}x^{i-2} + A_{-i-3}x^{-i-3}, \\ \frac{v}{r} &= B_i x^i + B_{-i-1}x^{-i-1} + B_{i-2}x^{i-2} + B_{-i-3}x^{-i-3}; \end{aligned} \tag{38}$$

in which the twelve constants must satisfy the eight equations

$$\begin{aligned} (\lambda+2\mu)C_i+(i+1)\mu D_i &= 0, \\ (i+3)A_i-i(i+1)B_i-C_i &= 0, \\ A_i-(i+2)B_i+D_i &= 0, \\ (\lambda+2\mu)C_{-i-1}-i\mu D_{-i-1} &= 0, \\ (i-2)A_{-i-1}+i(i+1)B_{-i-1}+C_{-i-1} &= 0, \\ A_{-i-1}+(i-1)B_{-i-1}+D_{-i-1} &= 0, \\ A_{i-2}-iB_{i-2} &= 0, \\ A_{-i-3}+(i+1)B_{-i-3} &= 0. \end{aligned} \tag{39}$$

The complete solutions will thus involve four arbitrary constants; by assigning values to these we may satisfy four equations of condition at the boundary.

This solution covers the case of a spherical shell with bounding surfaces free from stress, or with arbitrary surface stress of a restricted type (corresponding to the restriction imposed on the potential of the bodily forces), or with arbitrary surface displacements of a similarly restricted type.

For the case of the complete sphere, negative powers of x must be rejected, so that the only non-vanishing coefficients in (38) are A_{i-2}, A_i, B_{i-2}, B_i, C_i, D_i; these being connected by four relations included among (39). There remain two arbitrary constants for satisfying the surface conditions (26) and (27).

§ 6. *Case in which the Potential is a Harmonic Function.*

If W is a harmonic function, each term $R_i S_i$ in its development must satisfy Laplace's equation, so that

$$R_i = Ar^i + Br^{-i-1}.$$

Rejecting the negative power, we assume R_i or R to have the form Ar^i. Equations (32) and (33) then reduce to

$$(\lambda + 2\mu)y = -\frac{i}{(2i+1)x^{i+1}}\int \rho \frac{d(Rx^{i+1})}{dx}\,dx, \tag{40}$$

$$\mu z = -\frac{1}{(2i+1)x^{i+1}}\int \rho \frac{d(Rx^{i+1})}{dx}\,dx. \tag{41}$$

The further discussion will be restricted to the case in which $i = 2$.

§ 7. *Sphere Strained by Centrifugal Force.*

If the bodily forces are the centrifugal forces due to uniform rotation about a fixed axis, we have

$$W = -\frac{cg}{3a} r^2 P_2, \tag{42}$$

in which c denotes the ratio of equatorial centrifugal force to gravity, and P_2 is the zonal harmonic $\frac{3}{2}\cos^2\theta - \frac{1}{2}$. We now take

$$S_2 = -\tfrac{2}{3}P_2 = \tfrac{1}{3} - \cos^2\theta, \tag{43}$$

$$R_2 = R = \frac{cgr^2}{2a} = \frac{cga}{2}x^2. \tag{44}$$

In accordance with the notation of a former paper,* let e denote the ellipticity of a surface originally spherical and of radius r, and α the angular displacement of a radius vector for which $\theta = 45°$. With the above value of S_2, we have

$$e = \frac{u}{r}, \qquad \alpha = \frac{v}{r}. \tag{45}$$

In writing the complete solutions of the differential equations for this case, we take A_0 and A_2 as arbitrary constants, expressing B_0, B_2, C_2, D_2 in terms of them by means of (39). We thus obtain the equations

$$y = -\frac{7\mu}{\lambda}A_2x^2 - \frac{cga}{(\lambda+2\mu)x^3}\int_0^x \rho x^4\,dx, \tag{46}$$

* Transactions American Mathematical Society, vol. 11 (1910), p. 206.

$$z = \frac{7(\lambda + 2\mu)}{3\lambda} A_2 x^2 - \frac{cga}{2\mu x^3} \int_0^x \rho x^4 \, dx, \tag{47}$$

$$\begin{aligned} e = A_0 + A_2 x^2 + \frac{cga}{30\mu(\lambda + 2\mu)} \Bigg[& \frac{5(3\lambda + 5\mu)}{x^3} \int_0^x \rho x^4 \, dx \\ & - \frac{9(\lambda + \mu)}{x^5} \int_0^x \rho x^6 \, dx - 2(3\lambda + 8\mu) \int_0^x \rho x dx \Bigg], \end{aligned} \tag{48}$$

$$\begin{aligned} \alpha = \tfrac{1}{2} A_0 + \frac{5\lambda + 7\mu}{6\lambda} A_2 x^2 \\ + \frac{cga}{30\mu(\lambda+2\mu)} \left[\frac{5\mu}{x^3} \int_0^x \rho x^4 dx + \frac{3(\lambda+\mu)}{x^5} \int_0^x \rho x^6 dx - (3\lambda+8\mu) \int_0^x \rho x dx \right]. \end{aligned} \tag{49}$$

Substituting these values in (26) and 27), we obtain the following equations for determining A_0 and A_2:

$$\begin{aligned} 2A_0 - A_2 + \frac{cga}{30\mu(\lambda + 2\mu)} \Bigg[& -10(9\lambda + 10\mu) \int_0^1 \rho x^4 dx \\ & + 72(\lambda + \mu) \int_0^1 \rho x^6 \, dx - 4(3\lambda + 8\mu) \int_0^1 \rho x dx \Bigg] = 0, \end{aligned} \tag{50}$$

$$\begin{aligned} A_0 + \frac{8\lambda + 7\mu}{3\lambda} A_2 + \frac{cga}{30\mu(\lambda + 2\mu)} \Bigg[& 5(3\lambda + 2\mu) \int_0^1 \rho x^4 dx \\ & - 24(\lambda + \mu) \int_0^1 \rho x^6 \, dx - 2(3\lambda + 8\mu) \int_0^1 \rho x dx \Bigg] = 0. \end{aligned} \tag{51}$$

Solving, we find

$$\begin{aligned} A_0 = \frac{cga}{30\mu(\lambda + 2\mu)(19\lambda + 14\mu)} \Bigg[& 25(27\lambda^2 + 56\lambda\mu + 28\mu^2) \int_0^1 \rho x^4 \, dx \\ & - 504(\lambda + \mu)^2 \int_0^1 \rho x^6 \, dx + 2(3\lambda + 8\mu)(19\lambda + 14\mu) \int_0^1 \rho x dx \Bigg], \end{aligned} \tag{52}$$

$$A_2 = \frac{cga}{30\mu(\lambda + 2\mu)(19\lambda + 14\mu)} \cdot 360\lambda(\lambda + \mu) \left(\int_0^1 \rho x^6 \, dx - \int_0^1 \rho x^4 \, dx \right). \tag{53}$$

Of especial interest are the surface values of e and α:

$$e_1 = \frac{cga}{2\mu(19\lambda + 14\mu)} \left[5(8\lambda + 7\mu) \int_0^1 \rho x^4 \, dx - 21(\lambda + \mu) \int_0^1 \rho x^6 \, dx \right], \tag{54}$$

$$\alpha_1 = \frac{cga}{4\mu(19\lambda + 14\mu)} \left[5\lambda \int_0^1 \rho x^4 \, dx + 14(\lambda + \mu) \int_0^1 \rho x^6 \, dx \right]. \tag{55}$$

If ρ is constant it is easy to show that the foregoing equations give the following well-known results:

$$e = \frac{c}{2(19\lambda + 14\mu)}\left[2(4\lambda + 3\mu) - (3\lambda + 2\mu)x^2\right]\frac{\rho g a}{\mu}, \tag{56}$$

$$\alpha = \frac{c}{4(19\lambda + 14\mu)}\left[2(4\lambda + 3\mu) - (5\lambda + 4\mu)x^2\right]\frac{\rho g a}{\mu}. \tag{57}$$

§ 8. *Laplace's Law of Density.*

To illustrate the application of the above formulas to a case of variable density, assume Laplace's law, expressed by the formula

$$\rho = \rho_0 \frac{\sin \kappa x}{\kappa x} = \frac{\rho_m \kappa^3}{3(\sin \kappa - \kappa \cos \kappa)} \cdot \frac{\sin \kappa x}{\kappa x}, \tag{58}$$

in which ρ_m is the mean density and ρ_0 the value of ρ at the center. We then find

$$\int_0^x \rho x dx = \frac{\rho_m \kappa}{3(\sin \kappa - \kappa \cos \kappa)}(1 - \cos \kappa x), \tag{59}$$

$$\frac{1}{x^3}\int_0^x \rho x^4 dx = \frac{\rho_m \kappa}{3(\sin \kappa - \kappa \cos \kappa)}\left[\frac{6 - \kappa^2 x^2}{\kappa^2 x^2}\cos \kappa x - \frac{6 - \kappa^2 x^2}{\kappa^3 x^3}\sin \kappa x\right], \tag{60}$$

$$\frac{1}{x^5}\int_0^x \rho x^6 dx = \frac{\rho_m \kappa}{3(\sin \kappa - \kappa \cos \kappa)}\left[\frac{120 - 60\kappa^2 x^2 + 5\kappa^4 x^4}{\kappa^5 x^5}\sin \kappa x - \frac{120 - 20\kappa^2 x^2 + \kappa^4 x^4}{\kappa^4 x^4}\cos \kappa x\right]. \tag{61}$$

If the ratio of mean density to surface density is assumed to be 2, we find $\kappa = 2.461$, and

$$\int_0^1 \rho x^4 dx = .168\rho_m, \qquad \int_0^1 \rho x^6 dx = .108\rho_m. \tag{62}$$

These values substituted in (54) and (55) give

$$\frac{e_1}{c} = \frac{4.45\lambda + 3.61\mu}{2(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}, \tag{63}$$

$$\frac{\alpha_1}{c} = \frac{2.35\lambda + 1.51\mu}{4(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}. \tag{64}$$

Comparing with the surface values of e and α for a sphere of uniform density ρ_m,

$$\frac{e_1}{c} = \frac{5\lambda + 4\mu}{2(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}, \tag{65}$$

$$\frac{\alpha_1}{c} = \frac{3\lambda + 2\mu}{4(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}, \tag{66}$$

it appears that the assumption of variable density decreases e_1 by about 11 per cent and α_1 by about 22 per cent.

We may also compare the values at the center, which with Laplace's law of density are found to be

$$\frac{2\alpha_0}{c} = \frac{e_0}{c} = \frac{A_0}{c} = \frac{124.4\lambda^2 + 349.1\lambda\mu + 191.8\mu^2}{30(\lambda + 2\mu)(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}, \tag{67}$$

while the corresponding values for uniform density ρ_m are

$$\frac{2\alpha_0}{c} = \frac{e_0}{c} = \frac{4\lambda + 3\mu}{19\lambda + 14\mu} \frac{\rho_m g a}{\mu} = \frac{120\lambda^2 + 330\lambda\mu + 180\mu^2}{30(\lambda + 2\mu)(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}. \tag{68}$$

The central values of e and α are thus slightly increased by the assumption of variable density.

§ 9. *Gravitating Sphere.*

The results of the above solution cannot be applied directly to a gravitating body of size and mass comparable with those of the earth, since the changes in the gravitational forces caused by the strain will be of the same order of importance as the disturbing forces. There is no difficulty in expressing the gravitational forces in terms of the strain and the density and thus forming the differential equations which must replace (16) and (17) above given, but the solution of these equations appears to involve serious analytical difficulties.*

* In expressing the equations of interior equilibrium in terms of the strain, for the case of a gravitating body, it is necessary to assume that the changes of stress and strain from their initial values are related in the same way as if the initial condition were one of zero stress and strain. This assumption is the basis of the solution given for the case of a compressible sphere in the paper cited above, the reasoning being in this respect substantially identical with that previously applied to the case of incompressibility. (See A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*, 2d ed., p. 249.) It has sometimes been assumed that this reasoning is valid only in the case of incompressibility. Thus LOVE states that if the sphere is homogeneous and incompressible, the initial stress may be taken to be of the nature of a hydrostatic pressure, its value being then easily determined, and the terms which correspond to this initial stress in the equations of equilibrium being found to cancel with those representing the initial body-force. It seems clear, however, first that even in an incompressible body the initial condition may involve tangential stress, and secondly that the assumption of no initial tangential stress is not essential to the argument. Whatever values the initial stress-components may have, the terms representing them in the equations of equilibrium will cancel the terms representing the initial body-forces.

If the body is assumed incompressible and of uniform density, the effect of gravitation can be taken account of synthetically by a method employed by Kelvin.* This method consists in combining the results for the two cases of a gravitating fluid and a non-gravitating elastic solid. It does not appear that similar reasoning can be applied if either of the assumptions of incompressibility and uniform density is dropped, since the distribution of the forces of gravitation cannot be assumed to be the same for the strained elastic solid as for the fluid.

The method of applying this principle in the solutions cited above, both for the case of incompressibility and for that of compressibility, consists in comparing the forces acting upon a volume-element in the strained condition with those initially acting upon the same volume-element. It would perhaps be more reasonable to deal with an individual element of matter, using the increments of the stress-components acting on that element in expressing the stress-strain relations by the usual formulas. The increment of stress for an individual element would be in equilibrium with the increment of body-force acting upon that element, which would consist in part of the change in the gravitational force due to the displacement of the attracted element. This would introduce additional terms into the equations of equilibrium, but would add no difficulty to their solution.

* THOMSON and TAIT's *Natural Philosophy*, § 840, Ed. of 1890.

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